A new approach of fracture mechanics

Răzvan Răducanu

Abstract

The present paper is a first step towards modelling cracks within the frame of finite element anisotropic meshes. We will consider 2-dimensional models with straight cracks. The non-classical FE model is proposed, and it is noticed that the stiffness matrix is positive definite and symmetric. The proposed theory remains valid for kinked cracks as well.

M.S.C. 2000: 74B05, 74R99, 90C90.
Key words: finite element, anisotropic mesh, crack, numerical methods.

§1. Introduction

There are a great number of numerical methods used to solve boundary value problems that occur in fracture mechanics, like finite difference method, finite element method, finite volume method, boundary element method, meshless methods, etc. In this paper we will investigate a non-classical finite element method, the method of anisotropic finite elements. The solutions of a great number of boundary value problems could be called anisotropic because they exhibit different behaviour on certain directions, e.g., little variation on some directions and bigger variation on other directions. In order to characterize the anisotropic behaviour of the solution of the elliptic boundary value problems one can use anisotropic meshes. This kind of mesh violates the condition which characterizes isotropic meshes: the ratio between the diameters of the circumscribed and inscribed spheres of a finite element is bounded. Extensive work on anisotropic FE has been done by Zienkiewicz and Wu ([7]), Stein and Ohnimus ([5]), etc. In this paper we propose a first attempt in modelling cracks within the frame of anisotropic meshes. We will use barycentric coordinates and bubble functions in order to implement our theory.

§2. Basic equations

Let Ω be an open, bounded, polygonal domain in the 2-dimensional Euclidian space with an interior straight crack. Suppose that the domain D is occupied by an isotropic and homogenous medium. The boundary Γ is composed by mutually disjoint Γ_u, Γ_t and Γ_c, such that $\Gamma = \Gamma_u \bigcup \Gamma_t \bigcup \Gamma_c$. Prescribed displacements are imposed on

Applied Sciences, Vol.6, 2004, pp. 39-43.

Γ_u, while tractions are imposed on Γ_t. The crack surface Γ_c is assumed to be traction free. The basic equations of equilibrium of the linear elasticity are [6]:

The equilibrium equations

$$divT + f = 0 \text{ on D}; \tag{2.1}$$

the constitutive equations

$$T = C\left[E\right], \tag{2.2}$$

where

$$C\left[E\right] = \lambda tr(E)\delta + 2\mu E; \tag{2.3}$$

the strain-displacement relations

$$E(u) = \frac{1}{2}(\nabla u + \nabla u^T) \quad \text{on D}, \tag{2.4}$$

where u is the displacement vector, T is the Cauchy stress tensor, $E(u)$ is the strain tensor, f is the specific body force, λ and μ are constants characteristic to the material. We shall consider the following boundary conditions:

$$u = 0 \text{ on } \Gamma_u, \tag{2.5}$$

$$Tn = \bar{t} \text{ on } \Gamma_t, \tag{2.6}$$

$$Tn = 0 \text{ on } \Gamma_c, \tag{2.7}$$

where $\bar{t}$ are continuous functions given on the specified boundary parts, and n is the outward unit normal. We define the space of admissible displacement fields ([4]) by

$$U = \{v \in V : v = \overline{u} \text{ on } \Gamma_u; \;\; v \text{ discontinuous on } \Gamma_c\}, \tag{2.8}$$

where the space V is related to the regularity of the solution. A detailed discussion about the definition of V can be found in [2]. The test function space is defined by

$$U_0 = \{v \in V : v = 0 \text{ on } \Gamma_u; \;\; v \text{ discontinuous on } \Gamma_c\}. \tag{2.9}$$

In the following we take $V \equiv H^1$, where $H^1(\Omega)$ is the usual Sobolev space. The weak form of the equilibrium equations should provide $u \in U_0$ such that

$$\int_\Omega E(u) \cdot C\left[E(u)\right] d\Omega = \int_\Omega f \cdot v d\Omega + \int_{\Gamma_t} \bar{t} \cdot v d\Gamma, \forall v \in U_0. \tag{2.10}$$

It can be shown [1] that the weak form (2.10) is equivalent with the strong one (2.1)-(2.7).

§3. The finite element discretization

Let's begin with a classical finite element approach and consider a triangulation Im_h, which covers completely Ω, consisting of triangles T. Suppose that each edge of any triangle is entirely contained in Γ_u, Γ_t and Γ_c, respectively. We denote $M_h \subset C(\overline{\Omega})$ the space of piecewise linear, continuous functions over Im_h. Let $M_{0h} = M_h \bigcap H_0^1(\Omega)$.

Thus, the approximate finite element problem for (2.10) is requires to find $u_h \in M_{0,h}$ such that:

$$\int_{\Omega} E(u_h) \cdot C\,[E(u_h)]d\Omega = \int_{\Omega} f \cdot v_h d\Omega + \int_{\Gamma_t} \bar{t} \cdot v_h d\Gamma, \forall v_h \in M_{0h}. \tag{3.1}$$

In the following we propose a local finite element subspace, characteristic to the anisotropic meshes. In order to do this, let $T \in \mathrm{Im}_h$ be an arbitrary fixed triangle; let ω_T be the domain formed by T and all adjacent triangles that have at most a common edge E with T:

$$\omega_T = \bigcup_{T' \bigcap T = E} T'. \tag{3.2}$$

Let $\lambda_{T,1}, ...\lambda_{T,3}$ be the barycentric coordinates of T. We define the *element bubble function* $b_T \in P^4(T)$ via

$$b_T = 256 \cdot \lambda_{T,1} \cdot \lambda_{T,2} \cdot \lambda_{T,3} \text{ on } T, \tag{3.3}$$

where $P^3(T)$ represents the space of polynomials of order 3 or less. Let T_1 and T_2 be two triangles belonging to Im_h which have a common edge E. Let us denote:

$$\omega_E = T_1 \bigcup T_2. \tag{3.4}$$

In case that E is a boundary edge, let $\omega_E = T$, $T \supset E$. Let's define the *edge bubble function*

$$b_E = 27 \cdot \lambda_{T_\alpha,1} \cdot \lambda_{T_\alpha,2} \cdot \lambda_{T_\alpha,3} \text{ on } T_\alpha, \alpha = 1,2. \tag{3.5}$$

