An Instability of Packets of Short Gravity Waves in Waters of Finite Depth

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ABSTRACT

We consider the stability of a solitary wave packet of short gravity waves propagating in waters of moderate depth. It is shown that such wave packets are unstable to long transverse disturbances. The calculation indicates that steep swell may be more unstable to transverse disturbances when over the continental shelves than in deep waters.

1. Introduction

The evolution of modulated wave systems has generated considerable recent interest. For water waves the evolution equations depend on the fluid depth in two ways. First, a packet of waves forces fluid motion on the length scale of the modulation envelope. The amplitude of the forced wave increases as depth decreases and the group velocity of the carrier wave approaches the speed of a long wave. Second, as the depth parameter kh (k = wavenumber, h = water depth) diminishes and crosses a critical value of $kh = 1.363$ (Hasimoto and Ono, 1972) steep Stoke's waves change from moving faster to moving slower than linear gravity waves. A value of $kh = 1.363$ makes a boundary between two distinct modulation patterns for solitary wave packets. For $kh > 1.363$ there are envelope solitons. These are one-dimensional modulations that move in the same direction as the carrier wave crests. For $kh < 1.363$ envelope solitons are replaced by waveguide solitons. These propagate orthogonal to the wave crests, independent of the coordinate in the direction of propagation of the carrier wave. In this paper we focus attention on envelope solitons and the stability of these solitons to long transverse perturbations. It is shown that envelope solitons are more unstable in waters of moderate depth than their deep water counterparts.

2. Stability Analysis

We study the stability of solitary wave packets ($kh > 1.363$) to transverse disturbances. Previous studies have been made for deep water solitons by Vakhitov and Kolokolov (1973), Zakharov and Rubenchik (1974) and Saffman and Yuen (1977). The analysis we employ parallels that due to Zakharov and Rubenchik (1974). This approach has been criticized by Infeld and Rowlands (1977) because the perturbation quantities are not everywhere small compared with the unperturbed state. The objection has been discussed by Laedke and Spatschek (1978) who showed that an expansion assuming the norm of the perturbation quantities small compared with the norm of the basic state yields the Zakharov and Rubenchik (1974) results. Thus, in this discussion we employ the direct perturbation scheme using the analysis of Laedke and Spatshek to justify the procedure.

The basis for the calculations is the Davey-Stewartson equations for the behavior of the envelope of a gravity wave of carrier frequency ω. These equations are the leading terms in an expansion of the irrotational flow equations in wave slope $\epsilon(\epsilon = ak, a$ = wave amplitude) and are valid between a deep water limit $\epsilon kh \ll 1$ and a shoal water limit $\epsilon \ll (kh)^2$. Consider a slowly varying wave train depending on $\epsilon = \epsilon x$, $\eta = \epsilon y$ and $\tau = \epsilon^2 t$. The coordinate ϵ is in the direction of propagation of the carrier wave, η is a normal coordinate and τ is time.

The equations are

$$iA_\tau + \lambda A_{\xi\xi} + \mu A_{\eta\eta} = \nu|A|^2A + \nu_1 AQ, \quad (1)$$

$$\lambda_1 Q_{\xi\xi} + \mu_1 Q_{\eta\eta} = \kappa_1 \frac{\partial^2|A|^2}{\partial\eta^2}, \quad (2)$$

and the free surface displacement is

$$g\zeta = -i\epsilon\omega A \exp[i(kx - \omega t)] + \text{c.c.} + \epsilon^2\left[\frac{k^2}{g}Q - \frac{k\sigma + 2kh(1-\sigma^2)}{gh - c_g{}^2}|A|^2\right]. \quad (3)$$

The sea level variations described by (3) may be thought of as follows: On the left-hand side is the observed sea level fluctuation associated with the passage of a wave packet. It is composed of two

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parts. The last term in (3) is the plane wave radiation stress contribution to sea level [see Eq. (16.99), Whitham (1974)]. The Q contribution is a result of two-dimensional patterns of surface wave modulation. The equations are in a coordinate system stationary with respect to the group velocity c_g of the carrier wave. The coefficients are[1]

$$\left.\begin{aligned} &\lambda = \frac{1}{2}\omega''(k) \le 0, \qquad \mu = \frac{\omega'(k)}{2k} \equiv \frac{c_g}{2k} \ge 0 \\ &\nu = \frac{k^4}{4\omega\sigma^2}\left\{9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2}{gh - c_g^2}[4c_p^2 + 4c_pc_g(1-\sigma^2) + gh(1-\sigma^2)^2]\right\} \\ &\nu_1 = \frac{k^4}{\omega c_g}[2c_p + c_g(1-\sigma^2)], \qquad \lambda_1 = gh - c_g^2 \ge 0, \qquad \mu_1 = gh \\ &\kappa_1 = ghc_g\left[\frac{2c_p + c_g(1-\sigma^2)}{gh - c_g^2}\right], \qquad \sigma = \tanh kh \end{aligned}\right\} . \tag{4}$$

Larsen (1978) illustrates the general behavior of stationary one-dimensional soliton solutions of (1). Because the coefficient ν approaches zero as kh approaches 1.363, solitons of fixed amplitude approaching this depth have ever increasing widths.

