Standing Equatorial Wave Modes in Bounded Ocean Basins

PETER R. GENT

National Center for Atmospheric Research,[1] *Boulder, CO 80307*

(Manuscript received 9 August 1978, in final form 22 January 1979)

ABSTRACT

A new type of standing equatorial wave mode is described that exists in the semi-infinite ocean $0 \leq x \leq L$, $-\infty < y < \infty$. It consists of a *finite* sum of the meridionally trapped equatorial waves in an infinite x domain. The new mode is thus itself equatorially trapped and requires no energy sources or sinks at $|y| = \infty$. However, it exists only for a discrete, countable set of pairs of values of the frequency ω and the ocean zonal width L. Previously described standing modes exist for any ocean width, but are *infinite* sums of trapped equatorial waves and require a continuous energy source in the west at $|y| = \infty$ to balance the continuous energy sink in the east at $|y| = \infty$. Several examples of the new type of standing mode are given, and it is shown that as the standing mode period becomes very long, so the zonal scale becomes very short. The effect on the standing modes of bounding the basin meridionally is also described; energy is recycled round the basin by boundary-trapped Kelvin waves along the zonal walls. The amount of energy recycled in the new type of standing mode, however, is exponentially small compared to that recycled in the previously described standing modes.

1. Introduction

The theory of free equatorially trapped waves was first formulated in the mid-1960's, and applied to the equatorial stratosphere. Observational evidence of the existence of Kelvin waves and mixed Rossby-gravity, or Yanai, waves soon followed. [See Holton (1972, Chap. 12) for a short review and further references.]

An extension of equatorial wave theory to the oceans, where the effects of a zonally bounded basin have to be considered, was achieved by Moore (1968). He showed that in a meridionally infinite basin two standing modes exist for any values of the frequency ω and the ocean width L. The modes consist of an infinite sum of either the Kelvin and all odd m waves or the Yanai and all even m waves. For discrete sets of the frequency ω these standing modes consist of just the infinite sum of waves of order $m + 2M$ for $M = 0, 1, 2, \ldots$ The achievement of Moore (1968) was to show that, near the boundaries and far from the equator, these standing modes can be represented by meridionally propagating Kelvin waves. Thus, in order to maintain these standing modes, energy must be supplied from infinity along the western boundary because energy is lost to infinity along the eastern boundary. Moore went on to show that, in a bounded basin with zonal walls at $y = \pm l$, in which energy cannot be lost, the energy can be recycled by boundary trapped Kelvin waves propagating along $y = \pm l$ from east to west. These Kelvin waves must have the proper phase so that everything matches up smoothly, and this means that Moore's (1968) standing modes in a bounded basin only exist for a discrete set of frequencies.

Section 2 shows that another type of standing mode exists in a zonally bounded, but meridionally unbounded basin. These modes consist of a finite sum of either the Kelvin and odd m or the Yanai and even m waves. The modes themselves are equatorially trapped, contain no meridionally propagating Kelvin waves, and thus have no energy sinks and need no energy sources at infinity. However, this type of standing mode does not exist for all ocean zonal widths L, but only for a discrete set of pairs of the frequency ω and L. Examples of these new standing modes are given in Sections 3 and 4. In a bounded basin, no standing mode exists in which no energy is carried along the zonal walls at $y = \pm l$ by Kelvin waves. Section 5 shows, however, that if l is large enough, a small modification to the meridionally infinite standing mode ensures that it recycles the least possible amount of energy. The amount of energy is exponentially small compared to that carried round the walls in Moore's (1968) bounded basin modes, so that the coastal Kelvin waves play virtually no role in these new bounded basin standing modes.

There is some observational evidence of trapped equatorial waves in the ocean, but their exact role

[1] The National Center for Atmospheric Research is sponsored by the National Science Foundation.

0022-3670/79/040653-10$06.50

in the tropical oceanic circulation is presently uncertain. The possible relevance of these new standing modes to the ocean is briefly discussed in Section 6.

2. Standing equatorial wave modes in a zonally bounded basin

Moore and Philander (1976) show that the linear, inviscid, Boussinesq and hydrostatic equations of motion on an equatorial beta plane can be written in the form

$$\left.\begin{aligned} u_t - yv + p_x \\ v_t + yu + p_y \\ p_t + u_x + v_y \end{aligned}\right\} = 0. \tag{2.1}$$

The equations are nondimensionalized using a time scale of $\beta^{-1/2}(gH_n)^{-1/4}$, and a length scale of $\beta^{-1/2}(gH_n)^{1/4}$, and have had the vertical modal structure separated out in the variables u, v and p. H_n is the equivalent depth of the nth vertical baroclinic mode. Eqs. (2.1) can be combined into a single equation for the meridional velocity, *viz.*,

$$v_{ttt} - v_{xxt} - v_{yyt} + y^2 v_t - v_x = 0. \tag{2.2}$$

This equation possesses separable solutions of the form

$$v = [A_m \exp(ik_{m,1}x) + B_m \exp(ik_{m,2}x)]\psi_m(y)\exp(-i\omega t), \tag{2.3}$$

where

$$\psi_m = H_m(y)e^{-y^2/2}/(2^m m!\pi^{1/2})^{1/2}, \tag{2.4}$$

and H_m is the Hermite polynomial of degree m. The ψ_m's are a complete, orthonormal set of basis functions, and $k_{m,1}$ and $k_{m,2}$ are the roots of the dispersion relation

$$k^2 + \frac{k}{\omega} - \omega^2 + (2m+1) = 0, \tag{2.5}$$

viz.,

$$k_{m,1(2)} = -\frac{1}{2\omega} \pm \left[\omega^2 + \frac{1}{4\omega^2} - (2m+1)\right]^{1/2}, \tag{2.6}$$

where the 1 and 2 refer to the + and − signs, respectively. We also note the special cases of the dispersion relation (2.5): $m = 0$, $k_{0,1} = \omega - 1/\omega$, Yanai or mixed Rossby-Gravity wave; $m = -1$, $k_{-1,1} = \omega$, Kelvin wave with $v \equiv 0$. The second root for k from (2.6) in these two cases yields unacceptable solutions which are not bounded as $|y| \to \infty$, Matsuno (1966). The separable solutions can be written in the form

$$v = \sum_{m=0}^{\infty} [A_m \exp(ik_{m,1}x) + B_m \exp(ik_{m,2}x)]\psi_m(y)\exp(-i\omega t), \tag{2.7}$$

where

$$B_0 \equiv 0$$

and

$$\begin{aligned} u(p) = \Bigg\{ & \frac{A_{-1}}{\sqrt{2}}\exp(i\omega x)\psi_0(y) + \frac{A_0\omega}{\sqrt{2}}\exp[i(\omega - 1/\omega)x]\psi_1(y) \\ & + \sum_{m=1}^{\infty}\left[\frac{A_m}{\omega - k_{m,1}}\exp(ik_{m,1}x) + \frac{B_m}{\omega - k_{m,2}}\exp(ik_{m,2}x)\right]\left(\frac{m+1}{2}\right)^{1/2}\psi_{m+1}(y) \\ & \pm \sum_{m=1}^{\infty}\left[\frac{A_m}{\omega + k_{m,1}}\exp(ik_{m,1}x) + \frac{B_m}{\omega + k_{m,2}}\exp(ik_{m,2}x)\right](m/2)^{1/2}\psi_{m-1}(y)\Bigg\} i\exp(-i\omega t), \end{aligned} \tag{2.8}$$

where the + and − signs refer to u and p, respectively.

