Oceanic Internal Waves Are Not Weak Waves

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ABSTRACT

It is shown that the oceanic internal wave field is too energetic by roughly two orders of magnitude to be treated theoretically as an assemblage of weakly interacting waves. This may be seen both from recent weak wave theoretical calculations which contradict their premises and also from inspection of magnitudes of advection and wave propagation terms. Thus, much recent discussion of results of implications of weak wave theory should be questioned critically. Scaling arguments based on buoyant turbulence are reviewed briefly. The role of vertical mass flux as distinguishing weak wave interactions from stronger turbulence is discussed. Possible progress by renormalization of wave interaction equations is considered.

1. Observations

Spurred by a wide variety of observations of oceanic temperature and velocity fields and by the synthesis of these observations in the model variance spectra suggested by Garrett and Munk (1972, 1975) and by Cairns and Williams (1976) and Desaubies (1976), several theoretical efforts have attempted to account for the statistical state of the internal wave field. For the model spectrum, or GM spectrum, temperature fluctuations may be interpreted as vertical displacements which may be related to horizontal displacements or velocities according to the linear dynamics of internal gravity-inertial waves of given frequency. The dependence of total wave energy E on depth z due to variation in stratification or Väisälä-Brunt frequency $N(z)$ is assumed to follow WKBJ scaling, *viz.*, nearly that $E \propto N(z)$, implying that wave trains propagate almost unimpeded through the depth of the ocean. Because free waves do not propagate at frequencies $\omega > N$, frequency spectra are predicted to fall off sharply near N. Thus the model spectrum is based on an assumption of nearly linear wave dynamics. Consistency of observations with linear internal wave dynamics has been discussed by Fofonoff (1969) and by Müller *et al.* (1978). Overall, the good fit between various observations and the model spectrum suggests that the oceanic internal wave field may be treated in first approximation as a superposition of weakly interacting waves. Nonetheless, the point of this paper will be to argue that the oceanic internal wave field is too energetic by roughly two orders of magnitude to be so treated.

In discussing the above observations, at least two caveats are required. First, temperature fluctuations may represent not only internal wave displacements on a smooth temperature profile but may be due also to a variety of processes which generate thermohaline finestructure. Confusion or contamination of internal wave spectra by finestructure has been discussed, e.g., by Joyce and Desaubies (1977). A second remark is that relative nonlinearity depends on the length scale considered. Usually it is supposed that small-scale features (<1 m vertical) are more aptly described as turbulent rather than as wavelike. Thus, small-scale measurement of temperature gradient or of shear are not Gaussianly distributed but are more intermittent, as in Osborn (1978). However, even the larger internal wave scales (>10 m vertical, say) are argued below to be highly nonlinear as well.

Although the GM spectral model supposes linear dynamics, for example, to relate temperatures and velocities, linear dynamics cannot account for the distribution of energy among the waves, i.e., for the shape of the spectrum. Moreover, the GM model specifies the absolute amplitude as well as shape of the spectrum but only as an empirical fit to observations. A theoretical goal is to account for that empirical spectrum. There are three parts to the problem: 1) energy sources, 2) energy dissipation and 3) energy redistribution mechanisms. Energy sources, such as discussed by Thorpe (1975), are very irregularly distributed in space or time, and hence do not readily account for the universality of the GM spectrum. Rather, we focus on energy redistribution and dissipation mechanisms as likely to control the wave spectrum. Here there are three theoretical approaches: 1) saturation control, 2) weak wave-wave interactions and 3) turbulent cascade.

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2. Saturation

Especially, the universal amplitude of the GM spectrum suggests some saturation control whereby one imagines that a variety of possible energy sources pump energy into the wave field which becomes so energetic that "breaking" occurs, leading to rapid dissipation of any excess energy. The model is much as Phillips (1958) saturation hypothesis for wind-generated surface gravity waves. However, for internal waves it is more difficult to define the "breaking" process or even suitable criteria for breaking. In simple cases one may recognize either shear instability, as in Miles (1961) or Howard (1961), or gravitational instability, as in Orlanski and Bryan (1969). In fact, these mechanisms work together, as discussed by Thorpe (1978), while breaking in the open ocean, many-wave environment may be of a more amorphous, unrecognizable kind. Thus, while the absolute amplitude of the GM spectrum suggests some saturation control, the notion of internal wave breaking is vague and in particular does not predict a shape for the wave spectrum.