We simplify our calculations by restricting attention to stationary solitons. This defines a basic state which is a solution of (1) and (2) given by

$$A = \phi(\xi)e^{-i\gamma^2\tau}, \tag{5}$$

$$Q = 0, \tag{6}$$

where

$$\lambda\phi_{\xi\xi} + \gamma^2\phi - \nu\phi^3 = 0, \tag{7}$$

yielding the solution

$$\phi = \gamma\left(\frac{2}{\nu}\right)^{1/2} \operatorname{sech}\left[\gamma\left(\frac{-1}{\lambda}\right)^{1/2}\xi\right]. \tag{8}$$

We consider a perturbation of the basic soliton of the form

$$A = [\phi(\xi) + u + iv]\exp(-\gamma^2\tau) \tag{9}$$

and, furthermore, let u, v and Q be proportional to $\exp[i(l\eta - \Omega\tau)]$. Substitution into Eqs. (1) and (2) neglecting quadratic terms in the perturbed quantities yields

$$(L_1 - \mu l^2)u = \Omega v + \nu_1\phi Q, \tag{10}$$

$$(L_0 - \mu l^2)v = \Omega u, \tag{11}$$

$$\lambda_1 Q_{\xi\xi} - \mu_1 l^2 Q = -2\kappa_1 l^2\phi u. \tag{12}$$

This eigenvalue problem is to be solved subject to vanishing u, v and Q as $|\xi| \to \infty$. The operators L_0 and L_1 are defined

$$L_0 = \lambda\frac{\partial^2}{\partial\xi^2} + \gamma^2 - \nu\phi^2, \tag{13}$$

$$L_1 = \lambda\frac{\partial^2}{\partial\xi^2} + \gamma^2 - 3\nu\phi^2. \tag{14}$$

The basic soliton is a solution of $L_0\phi = 0$. Considering the auxiliary functions

$$u_0^- = \phi_\xi, \quad u_0^+ = -\frac{\partial\phi}{\partial\gamma^2}, \qquad v_0^- = -\xi\phi, \quad v_0^+ = \phi, \tag{15}$$

it follows from (13) and (14) that

$$L_0v_0^+ = 0, \quad L_0v_0^- = 2\lambda u_0^-, \qquad L_1u_0^- = 0, \quad \text{and} \quad L_1u_0^+ = v_0^-. \tag{16}$$

Thus, $L_0L_1u_0 = 0$ and $L_1L_0v_0 = 0$, where u_0 and v_0 are linear combinations of the even and odd parity auxiliary functions defined in (15). It also follows that u_0 is a solution of the eigenvalue equation

$$(L_0 - \mu l^2)(L_1 - \mu l^2)u = \Omega^2 u + \nu_1(L_0 - \mu l^2)\phi Q, \tag{17}$$

in the limiting case where

$$\mu l^2 \to 0; \quad \Omega^2 = 0, \quad Q = 0.$$

On expanding the dependent variables in a power series in μl^2 (μl^2 small compared with γ^2)

$$\left.\begin{aligned} v &= v_0 + \mu l^2 v_1 + \ldots \\ u &= u_0 + \mu l^2 u_1 + \ldots \\ \Omega^2 &= \mu l^2\Omega_1^2 + \ldots \\ Q &= \mu l^2 Q_1 + \ldots \end{aligned}\right\}, \tag{18}$$

we find

$$L_0L_1u_1 = (L_0 + L_1)u_0 + \Omega_1^2 u_0 + \nu_1 L_0\phi Q_1. \tag{19}$$

Eq. (19) will have a solution only if the right-hand side is orthogonal to the conjugate equation $L_1L_0v_0$

[1] A misprint omitting the ω in the expression for ν_1 in the Davey-Stewartson paper is corrected in the above.