We now consider the standing modes that are possible in a semi-infinite ocean strip bounded by meridional walls at $x = 0, L$. We must apply the boundary conditions $u(0) = u(L) = 0$, and since the ψ_m's are orthonormal, this implies that the coefficient of each ψ_m must vanish at $x = 0, L$. The standing modes are of three types:

a. Type one

For any frequency ω, there exist two standing modes, one with a meridional velocity consisting of a Yanai wave and all even m waves, and another consisting of all odd m waves in v plus a Kelvin wave component in u and p. The relations between wave amplitudes are easily obtained by imposing $\psi_m = 0$ at $x = 0, L$ in Eq. (2.8). The existence of these two standing modes was first shown by Moore (1968), who also pointed out that for any given $\omega \neq 0$, these standing modes contain a finite number of waves with real zonal wavenumbers, plus an infinite sum of boundary-trapped waves with imaginary zonal wavenumbers. These boundary-trapped waves occur for all $m \geq M$, where $2M > (\omega - 1/2\omega)^2$. Moore (1968) went on to show that the infinite sum of eastern (western) boundary-trapped waves can be represented by outgoing (incoming) meridionally propagating Kelvin waves within a distance of the boundary proportional to y^{-1} as $|y| \to \infty$. Thus, in the absence of ocean boundaries to the north and south, these standing modes need a continual supply of energy from $|y| = \infty$ along the western boundary, which is balanced by an equal energy loss to $|y| = \infty$ along the eastern boundary.

b. Type two

For the countable set

$$\omega - 1/2\omega = \pm[2m + n^2\pi^2/L^2]^{1/2}, \quad (2.9)$$

where n is any integer, it is easy to show that

$$\exp(ik_{m,1}L) = \exp(ik_{m,2}L). \quad (2.10)$$

Thus, if

$$\frac{A_m}{\omega + k_{m,1}} + \frac{B_m}{\omega + k_{m,2}} = 0, \quad (2.11)$$

the coefficient of ψ_{m-1} in Eq. (2.8) vanishes at $x = 0, L$. Then, for every $m \geq 1$, a standing mode exists in which the meridional velocity is a combination of all waves of order $m + 2M, M = 0, 1, 2, \ldots,$ and the recurrence relation on the amplitudes is again easily obtained from (2.8). Moore (1968) also noted the existence of this type of standing mode. With ω given by Eq. (2.9), the k_m's are real but almost all the larger m wavenumbers are complex, so that the infinite sum of waves again represents a standing mode which requires an energy source along the western boundary at $|y| = \infty$ and a corresponding energy sink along the eastern boundary.

c. Type three

For the countable set of pairs of ω and L, where ω still satisfies Eq. (2.9), but now

$$\frac{A_m}{\omega - k_{m,1}} + \frac{B_m}{\omega - k_{m,2}} = 0, \quad (2.12)$$

then the coefficient of ψ_{m+1} in (2.8) vanishes at $x = 0, L$. When m is an even integer, let v be a combination of the Yanai wave and all even waves up to and including the mth wave. From (2.8), imposing $u = 0$ at $x = 0, L$ requires

$$M^{1/2}\left[\frac{A_M}{\omega + k_{M,1}} + \frac{B_M}{\omega + k_{M,2}}\right] + (M - 1)^{1/2}\left[\frac{A_{M-2}}{\omega - k_{M-2,1}} + \frac{B_{M-2}}{\omega - k_{M-2,2}}\right] = 0, \quad (2.13)$$

$$M^{1/2}\left[\frac{A_M}{\omega + k_{M,1}}\exp(ik_{M,1}L) + \frac{B_M}{\omega + k_{M,2}}\exp(ik_{M,2}L)\right] + (M - 1)^{1/2}\left[\frac{A_{M-2}}{\omega - k_{M-2,1}}\exp(ik_{M-2,1}L) + \frac{B_{M-2}}{\omega - k_{M-2,2}}\exp(ik_{M-2,2}L)\right] = 0, \quad (2.14)$$

with M taking the values of all even integers between 4 and m. Finally, the coefficient of ψ_1 in the expression for u must vanish at $x = 0, L$, requiring

$$\frac{A_0\omega}{\sqrt{2}} + \frac{A_2}{\omega + k_{2,1}} + \frac{B_2}{\omega + k_{2,2}} = 0, \quad (2.15)$$

$$\frac{A_0\omega}{\sqrt{2}}\exp[i(\omega - 1/\omega)L] + \frac{A_2}{\omega + k_{2,1}}\exp(ik_{2,1}L) + \frac{B_2}{\omega + k_{2,2}}\exp(ik_{2,2}L) = 0. \quad (2.16)$$

Eqs. (2.12)–(2.16) comprise $m + 1$ relations between the $m + 1$ coefficients $A_0, A_2, B_2, \ldots, A_m, B_m$, so that the system is singular. In order for a nontrivial solution to exist, the determinant of the matrix multiplying the column vector of the wave amplitudes must vanish, and this puts a necessary condition on the ocean width L for which this standing mode can exist. This condition is tedious to evaluate for general m but examples when m is small are given in the next sections. If m is odd, a comparable standing mode is possible, with v being a combination of all odd waves up to and including the mth wave, and u and p having a Kelvin wave component. The equations to be satisfied are (2.9) and (2.12), (2.13) and (2.14) for M all odd integers between 3 and m, but with (2.15) and (2.16) replaced by

$$A_{-1} + \frac{A_1}{\omega + k_{1,1}} + \frac{B_1}{\omega + k_{1,2}} = 0, \quad (2.17)$$

$$A_{-1}\exp(i\omega L) + \frac{A_1}{\omega + k_{1,1}}\exp(ik_{1,1}L) + \frac{B_1}{\omega + k_{1,2}}\exp(ik_{1,2}L) = 0. \quad (2.18)$$

Note that Eq. (2.9) implies that the k_m's in these solutions are real, and the dispersion relation [Eq. (2.6)] then shows that all the zonal wavenumbers in these standing modes are real. Thus these standing modes contain no boundary trapped waves, and consist entirely of waves that are purely equatorially trapped, so that no energy sources or sinks at $|y| = \infty$ are required for these standing modes once they are excited. However, it must be emphasized that these equatorially trapped standing modes only exist for a *countable set of pairs* of discrete values of the frequency ω and the ocean width L.