3. Weak wave interaction

Whereas saturation control is a suggestive but vague idea, theories of weak wave interaction are rich in detailed prediction. However, in their presentations of weak wave calculations for the internal wave field, Olbers (1976) and McComas and Bretherton (1977) do not discuss extensively the theoretical basis for these calculations which was developed by Hasselmann (1962) or Benney and Saffman (1966). It is important to review briefly that basis in order to assess the subsequent calculations.

Let a physical space field $\phi(\mathbf{x},t)$ have a Fourier representation

$$\phi(\mathbf{x},t) = \int d\mathbf{k}\hat{\phi}(\mathbf{k},t)\exp(i\mathbf{k}\cdot\mathbf{x}). \qquad (1)$$

Then suppose we have obtained a quadratically nonlinear equation of motion for $\hat{\phi}(k,t)$, i.e.,

$$(\partial_t + i\Omega_l)\hat{\phi}_l^* = \delta\int_\Delta dk_m dk_n A_{lmn}\hat{\phi}_m\hat{\phi}_n, \qquad (2)$$

where subscripts l, m, n, denote both wave vector $\mathbf{k}$ and an index s which labels distinct free wave modes propagating for any $\mathbf{k}$. The asterisk denotes complex conjugation, while $\int_\Delta$ denotes integration over wave vectors satisfying $\mathbf{k}_l + \mathbf{k}_m + \mathbf{k}_n = 0$. A_{lmn} is a coupling coefficient whose form depends on the kind of nonlinearity in the equation of motion of $\phi(\mathbf{x},t)$, while δ is included as a factor scaling the strength of nonlinear coupling. The usual object is to predict the time evolution of the average variance spectrum $\Phi(l,t) = \langle\hat{\phi}_l^*\hat{\phi}_l\rangle$, where angle brackets denote a hypothetical average over an ensemble of similar but not identical initial conditions.

Weak wave or "weak turbulence" theory then considers the case where δ is small in some sense. For very small δ, the system is close to free wave propagation

$$\hat{\phi}_l = a_l\exp(i\Omega_l t), \qquad (3)$$

with a_l a constant, complex amplitude. Nonlinearity is included by expanding a_l in a power series in δ, hence one may write $\Phi(l,t) = \langle a_l^* a_l\rangle$ as a power series in δ where the lowest order term Φ_0 is constant and higher order terms $\delta^n\Phi_n$ depend on t. For most systems with quadratic nonlinearity, Φ_2 contains terms proportional to t due to interactions among triads of waves satisfying the frequency resonance condition

$$\Omega_l + \Omega_m + \Omega_n = 0. \qquad (4)$$

Thus, no matter how small the wave amplitude or, equivalently, how small δ is chosen, there comes a time when $\delta^2\Phi_2 > \Phi_0$ and the power series breaks down. This difficulty is resolved by supposing that Φ_0 which is constant in the "fast" time t may gradually vary over a "slow" time $\tau = \delta^2 t$. Then one absorbs the secular growth of Φ_2 into the slow variation of Φ_0 as

$$\frac{\partial\Phi_0}{\partial\tau} = C_t^\infty\Phi_2, \qquad (5)$$

where $C_t^\infty\Phi_2$ denotes the coefficient of t in Φ_2 in the limit $t\to\infty$. There remains a difficulty in evaluating $C_t^\infty\Phi_2$ which integrates over averages of four complex wave amplitudes, say, $\langle a_l a_m a_p a_q\rangle$. This is resolved by the random phase approximation that the phase of each wave is statistically independent of all other waves and hence, e.g.,

$$\langle a_l a_m a_p a_q\rangle = \Phi_l\Phi_m[\delta(l-p)\delta(m-q) + \delta(l-q)\delta(m-p)] \qquad (6)$$

for fixed l and m. Substitution of (6) into (5) is also called the quasi-gaussian approximation whose consistency in the limit $\delta\to 0$ was demonstrated by Benney and Newell (1969). The result is a transport equation which usually has a form

$$\frac{\partial\Phi_l}{\partial\tau} = T_l = \int_\Delta dk_m dk_n B_{lmn}(\Phi_m\Phi_n - \Phi_l\Phi_n)\pi\delta(\Omega_l + \Omega_m + \Omega_n), \qquad (7)$$

where B_{lmn} are geometric coefficients obtained from the A_{lmn} in (2). For simplicity I also do not distinguish sum and difference interactions between left- and right- going waves, a distinction which formally is absorbed in the subscript indexing. Strictly, Eq. (7) is valid as $\delta\to 0$. One hopes that (7) is approximate if δ is small enough.