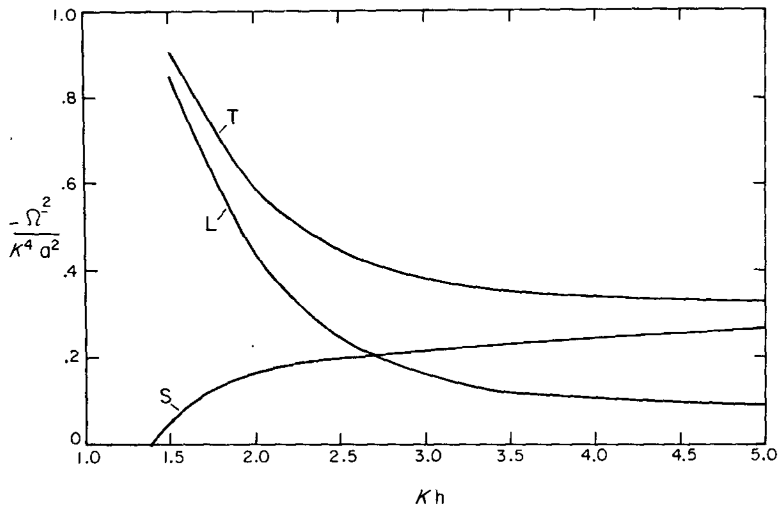

FIG. 1. The unstable eigenvalue as a function of kh.

= 0. It follows that

$$\Omega_1^{2+} = \frac{-1}{\langle v_0^+ | u_0^+ \rangle} \times \{\langle v_0^+ | (L_1 + L_0) u_0^+ \rangle + \nu_1 \langle v_0^+ | L_0 \phi Q_1^+ \rangle\}, \quad (20)$$

$$\Omega_1^{2-} = \frac{-1}{\langle v_0^- | u_0^- \rangle} \times \{\langle v_0^- | (L_1 + L_0) u_0^- \rangle + \nu_1 \langle v_0^- | L_0 \phi Q_1^- \rangle\}, \quad (21)$$

since inner products of functions of opposite parity vanish. Inner products are defined $\langle a | b \rangle = \int_\Gamma ab d\Gamma$. The inner products may be readily evaluated (details are in Appendix A) with the result

$$\Omega_1^{2+} = \frac{I}{\dfrac{\partial I}{\partial \gamma 2}} = 2\gamma; \quad I = \int_{-\infty}^{\infty} \phi^2 d\xi, \quad (22)$$

for the symmetric stability eigenvalue. This is an identical result to the deep water soliton equation ($Q = 0$); thus, there is no depth influence on the symmetric perturbation eigenvalues. The situation is different for the asymmetric perturbation. Here we find

$$\Omega_1^{2-} = \frac{-2}{I}\left[\frac{2}{3}\gamma^2 I - \frac{4}{3}\frac{\lambda}{\lambda_1}\frac{\kappa_1}{\mu}\frac{\gamma^2}{\nu} I \nu_1\right]. \quad (23)$$

It is convenient to rewrite the unperturbed soliton

$$\phi = a \operatorname{sech} B\xi, \quad (24)$$

where

$$a = \gamma\left(\frac{2}{\nu}\right)^{1/2}, \quad B = \left(-\frac{\nu}{2\lambda}\right)a, \quad (25)$$

in order to fix the soliton amplitude and allow its width to vary as the depth diminishes ($\nu \to 0$). This yields for the unstable asymmetric mode

$$\Omega^{2-} = -\frac{2}{3}\mu\nu l^2 a^2 + \frac{4}{3}\frac{\lambda}{\lambda_1} k_1 \nu_1 l^2 a^2 < 0. \quad (26)$$

The leading term in Eq. (26) is directly related to the Stokes correction for the phase speed of the wave. The second term comes from the Q contribution to sea level. This term relates to that portion of the sea level fluctuation not resulting from the plane-wave radiation stress. The relative contributions of each of these terms to the total instability is depicted in Fig. 1. In this figure we show the nondimensional quantity Ω^{2-}/K^4a^2 as a function of kh. The Stokes term is labeled S and the Q contribution is labeled L. The curve labeled T illustrates the total growth rate. For kh values <2.7 the instability is dominated by the two-dimensional contribution Q. The asymptote, kh large, yields the results of Zakharov and Rubenchik (1974).

In deep water the stability diagram for envelope solitons indicates both a maximum growth rate and a cutoff wavenumber above which solitons are stable (Saffman and Yuen, 1977). Presumably, there is a similar behavior for the Davey-Stewartson model. The presence of the terms in the eigenvalue equation proportional to Q complicates the study. Thus, we have not determined that there exists a maximum growth rate nor have we established a cutoff wavenumber for the instability. Nevertheless, for sufficiently small depths, instability is dominated by interaction with the two-dimensional structure of the induced sea level fluctuations and for long transverse disturbances it grows more rapidly than in deep water.

Note in Proof: After this paper was accepted a paper was published by Mark J. Ablowitz and Harvey Segur (On the evolution of packets of water waves. *J. Fluid Mech.*, **92**, 691–716). This paper treats the problem discussed in this paper and a shoal water wave system. They arrive at an identical stability criteria to that in this paper. However, the details of the soliton stability as a function of depth are unique to this paper.