Rattray and Charnell (1966) found some trapped standing modes consisting of a finite sum of equatorial waves for any ocean width and a countable set of frequencies. Moore (1968) pointed out that these solutions are incorrect because of the unjustified neglect of some terms in the expressions for u and p.

3. Examples of the simplest trapped standing modes

The standing modes naturally divide into those involving odd m waves and the Kelvin wave, and those involving even m waves and the Yanai wave. In these examples the arbitrary amplitude of these two groups will be specified by setting either A_{-1} or A_0 equal to 1. The simplest trapped standing mode involves only two first waves and the Kelvin wave. In this case we require

$$\exp(ik_{1,1}L) = \exp(ik_{1,2}L) \tag{3.1}$$

or

$$k_{1,1} - k_{1,2} = 2n\pi/L, \quad n = 1, 2, 3 \ldots . \tag{3.2}$$

This gives

$$\omega + 1/2\omega = \pm\left[4 + \frac{n^2\pi^2}{L^2}\right]^{1/2}. \tag{3.3}$$

We also require

$$\frac{A_1}{\omega - k_{1,1}} + \frac{B_1}{\omega - k_{1,2}} = 0, \tag{3.4}$$

$$1 + \frac{A_1}{\omega + k_{1,1}} + \frac{B_1}{\omega + k_{1,2}} = 0, \tag{3.5}$$

$$\exp(i\omega L) + \left[\frac{A_1}{\omega + k_{1,1}} + \frac{B_1}{\omega + k_{1,2}}\right] \times \exp(ik_{1,1}L) = 0. \tag{3.6}$$

In this case it is very easy to see that (3.5) and (3.6) are the same equation, thus giving two equations to determine A_1 and B_1, if and only if

$$\exp(i\omega L) = \exp(ik_{1,1}L) \tag{3.7}$$

or

$$\omega - k_{1,1} = 2p\pi/L, \quad p = 1, 2, 3 \ldots . \tag{3.8}$$

Using (2.6), (3.3) and (3.8), it can be shown that the necessary restriction on the ocean width is

$$L = [p(p + n)]^{1/2}\pi \tag{3.9}$$

for any nonzero integers n and p.

This standing mode can be thought of in the following way. There is one wave carrying energy westward, as shown by the dispersion relation in Fig. 1. At a western boundary it is reflected by two waves carrying energy eastward, whose amplitudes are known compared to the amplitude of the incoming wave. Individually each wave carrying energy eastward would be reflected at an eastern boundary by an infinite series of odd m waves. For specific values of ω and L, however, the reflection of the sum of the two waves is accomplished by a single wave carrying energy westward whose amplitude is equal to that of the original wave, so that a standing mode exists. The u velocity of the sum of the waves carrying energy eastward has nonzero components for only two of the meridional basis set, the same two components as the single mode carrying energy westward. Thus making u vanish at the eastern boundary requires two conditions: those on ω and L. For this simplest mode the conditions on ω and L are that the relative phase of the two first waves has n complete wavelengths across the basin [Eq. (3.2)], and the relative phase of the Kelvin and first wave with larger wavenumber has p complete wavelengths across the basin [Eq. (3.8)].

A similar argument shows that (3.9) is also the condition on L for the existence of the standing mode in which v is a combination of the Yanai and two second waves. In this case, however, ω, A_2 and B_2 are given by

$$\omega - 1/2\omega = \pm[4 + n^2\pi^2/L^2]^{1/2}, \tag{3.10}$$

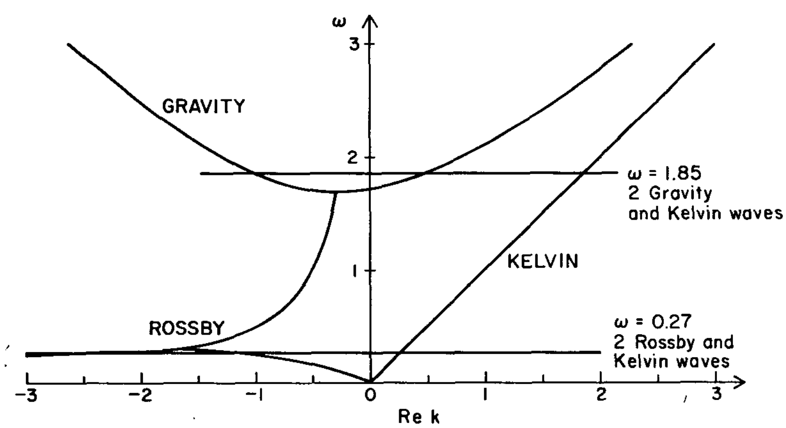

FIG. 1. Dispersion relation (2.5) for $m = 1$, showing the two values of ω corresponding to the standing modes consisting of two first gravity or two first Rossby waves plus the Kelvin wave, $n = p = 1$.

$$\frac{A_2}{\omega - k_{2,1}} + \frac{B_2}{\omega - k_{2,2}} = 0, \quad (3.11)$$

$$\frac{\omega}{\sqrt{2}} + \frac{A_2}{\omega + k_{2,1}} + \frac{B_2}{\omega + k_{2,2}} = 0. \quad (3.12)$$

It is only for these $m = 1$ and 2 modes that the necessary condition on the ocean width L is the same; for higher values of m it is different.

The gravest standing mode consists of first waves and the Kelvin wave with $n = p = 1$. Then $L = \sqrt{2}\pi$, $\omega = (3 \pm \sqrt{5})/2\sqrt{2}$ and $k_{1,1(2)} = -1/2\omega \pm 1/\sqrt{2}$. When the plus sign is taken in the expression for ω, the mode consists of the Kelvin and two first gravity waves, whereas with the minus sign it consists of the Kelvin and two first Rossby waves; see Fig. 1 which is a plot of the dispersion relation (2.5) for $m = 1$. With $\omega = 0.27$, u, v and p at times $t = 0$ and $\pi/2\omega$ are shown in Fig. 2 as functions of x and y. Note that u and p are symmetric, and v is antisymmetric about the equator. This is one of the most equatorially confined of these trapped standing modes.

Fig. 3 shows a time sequence of the zonal velocity at the equator in time increments of $\pi/6\omega$ for t between 0 and $5\pi/6\omega$. Each node can be traced through the solution, seen to be propagating westward, and crosses the ocean in a transit time of $10.8/\omega$. However, the propagation is not spatially uniform in the sense that the phase speed of a node is larger near the center and boundaries of the ocean than it is near $x = L/4$ and $3L/4$. For this standing mode the ratio of phase speed at $x = L/2$ to that at $x = L/4$, and $3L/4$ is about 1.5. The average phase speed is 0.11 that of the Kelvin wave, or 0.47 that of the long Rossby wave which carries energy westward. Fig. 3 shows that the nodal amplitude also varies with x.