An equation of motion of the form (2) for internal

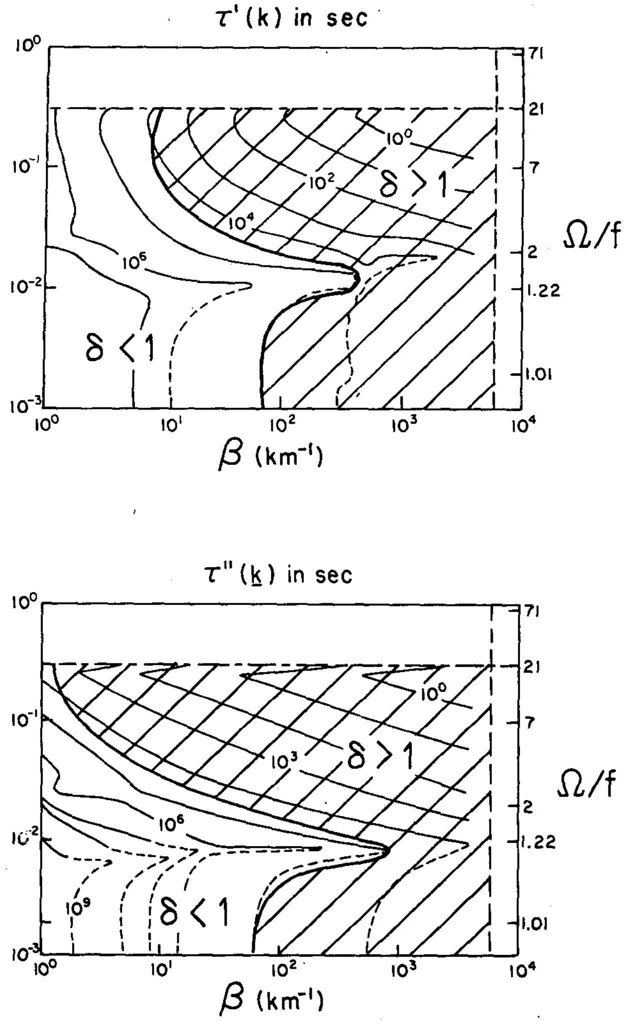

FIG. 1. Based on the model spectrum of Garrett and Munk (1975), McComas (1977) has computed relaxation times for perturbations to that model spectrum by assuming the weak wave or resonant interaction hypothesis. This hypothesis requires that the ratio $\delta = \tau_w/\tau_I$ be small compared to unity, where $\tau_w = 2\pi/\Omega$ is the wave period and τ_I the wave interaction time. Here I sketch approximately the curve $\delta = 1$ on McComas' figures. Over most of the wavenumber-frequency space, weak wave theory computes $\delta > 1$, contradicting its premises.

gravity-inertial waves has been obtained by Olbers (1976) and by McComas and Bretherton (1977). Their method considers the Lagrangian particle displacement field $\xi(t) = \mathbf{x}(t) - \mathbf{x}_0$, where $\mathbf{x}_0$ is a rest position. The particle motion is governed by a Hamiltonian H which is given by a power series in the small displacement $|\xi|$. Retaining terms up to cubic in $|\xi|$, Hamilton's equations yield (2). Then an equation of the form of (7) has been numerically evaluated to give the evolution rate for the wave action spectrum, i.e., the wave energy divided by frequency.

Results of these numerical evaluations show that the evolution time of the spectrum is longer than the wave period for frequencies $\lesssim 0.3\,N$ and vertical scales $\gtrsim 10$ m. It is then sometimes implied that, since the spectral evolution time is longer than the wave period, the weak wave premise is satisfied. It is *most important* to realize that this certainly is *not true*. For example, a theory of fully developed turbulence might predict a stationary spectrum, hence with an infinitely long evolution time which is no indication of weak interaction.