Suppose that both the previously defined bubble functions are zero outside their original domain of definition. We define the space

$$H_0^1(\omega_T) = \{v \in H^1(\omega_T) | v = 0 \text{ on } (\partial\omega_T - \Gamma) \bigcup \Gamma_u\}. \tag{3.6}$$

Thus, the local finite element space $V_T \subset H_0^1(\omega_T)$ will be given by:

$$V_T = span\{b_T, b_{E_1}, ..., b_{E_k}, b_{E_{k+1},1}, b_{E_{k+1},2}, ..., b_{E_m,1}, b_{E_m,2}\}, \tag{3.7}$$

where $E_1, ..., E_k$ denote all interior and Neumann edges (Γ_t), $E_{k+1}, ..., E_m$ denote all the edges that belong to the crack $(\subset \Gamma_c)$, $E_{m+1}, ...E_3$ denote all the Dirichlet edges (Γ_u), $1 \le k \le m \le 3$, and

$$b_{E,i} = b_E \cdot \lambda_{T,\alpha}, \alpha = 1,2. \tag{3.8}$$

In this way, an arbitrary function from V_T can be written as

$$v_T = \xi_0 \cdot b_T + \sum_{i=1}^{k} \xi_i \cdot b_{E_i} + \sum_{i=k+1}^{m} \sum_{\beta=1}^{2} \xi_{i,\beta} \cdot b_{E_i,\beta}, \quad \xi_i, \xi_{i,\beta} \in \mathbb{R}. \tag{3.9}$$

We define the vector

$$\Phi = \{b_T, b_{E_1}, ..., b_{E_k}, b_{E_{k+1},1}, b_{E_{k+1},2}, ..., b_{E_m,1}, b_{E_m,2}\} \tag{3.10}$$

and let $E_1, ..., E_k$ denote all the interior and Neumann edges; let $E_{k+1}, ..., E_m$ denote all the edges that belong to the crack ($\subset \Gamma_c$), and let $E_{m+1}, ...E_3$ denote all the Dirichlet edges, $1 \leq k \leq m \leq 3$.

Consider the vector

$$v = (\xi_0, \xi_1, ..., \xi_k, \xi_{k+1,1}, \xi_{k+1,2}, ..., \xi_{m,1}, \xi_{m,2}) \in R^{1+k+2(m-k)}. \tag{3.11}$$

We shall look for a solution of (3.1) in V_T of the form

$$v_T = \Phi \cdot v \in V_T, \tag{3.12}$$

with the unknown v. The formula (3.12) has the aspect of a classical FE solution. Further on the algorithm follows exactly the FE classical ideas, i.e. we shall finally have the algebraic equation

$$Kv = g, \tag{3.13}$$

where K is the stiffness matrix, having the entries

$$K_{IJ} = \int_\Omega B_I^T C B_J d\Omega, \tag{3.14}$$

and

$$g = \int_{\Gamma_t} \Phi \cdot t d\Gamma + \int_\Omega \Phi \cdot f d\Omega, \tag{3.15}$$

where B is related to Φ, like in the classical FE algorithm. One can show that the finite element stiffness matrix K is symmetric and positive definite and the conditioning number of the same matrix is bounded, independently of T. The demonstration follows immediately as in [3], for the 3-dimensional case. Some numerical examples, which will compare the present solution with, the ones existing in literature will follow in another paper.

§4. Conclusions

In this paper we have presented a first attempt to model cracks within the frame of finite element anisotropic meshes. We have studied 2-dimensional models with one straight crack. It was noticed that the theory works for kinked cracks as well. We have used barycentric coordinates and bubble functions in order to implement our theory. As well, to implement the theory, the FE model with an anisotropic mesh is proposed, and it is noticed that the stiffness matrix is positive definite and symmetric.

References

[1] T. Belytschko, T. Black, *Elastic crack growth in finite elements with minimal remeshing*, Int.J.Num.Meth. in Engrg., 45 (1999).

[2] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Publishing Inc., Boston, 1985.

[3] G.Kunert, *Error estimation for anisotropic tetrahedral and triangular finite element meshes*, Preprint SFB393/97-16, TU Chemnitz, 1997.

[4] N.Moes & al, *A finite element method for crack growth without remeshing*, Int. J. for Num. Meth. In Engrg. 46 (1999).

[5] E. Stein and S. Ohnimus, *Coupled model- and solution-adaptivity in the finite-element method*, Symp. on Advances in Comp. Mech., Vol. 2 (Austin, TX, 1997). Comput. Methods Appl. Mech. Engrg. 150 (1997), no. 1-4, 327-350.

[6] P.P.Teodorescu, *Plane Elasticity Theory Problems* (in Romanian), Vol. 1, Acad. Eds., R.P.R., 1961.

[7] O.C.Zienkiewicz and J. Wu, *Automatic directional refinement in adaptive analysis of compressible flows*, Int.J.Num.Meth. in Engrg. 37 (1994), 2189-2210.

Author's address:

Răzvan Răducanu
Department of Applied Mathematics,
Al. I. Cuza University, Iaşi, Romania
E-mail: rrazvan@uaic.ro