Fig. 4 shows u, v and p as functions of x and y for the standing mode with second Rossby waves and the Yanai wave at times $t = 0$ and $\pi/2\omega$ for $n = p = 1$. Again, $L = \sqrt{2}\pi$, but $\omega = 0.214$, $k_{2,1} = -1.63$, $k_{2,2} = -3.04$, and the Yanai wavenumber $\omega - 1/\omega = -4.46$. For this mode, u and p are antisymmetric and v is symmetric about the equator, but the zonal structure is similar to the first Rossby and Kelvin wave mode. This mode also propagates spatially nonuniformly to the west in the manner described above, with an average phase speed of about 1.5 that of the Yanai wave, or 0.55 that of the long Rossby wave.

These standing modes were found when considering the response of the Pacific Ocean to forcing on very long space and time scales. We now give an example of a mode with a long x scale, but which has a relatively short time scale; and then show that as the period of these modes increases, their spatial scale decreases. Thus there are no trapped modes with very long space and time scales. If we take the equivalent depth of the first vertical baroclinic mode as 40 cm, which is typical of the Pacific Ocean, then using $g = 10^3$ cm^2 s^{-1}, the phase speed of the corresponding Kelvin wave is 2 m s^{-1}. Using β at the equator equal to 2×10^{-13} cm^{-1} s^{-1} gives, for the first baroclinic mode, a time scale of 1.83 days and a length scale of 316 km. Thus, for the first Rossby and Kelvin wave mode with $p = 8$, $n = 10$, we obtain $L = 12\pi$, so that the ocean width is almost 12 000 km, similar to the Pacific. The meridional width $-4 \leq y \leq 4$ in Figs. 2 and 4 corre-

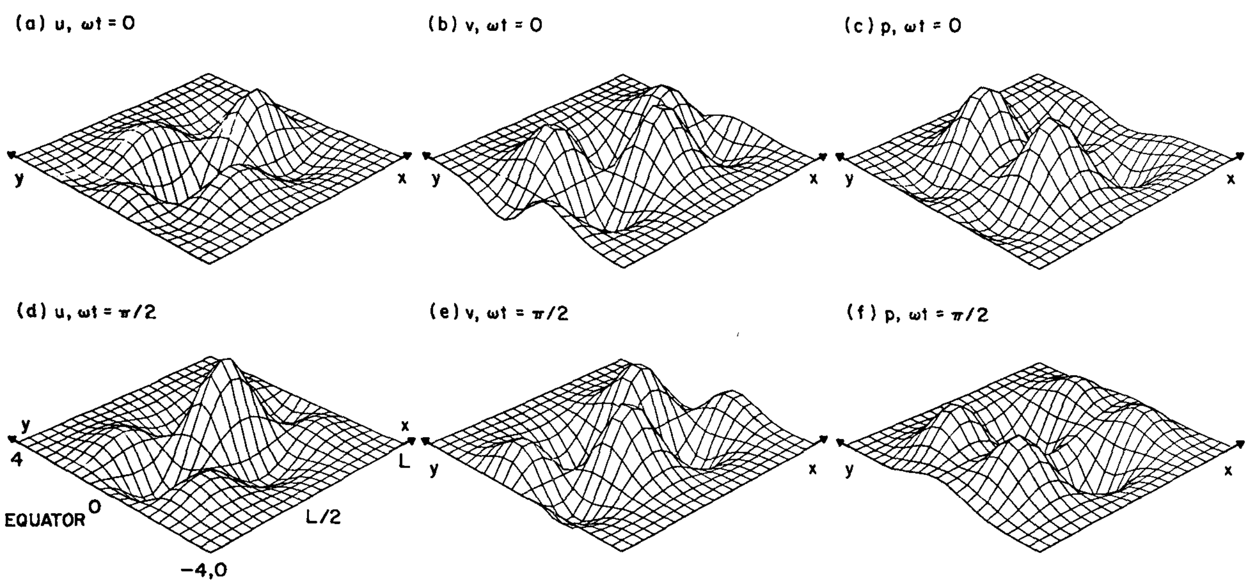

FIG. 2. Mode consisting of first Rossby and Kelvin waves, $n = p = 1$. Plots of u, v and p at times $\omega t = 0$ and $\pi/2$.