Another numerical evaluation has been performed by McComas (1977) who considered energy transfer according to (7) for a smooth GM spectrum with a small, narrow-band perturbation. Perturbations could consist either of additional energy in a narrow band or of an asymmetry between upward and downward propagating waves. (The GM spectrum is symmetric in the vertical.) McComas then plotted the relaxation time for the perturbation as a function of frequency and vertical wavenumber.

In this context it becomes more nearly possible to address the question of weakness of interaction. Wave-wave interactions may be viewed as two competing processes, one which forces or generates wave variance at any scale and another which scatters wave variance out of that scale. On a smooth spectrum the two processes nearly cancel (exactly cancel if stationary). By perturbing the spectrum in a narrow band, McComas has enhanced the scattering process locally, thereby obtaining a relaxation time for the perturbation which may be orders of magnitude more rapid than the evolution time of the smooth spectrum. As waves are destroyed and regenerated by nonlinear interaction, it is this relaxation time of perturbations which may properly measure the residence time of any particular wave component. Thus, it is *this relaxation time which must be long compared to wave periods* in order to employ the weak wave theory leading to (7). Confusion between this very short relaxation time and the longer spectral evolution time figures, in part, in Müller's (1976) erroneous estimate of a possible wave-induced viscosity (see Ruddick and Joyce, 1979). The same confusion is evident in claims that the weak wave method is justified when the spectral evolution time is longer than the wave period.

In Fig. 1 I have sketched on McComas (1977) results a solid curve corresponding to relaxation time equal to wave period, denoted $\delta = 1$. Then one observes that over most of the GM spectrum, i.e., over vertical wavelengths $\lesssim 100$ m and frequencies $> 2f$, *the relaxation time is shorter than the wave period*, i.e., $\delta > 1$. This suggests that nonlinear interaction destroys and regenerates these waves in a fraction of their wave period. In fact, the result contradicts the premises of the calculation and so it is not reliable. However, the result is a clear, indirect proof: the assumption that oceanic internal waves are weak waves leads to a result which contradicts the assumption.

> Therefore oceanic internal waves are not weak waves.

The same question can be approached in terms of the functional derivative of the transfer (7) with respect to the action spectrum, i.e.,

$$\eta_l \equiv \frac{\delta T_l}{\delta \Phi_l} = \int_\Delta B_{lmn}\Phi_m \pi \delta(\Omega_l + \Omega_m + \Omega_n) dk_m dk_n, \quad (8)$$

where η_l is the relaxation rate discussed above. Effectively this is the operation carried out numerically by McComas (1977). Direct numerical evaluations of (8) by Borchardt (private communication) and by Pomphrey *et al.* (1980) are in rough agreement with McComas. In particular, if the GM spectrum is substituted for Φ_m in (8), then one obtains n_l^{-1} less than the wave period. However, we also see in (8) that $\eta \propto \Phi$. Thus, if we were to require that the weak wave hypothesis be satisfied, i.e., that $\eta_l^{-1} > 2\pi\Omega_l^{-1}$, for waves of vertical scales > 10 m and frequencies < 0.3 N, it would be necessary to reduce the variance spectrum Φ (i.e., the GM energy level) by a factor of ~100. This estimate may be seen from Fig. 1. Quantitatively,

> The oceanic internal wave field is too energetic by a factor of ~100 to be treated as weakly interacting waves.

4. Interpretation

There is a reservation to be addressed: η_l may not be an altogether suitable measure of the interaction strength. Indeed, η_l almost certainly overestimates the interaction rate. The problem occurs when l denotes some high-wavenumber mode, while the value of η_l is dominated by contributions from Φ_p where p denotes a low-wavenumber mode. Just such interactions give rise to diffusion-like terms in wavenumber space. As the ratio of wave scales becomes very large, η_l represents in part a random Doppler shift in the random phase velocity field due to low-wavenumber modes. Moreover, if we consider interaction among a triad of high-wavenumber modes, the relative Doppler shift among the three waves tends to vanish as $\mathbf{k}_l + \mathbf{k}_m + \mathbf{k}_n = 0$. Thus, one might hope to employ a resonant interaction formulation even if $\eta_l > \Omega_l$. Quantitatively the fraction of η_l associated with random Doppler shifting is difficult to assess. In any case that fraction only becomes large as the ratio of length scales becomes large. For such high-wavenumber modes, the ratio η_l/Ω_l is quite large (as computed by weak wave theory from the GM spectrum) and so is expected to remain large even after discounting some random Doppler shift.