Acknowledgments. I wish to thank both reviewers of the original paper for their incisive comments. At the suggestion of one of the reviewers I am omitting a section in the original paper comparing observed sea level fluctuations and wave trains which was conjecture. I will discuss these observations in a future report when a more detailed analysis of the data has been completed. This study was supported by the National Science Foundation under Grant OCE 78-11578 and represents Contribution No. 1105 from the Department of Oceanography, University of Washington.

APPENDIX

Details of the Stability Calculations

The evaluation of the eigenvalues Ω requires the determination of a number of inner products. We consider these inner products in the Appendix.

The term $\langle v_0^+ | u_0^+ \rangle$. This term is by definition

$$\int_{-\infty}^{\infty} \phi\left(-\frac{\partial \phi}{\partial \gamma^2}\right) d\xi$$

which has the value

$$\langle v_0^+ | u_0^+ \rangle = -\frac{\partial I}{\partial \gamma^2}, \tag{A1}$$

with

$$I = \int_{-\infty}^{\infty} \phi^2 d\xi.$$

The term $\langle v_0^- | u_0^- \rangle$. This term is by definition

$$-\int_{-\infty}^{\infty} \xi \phi \phi_\xi d\xi$$

which has the value

$$\langle v_0^- | u_0^- \rangle = \frac{I}{2}. \tag{A2}$$

The term $\langle v_0^+ | (L_1 + L_0) u_0^+ \rangle$. We note that $L_0 = L_1 - 2\nu\phi^2$ and that $L_1 u_0^+ = \phi$, then we calculate

$$\langle v_0^+ | (L_1 + L_0) u_0^+ \rangle = \int_{-\infty}^{\infty} \left(2\phi^2 + 2\nu\phi^3 \frac{\partial \phi}{\partial \gamma^2}\right) d\xi = I. \tag{A3}$$

The term $\langle v_0^- | (L_1 + L_0) u_0^- \rangle$. Since $L_1 u_0^- = 0$, we have

$$\langle v_0^- | (L_1 + L_0) u_0^- \rangle = 2\langle v_0^- | \phi^2 u_0^- \rangle = \tfrac{2}{3}\gamma^2 I. \tag{A4}$$

The term $\langle v_0^+ | L_0 \phi Q_1^+ \rangle$. We note that the operator L_0 may be written

$$L_0 = \frac{\lambda}{\phi}\frac{\partial}{\partial \xi}\left(\phi^2 \frac{\partial}{\partial \xi}\frac{1}{\phi}\right),$$

then it follows that the inner product is

$$\langle v_0^+ | L_0 \phi Q_1^+ \rangle = \lambda \int_{-\infty}^{\infty} \frac{\partial}{\partial \xi}\left(\phi^2 \frac{\partial}{\partial \xi} Q_1^+\right) d\xi = 0. \tag{A5}$$

The term $\langle v_0^- | L_0 \phi Q_1 \rangle$. The solution of Eq. (12) may be written

$$Q_1^- = \frac{\kappa_1}{\mu\lambda_1\left(\dfrac{\mu_1 l}{\lambda_1}\right)^{1/2}} \times \left\{\int_{-\infty}^{\xi} \exp[(\mu_1 l/\lambda_1)^{1/2}(\chi - \xi)]\phi\phi_\chi d\chi + \int_{\xi}^{\infty} \exp[(\mu_1 l^2/\lambda_1)^{1/2}(\xi - \chi)]\phi\phi_\chi d\chi\right\}.$$

Using the differential expression for the operator L_0 we find

$$\langle v_0^- | L_0 \phi Q_1^- \rangle = -\lambda \int_{-\infty}^{\infty} \xi \frac{\partial}{\partial \xi}(\phi^2 Q_{1_\xi}^-) d\xi = \lambda \int_{-\infty}^{\infty} \phi^2 Q_{1_\xi}^- d\xi.$$

The derivative of Q_1^- has the value,

$$Q_{1_\xi}^- = -\left(\frac{\kappa_1}{\mu\lambda_1}\right) \times \left\{\int_{-\infty}^{\xi} \exp[-(\mu_1 l^2/\lambda_1)^{1/2}(\xi - \chi)]\phi\phi_\chi d\chi - \int_{\xi}^{\infty} \exp[-(\mu_1 l^2/\chi_1)^{1/2}(\chi - \xi)]\phi\phi_\chi d\chi\right\},$$

and because ϕ^2 vanishes at $\pm\infty$ we may take the limit as $\mu l^2 \to 0$ before integration. Thus, we have

$$\langle v_0^- | L_0 \phi Q_1^- \rangle = -\frac{\lambda}{\lambda_1}\frac{\kappa_1}{\mu}\frac{4}{3}\gamma^2 I. \tag{A6}$$

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