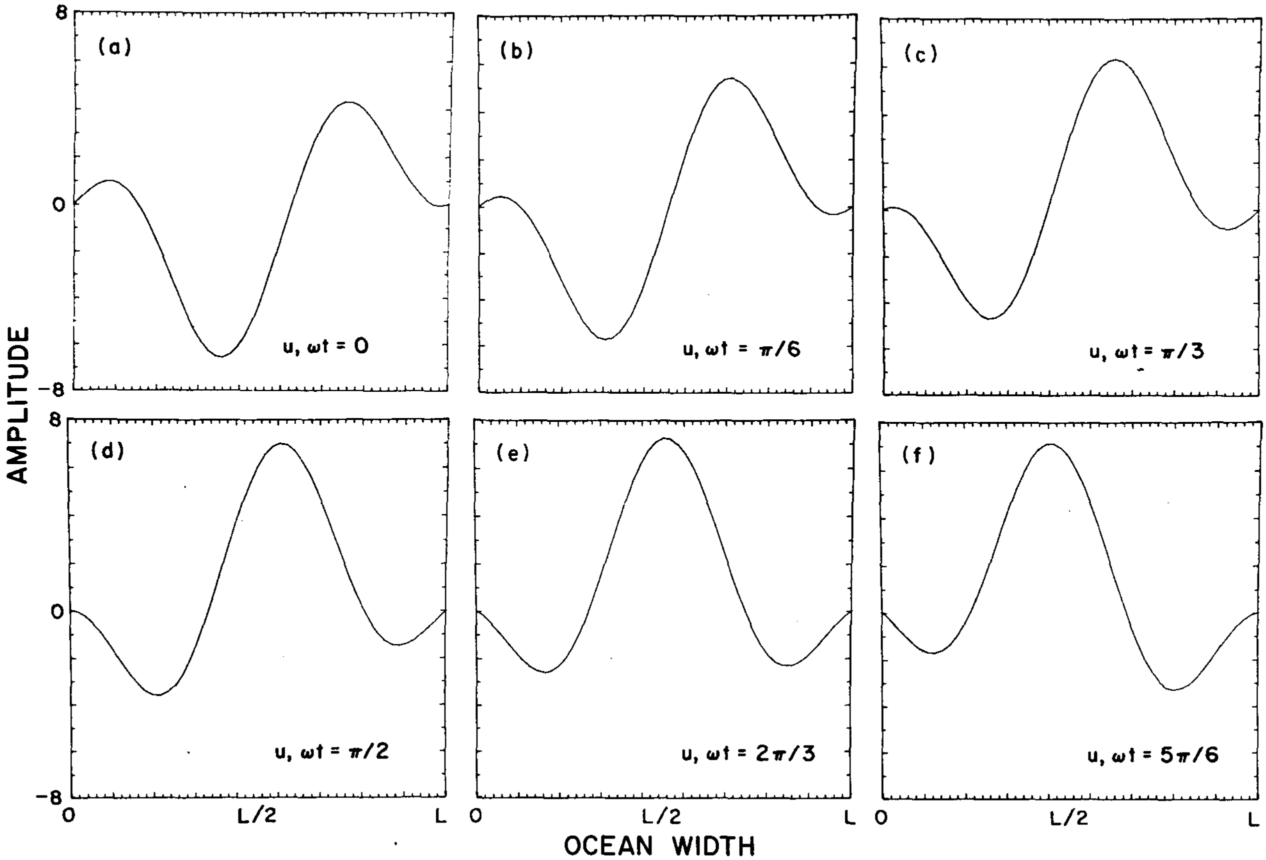

FIG. 3. Same mode as Fig. 2. Time sequence of u at the equator for $\omega t = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$.

sponds to 2530 km; with $\omega = 0.263$ the mode time scale is 7 days. For this mode $k_{1,1} = -1.07$ and $k_{1,2} = -2.74$, and a time sequence of the zonal velocity at the equator in time increments of $\pi/6\omega$ for t between 0 and $5\pi/6\omega$ is shown in Fig. 5. As is to be expected from the larger values of n and p, this mode has many more nodes than the $n = p = 1$ modes. It still propagates westward, however, in

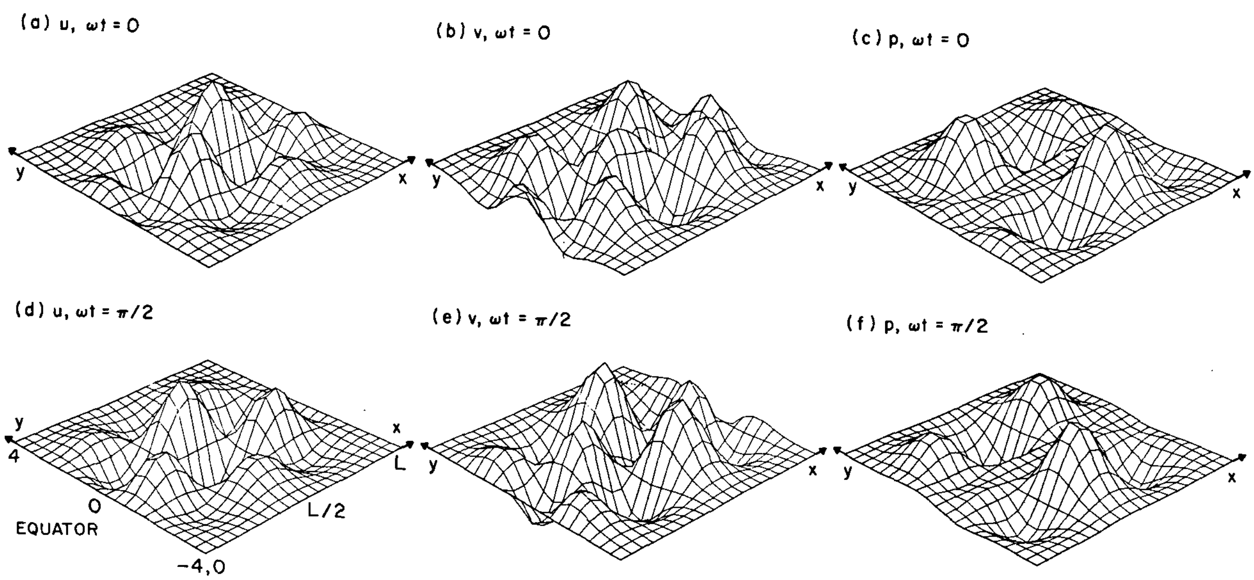

FIG. 4. Mode consisting of second Rossby and Yanai waves, $n = p = 1$. Plots of u, v and p at times $\omega t = 0$ and $\pi/2$.

a spatially nonuniform manner since the nodes are not equally spaced in Fig. 5. The average phase velocity is about 0.1 that of the Kelvin wave, or 0.4 that of the long Rossby wave.

If we consider the very low frequency limit as $\omega \to 0$, then

$$\omega \to \pm L/2n\pi \quad \text{so that} \quad L/n\pi \to 0. \quad (3.13)$$

Thus for finite L/π, $n \to \infty$, so the zonal mode scale tends to zero. For the standing mode consisting of first Rossby and then Kelvin wave

$$L^2/\pi^2 = p(p + n), \quad \text{so} \quad p/n \to 0. \quad (3.14)$$

As

$$\omega \to 0,\ k_{1,1} \to -3\omega \quad \text{and} \quad k_{1,2} \to -1/\omega. \quad (3.15)$$

In this limit

$$A_1 \to 4\omega \quad \text{and} \quad B_1 \to -1/\omega. \quad (3.16)$$

so that $A_1/B_1 \to 0$, but $A_1/(\omega \pm k_{1,1})$ and $B_1/(\omega \pm k_{1,2})$ are of the same order of magnitude. Thus, in the zonal velocity and pressure, the low-wavenumber components $\exp(i\omega x)$ and $\exp(ik_{1,1}x)$ and the high-wavenumber component $\exp(ik_{1,2}x)$ have comparable magnitudes, but in the meridional velocity the high-wavenumber component dominates the low-wavenumber component. The same is true of the standing mode consisting of second Rossby waves and the Yani wave. To try to match long time-scale variations in the Pacific, we put $p = 1$ and $n = 100$. Then $L = \sqrt{101}\pi$ and $\omega = 0.05$, corresponding to an ocean width of about 10 000 km and a mode period of 230 days. The mode zonal length scale will be 100 km, decreasing further if the period increases. In this example the amplitude of the high-wavenumber component of v is 100 times that of the low-wavenumber component.