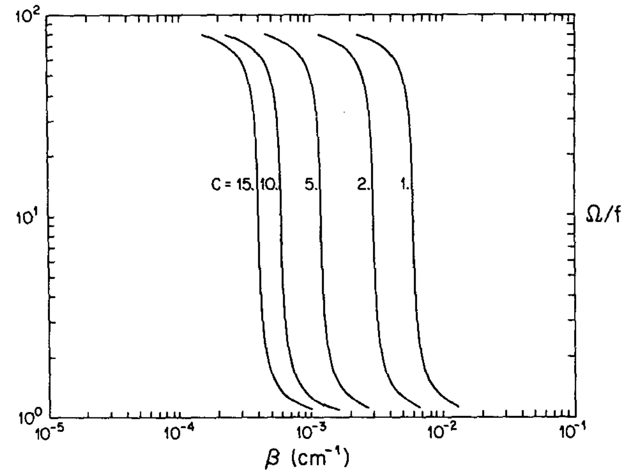

FIG. 2. Contours of horizontal phase speed c as function of vertical wavenumber β and frequency Ω/f for $N = 6 \times 10^{-3}$ s^{-1}, $f = 7 \times 10^{-5}$ s^{-1}.

Rather than attempting further to assess the weak wave formalism *a posteriori*, we turn to an *a priori* assessment by comparing horizontal phase speeds of internal waves with horizontal fluid speeds associated with quasi-inertial shear. For typical $f^2 \ll \Omega^2 \ll N^2$, horizontal phase speed may be approximated

$$c = \frac{\Omega}{\alpha} = \frac{\Omega}{\beta}\left(\frac{N^2 - \Omega^2}{\Omega^2 - f^2}\right)^{1/2} \approx \frac{N}{\beta}. \quad (9)$$

For $N = 6 \times 10^{-3}$ s^{-1}, $f = 7 \times 10^{-5}$ s^{-1}, the complete relations $c(\Omega,\beta)$ is contoured in Fig. 2. The approximation $c = N/\beta$ is readily seen. Total kinetic energy is somewhat variable in near-inertial frequencies. However, consistently with the GM models or observations such as Sanford (1975) one may take $KE = 5N$, where KE is in erg cm^{-3} and N in cph (cycles per hour). Then for $N = 6 \times 10^{-3}$ s$^{-1} \approx 3.6$ cph, the rms horizontal fluid speeds are

$$u_{\rm rms} = (2KE)^{1/2} = 6 \text{ cm s}^{-1}.$$

Now the condition $c = u_{\rm rms}$ defines a transition wavenumber $\beta_c = N/u_{\rm rms} \approx 10^{-3}$ cm^{-1}, corresponding to a vertical wavelength of over 60 m. The result for $N = 6 \times 10^{-3}$ s^{-1} is that waves of vertical wavelength < 60 m propagate more slowly than typical fluid velocities. If one chooses smaller N to represent the main thermocline or deeper water, the transition wavelength increases as $N^{-1/2}$.

Turning to the question of the relative vertical length scales of buoyancy waves and of inertial shear we are confronted by velocity spectra which are continuous and red in vertical wavenumber. However, we see from observations such as Sanford (1975) that velocity differences of 6 cm s^{-1} are encountered over vertical separations of ~60 m at $N = 3.6$ cph. Thus, there is no disparity between the vertical length scale of buoyancy waves and the vertical scale of the shear with which the waves strongly interact. If we considered buoyancy waves of still higher vertical wavenumber, their horizontal phase speeds would be proportionately less and they would encounter critical velocity differences over still shorter distances.

This *a priori* assessment is altogether consistent with the *a posteriori* self-contradiction of the weak wave calculations.

> Most oceanic internal waves encounter fluid velocity differences greater than their phase speeds over distances of their own vertical wavelengths.

Quantitatively, to reduce the wavelength of these strongly affected waves by an order of magnitude, i.e., to increase β_c, one should reduce $u_{\rm rms}$ by an order, thus reducing energy levels by a factor of ~100 as we have seen previously.