4. Example of the next simplest trapped standing modes

The next simplest standing mode has v consisting of first and third waves, and u and p having a Kelvin wave component. In this case

$$\omega + 1/2\omega = \pm[8 + n^2\pi^2/L^2]^{1/2}, \quad n = 1, 2, 3 \ldots, \quad (4.1)$$

$$\frac{A_3}{\omega - k_{3,1}} + \frac{B_3}{\omega - k_{3,2}} = 0, \quad (4.2)$$

plus four more equations relating A_1, B_1, A_3 and B_3. The condition on L can be reduced to the form

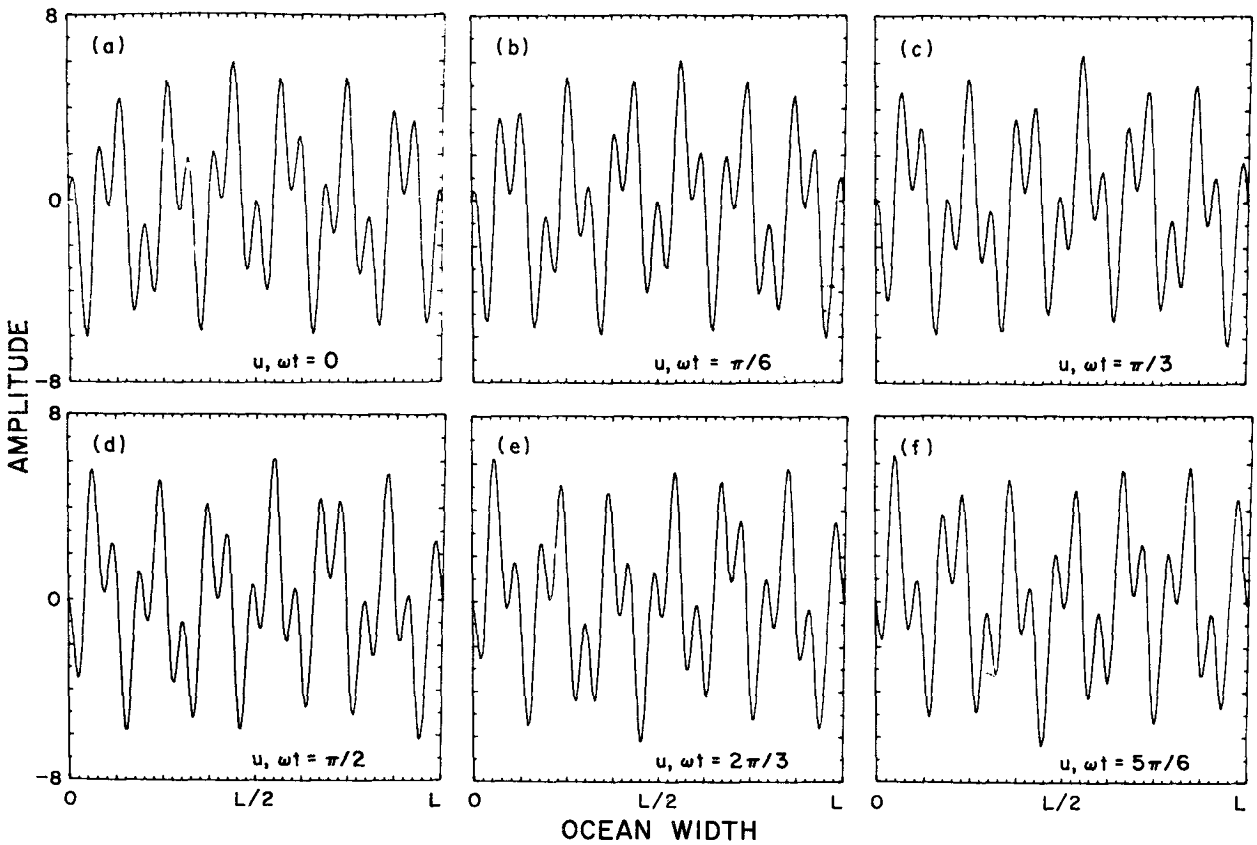

FIG. 5. Mode consisting of first Rossby and Kelvin waves, $n = 10$, $p = 8$. Time sequence of u at the equator for $\omega t = 0$, $\pi/6$, $\pi/3$, $\pi/2$, $2\pi/3$ and $5\pi/6$.

$$(\omega - k_{1,2})(\omega + k_{1,1})[\exp(ik_{1,1}L) - \exp(ik_{3,1}L)] \times [\exp(ik_{1,2}L) - \exp(i\omega L)] = (\omega - k_{1,1})(\omega + k_{1,2})[\exp(ik_{1,2}L) - \exp(ik_{3,2}L)][\exp(ik_{1,1}L) - \exp(i\omega L)]. \quad (4.3)$$

After some algebra, this can be written in the form

$$T \tan(\pi Y/2) + S \tan(\pi Z/2) = 0, \quad \text{if } n \text{ is odd}$$

or (4.4)

$$S \tan(\pi Y/2) + T \tan(\pi Z/2) = 0, \quad \text{if } n \text{ is even},$$

where

$$S = (8 + n^2\pi^2/L^2)^{1/2}(6 + n^2\pi^2/L^2)^{1/2} + 4 + n^2\pi^2/L^2, \quad (4.5)$$

$$T = (4 + n^2\pi^2/L^2)^{1/2}[(8 + n^2\pi^2/L^2)^{1/2} + (6 + n^2\pi^2/L^2)^{1/2}], \quad (4.6)$$

$$Y = (n^2 + 8L^2/\pi^2)^{1/2}, \quad (4.7)$$

$$Z = (n^2 + 4L^2/\pi^2)^{1/2}. \quad (4.8)$$

For the standing mode where v consists of a Yanai wave and second and fourth waves, ω is given by

$$\omega - 1/2\omega = \pm[8 + n^2\pi^2/L^2]^{1/2}, \quad n = 1, 2, 3 \ldots \quad (4.9)$$

and the condition on L is again given by Eq. (4.4), but with $(6 + n^2\pi^2/L^2)$ replaced by $(10 + n^2\pi^2/L^2)$ in the definition of S and T. It can be shown that T/S is a monotonically increasing function of $n^2\pi^2/L^2$ varying in the range $0.966 \leq T/S \leq 1$ for the mode which includes the Kelvin wave, and $0.926 \leq T/S \leq 1$ for the mode which includes the Yanai wave as $n^2\pi^2/L^2$ increases from zero to infinity. Therefore, a good approximation to the condition on L is

$$\tan(\pi Y/2) + \tan(\pi Z/2) = 0 \quad (4.10)$$

or

$$Y + Z = 2p \text{ for any integer } p > n. \quad (4.11)$$

A little algebra gives this approximate condition on L as

$$L^2/\pi^2 = 3p^2 - p(8p^2 + n^2)^{1/2}. \quad (4.12)$$

These standing modes have two waves carrying energy westward. At a western boundary they are reflected by three waves carrying energy eastward, whose amplitudes are known relative to the amplitudes of the incoming waves. At an eastern boundary the reflection of these three waves need not be an infinite series of waves, but can be accomplished by two waves carrying energy westward whose amplitudes equal those of the original waves. This occurs for specific values of ω, L and the amplitude ratio of the two waves carrying energy westward. These three conditions are used to make $u = 0$ at the eastern boundary, because, in these modes, u has three nonzero components in the meridional basis set. Eqs. (4.1) and (4.9) show that the relative phase of the two highest order waves has n complete wavelengths across the basin. Eq. (4.3) does not have such a simple interpretation but relates the relative phases of the waves to their u velocity amplitudes. Higher order standing modes can be viewed in a similar way.

We have computed one example of a standing mode in which v has first and third gravity wave components, and u and p have a Kelvin wave component. When $n = 1$ and $p = 2$, $\omega = 2.99$ corresponding to $L = 0.716\pi$, which compares with the estimate of 0.715π from (4.12). Fig. 6 shows u, v and p as functions of x and y at times 0 and $\pi/2\omega$ for this mode; u and p are symmetric and v is anti-

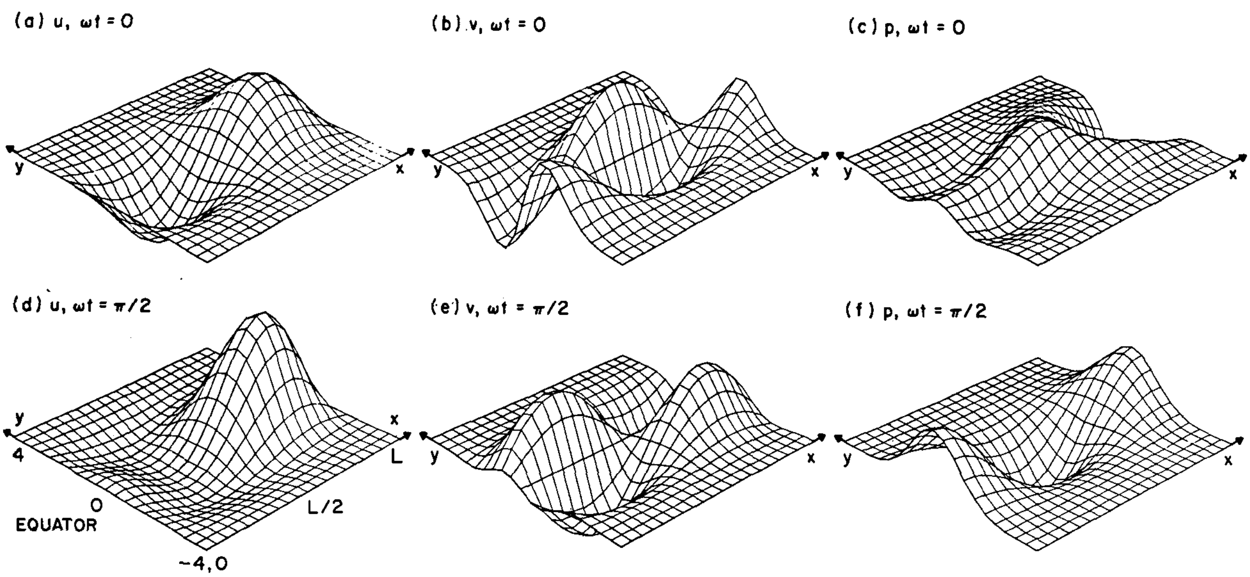

FIG. 6. Mode consisting of third and first gravity and Kelvin waves, $n = 1, p = 2$. Plots of u, v and p at times $\omega t = 0$ and $\pi/2$.

symmetric about the equator. Modes of this type are less equatorially confined than those shown in Figs. 2 and 4, but this is not apparent from Fig. 6 because, in this particular mode, the amplitudes of the Kelvin wave and the first gravity wave associated with wavenumber $k_{1,2}$ are much larger than the remaining amplitudes. This mode propagates eastward in the spatially nonuniform manner described for the first Rossby and Kelvin $n = p = 1$ mode, but the nonuniformity of the propagation is much more pronounced.

For the examples given in this paper, the standing modes containing Rossby waves propagate westward, and the mode containing gravity waves eastward. For the simplest modes it can be shown that as $\omega \to 0$ the standing modes propagate slowly westward, so that this rule may be true, in general.

5. Standing equatorial wave modes in a bounded basin

In this section the effect of bounding the basin on the standing modes described in Sections 2–4 will be examined. $v = 0$ is imposed on the zonal boundaries at $y = \pm l$, and this means that the dispersion relation (2.5) becomes

$$k^2 + k/\omega - \omega^2 + 2\mu_m + 1 = 0. \quad (5.1)$$

In this section we shall only consider basins symmetric about the equator, and which have large l in the sense that

$$l \gg (2\mu_M + 1)^{1/2}, \quad (5.2)$$

where the $l \to \infty$ standing mode consists of waves of order M or less. This implies that l is much larger than the turning latitudes of all the waves in the standing mode. When this criterion is satisfied, Moore (1968) showed that the energy flowing north and south along the eastern boundary in his meridionally infinite standing modes can be recycled around the basin by means of boundary-trapped Kelvin waves propagating westward along $y = \pm l$. He calculated the order 1 solution because the amount of energy recycled around the basin is of order 1. For everything to match up smoothly requires that these Kelvin waves have the correct phase, and this implies a discrete set of frequencies for Moore's standing modes in a bounded basin.

When l satisfies (5.2), Cane and Sarachik (1979) show that μ_m can be approximated by

$$\mu_m \approx m + \sqrt{2/\pi}\,\frac{(\sqrt{2}l)^{2m+1}e^{-l^2}}{m!} \equiv m + \epsilon. \quad (5.3)$$

The complete orthogonal basis set ϕ_{μ_m} is given by

$$\phi_{\mu_m} = U_m + b_m V_m, \quad (5.4)$$

where $U_m(V_m)$ is exponentially decreasing (increasing) away from equator and $b_m \approx -\pi\epsilon m!/2$. If $v \propto \phi_{\mu_m}$, u can be expressed in terms of $\phi_{\mu_{m\pm1}}$, but these are not equal to $\phi_{\mu_{m\pm1}}$ since μ_m is no longer an integer. Thus, u is now not a simple sum of just two of the basis functions, but is an infinite sum over all the basis functions. Thus, if v is a finite sum of the ϕ_{μ_m}'s, there are an infinite number of coefficients to be set to zero to satisfy $u = 0$ on $x = 0, L$. This is impossible with only a finite number of wave amplitude ratios and conditions on ω and L at one's disposal. This new type of standing mode in a meridionally infinite basin depends crucially for its existence on u being expressed as the sum of just two basis functions. Thus, in a bounded basin, no standing modes exist in which v is a finite sum of ϕ_{μ_m}'s, which implies there are no standing modes which recycle no energy around the basin.

We define v_∞ to be the form of a trapped standing mode in a meridionally infinite basin, but ω and k satisfy the dispersion relation (5.1). In a bounded basin we let the corresponding standing mode be denoted by

$$\left.\begin{aligned} v &= v_\infty + v' \\ u &= u_\infty + u' \end{aligned}\right\}, \quad (5.5)$$

where v' represents the infinite sum which recycles energy around the basin. Imposing

$$u_\infty = 0 \quad \text{at} \quad x = 0, L \quad \text{for all } y \quad (5.6)$$

ensures that the mode recycles the least amount of energy because it makes u' as small as possible at $y = 0$. It is $O(\epsilon)$ there, and ϵ must now be considered as the maximum ϵ defined by (5.3) over all the m's contained in the v_∞ standing mode. The amount of recycled energy is $O(\epsilon)$ because $u = O(\epsilon^{1/2})$ on $y = \pm l$, and is exponentially small compared to that recycled in Moore's (1968) bounded basin modes. Boundary condition (5.6) implies an $O(\epsilon^{1/2})$ correction to ω and L compared to the $l \to \infty$ solution because by Eq. (5.1) the k's are changed by $O(\epsilon^{1/2})$. Leaving ω and L as the $l \to \infty$ values would make $u' = O(\epsilon^{1/2})$ at $x = 0, L, y = 0$ so that more energy would have to be recycled around the basin.

For the mode consisting of the Kelvin and two first waves, boundary condition (5.6) implies

$$\left.\begin{aligned} L &= \left[\frac{2p(p+n)}{\mu_1 + 1}\right]^{1/2}\pi \\ \omega + 1/2\omega &= \pm\left[2(\mu_1 + 1) + \frac{n^2\pi^2}{L^2}\right]^{1/2} \end{aligned}\right\}. \quad (5.7)$$

For the standing mode consisting of the "Yanai" and two second waves, the condition on L is slightly different from (5.7), *viz.*,

$$\left.\begin{aligned} L &= \left[\frac{2p(p+n)}{\mu_2 - \mu_0}\right]^{1/2}\pi \\ \omega - 1/2\omega &= \pm\left[2\mu_2 + \frac{n^2\pi^2}{L^2}\right]^{1/2} \end{aligned}\right\}, \quad (5.8)$$

TABLE 1. Coefficients in Eq. (4.4) for bounded basin standing modes.

Kelvin plus two 1st and two 3rd waves	"Yanai" plus two 2nd and two 4th waves
$\omega + 1/2\omega = \pm[2(\mu_3 + 1) + n^2\pi^2/L^2]^{1/2}$	$\omega - 1/2\omega = \pm[2\mu_4 + n^2\pi^2/L^2]^{1/2}$
$S = [2(\mu_3 + 1) + n^2\pi^2/L^2]^{1/2}[2\mu_3 + n^2\pi^2/L^2]^{1/2} + 2(\mu_3 - \mu_1) + n^2\pi^2/L^2$	$S = [2(\mu_4 + 1) + n^2\pi^2/L^2]^{1/2}[2\mu_4 + n^2\pi^2/L^2]^{1/2} + 2(\mu_4 - \mu_2) + n^2\pi^2/L^2$
$T = [2(\mu_3 - \mu_1) + n^2\pi^2/L^2]\{[2(\mu_3 + 1) + n^2\pi^2/L^2]^{1/2} + [2\mu_3 + n^2\pi^2/L^2]^{1/2}\}$	$T = [2(\mu_4 - \mu_2) + n^2\pi^2/L^2]^{1/2}\{[2(\mu_4 + 1) + n^2\pi^2/L^2]^{1/2} + [2\mu_4 + n^2\pi^2/L^2]^{1/2}\}$
$Y = [n^2 + 2(\mu_3 + 1)L^2/\pi^2]^{1/2}$	$Y = [n^2 + 2(\mu_4 - \mu_0)L^2/\pi^2]^{1/2}$
$Z = [n^2 + 2(\mu_3 - \mu_1)L^2/\pi^2]^{1/2}$	$Z = [n^2 + 2(\mu_4 - \mu_2)L^2/\pi^2]^{1/2}$

and approximate values of μ_0, μ_1 and μ_2 can be found from (5.3). "Yanai" is put in quotes because, in a bounded basin, this wave splits into a gravity anti-Kelvin wave and a Rossby anti-Kelvin wave; [see Cane and Sarachik (1979), their Fig. 1]. The appropriate wave for each standing mode is obvious from the meridionally infinite basin mode. For the modes consisting of the Kelvin plus two first and two third waves, and the "Yanai" plus two second and two fourth waves, the condition on L is still given by Eq. (4.4), and the appropriate values of ω, S, T, Y and Z are given in Table 1.

6. Discussion

The trapped equatorial standing modes described in this paper exist for a discrete, countable set of pairs of values of ω and L. Are they excited in the real oceans? For a typical first vertical baroclinic mode, π times the zonal length scale is ~1000 km so that, for application to the equatorial Pacific, $[p(p + n)]^{1/2}$ must be between about 10 and 12, if either of the gravest standing modes is to be excited; $[p(p + n)]^{1/2}$ lies within the range [10, 12] for 46 different pairs of (p,n) so that the values are fairly dense. This does not include the values of L/π between 10 and 12 that are appropriate for the excitation of more complicated modes, involving higher order waves in the solution. The equivalent depths of the higher vertical baroclinic modes are smaller than for the first, so that the appropriate values of L/π for the Pacific will be larger than for the first mode. At larger values of L/π, the set of values of L/π for which some standing mode exists is denser. It is also likely that, if the value of L/π is very near but not quite equal to the exact value for a standing mode, then the standing mode may well be excited, but will be "leaky" in the sense that it will lose energy at a rate depending upon how close L/π is to the required exact value. One can argue, therefore, that for any value of the ocean width L, one or more of these trapped equatorial standing modes will be excited, even if they are "leaky." This hypothesis can be tested using the results of Section 5 and a numerical model. When this is done the possibility that these trapped standing modes are excited in the oceans can be assessed.

Acknowledgment. I would like to thank James C. McWilliams for several helpful discussions and a critical review of this work.

REFERENCES

Cane, M. A., and E. S. Sarachik, 1979: Forced baroclinic ocean motions III: The linear equatorial basin case. *J. Mar. Res.*, **37**, 355–398.

Holton, J. R., 1972: *An Introduction to Dynamic Meteorology*. Academic Press, 319 pp.

Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, **44**, 25–43.

Moore, D. W., 1968: Planetary gravity waves in an equatorial ocean. Ph.D. thesis, Harvard University, 207 pp.

——, and S. G. H. Philander, 1977: Modeling of the tropical ocean circulation. *The Sea*, Vol. 6, Interscience, 319-361.

Rattray, M., and R. L. Charnell, 1966: Quasigeostrophic free oscillations in closed basins. *J. Mar. Res.*, **24**, 82–102.