Two further remarks: The absence of a scale separation between waves and shear ought to dissuade proponents of critical level mechanisms, especially if those mechanisms are understood from the viewpoint of ray tracing. It would remain possible to consider singular modes of a Taylor-Goldstein problem but this would still be troubled by time-dependent "mean" motion and the presence of a broad spectrum of finite perturbations.

The second remark is that our discussion above may be cast as a Richardson number condition. Velocity variance which yields differences of 6 cm s^{-1} over 60 m vertical is dominated by somewhat larger scale waves for which a representative wavenumber could be $\frac{1}{2}\beta_c$, say. Shear variance is then $S^2 \approx \frac{1}{4}\beta_c^2 u_{\rm rms}^2$. With $\beta_c = N/u_{\rm rms}$ the Richardson number is

$$\mathrm{Ri} = \frac{N^2}{S^2} \approx 4. \tag{10}$$

Here the value of 4 is obviously only approximate and (10) should be read $1 < \mathrm{Ri} < 10$. This suggests a novel kind of saturation hypothesis. Perhaps oceanic energy levels are limited not by transition to three-dimensional turbulence near the classical $\mathrm{Ri} = \frac{1}{4}$ but rather by a transition from weak to strong wave-wave interactions nearer $\mathrm{Ri} \approx 4$.

5. Turbulence

The indicated failure of weak wave theories may prompt more attention to rather simpler kinds of dimensional scaling arguments which are usually associated with turbulent cascades. On vertical scales of tens to hundreds of meters, the ocean is manifestly not "turbulent," i.e., neither overturning nor nearly isotropic nor strongly dissipative. Temperature-velocity correlations seem consistent with internal wave relations rather than turbulence. Nonetheless, one might expect to realize turbulence scaling in smaller vertical scales. Moreover from the very limited dynamical basis for such scalings, one cannot easily define their limits (if any!) of validity. In this section I try to bring together a couple of different scaling arguments in order to round out a picture.

Dimensional scaling arguments for stably stratified turbulence predate observations of the GM spectrum, having originally developed around the problem of radio wave propagation in the stably stratified upper atmosphere. A common starting point for turbulent cascade theories is Kolmogorov's (1941) theory for the structure of unstratified, high Reynolds number turbulence. Assuming an inertial subrange in which details of the large eddy field are erased yet the direct effect of very small viscosity is not felt, only the rate $\epsilon(k) = \epsilon_0$ at which kinetic energy passes from larger to smaller scales is important. Then, dimensionally, a spectrum of kinetic energy can only have the form

$$K(k) = c_1 \epsilon_0^{2/3} k^{-5/3}, \tag{11}$$

with c_1 an empirical constant. If a stable mean stratification is present, it becomes possible to exchange kinetic and potential energy by doing work against the stratification, i.e., against gravity. Then $\epsilon(k) \neq \epsilon_0$, and Eq. (11) is not valid. Bolgiano (1959, 1962) argued that in larger scales of motion, the rate of working against gravity becomes more important then ϵ_0. This rate of working depends on the rate of production of density variance

$$\chi = \overline{\rho' w'} \frac{d\bar{\rho}}{dz}, \tag{12}$$

where ρ' is the fluctuation about mean density $\bar{\rho}$ and w' is vertical velocity. It may be argued that g, $\bar{\rho}$ and χ must appear in a group

$$r = \frac{g^2}{\bar{\rho}^2} \chi = g \frac{\overline{\rho' w'}}{\rho} N^2, \tag{13}$$

whence

$$K(k) = c_2 r^{2/5} k^{-11/5}, \tag{14}$$

with c_2 another empirical constant.

An alternative approach due to Shur (1962) and Lumley (1964) considers that Kolmogorov scaling (11) will hold even if $\epsilon(k)$ varies slowly with k, i.e., if

$$|\partial \ln\epsilon/\partial \ln k| < 1. \quad (15)$$

For a stationary spectrum, variations in $\epsilon(k)$ will be given by the buoyancy flux $-g\overline{\rho' w'}\bar{\rho}^{-1}$ which has a spectrum $N^2B(k)$. Thus

$$\frac{\partial\epsilon}{\partial k} = N^2B(k). \quad (16)$$

Assuming Kolmogorov scaling

$$B(k) = c_3\epsilon^{1/3}k^{-7/3}, \quad (17)$$

Eq. (16) may be solved for ϵ, requiring $\epsilon \to \epsilon_0$ at large k. The resulting kinetic energy spectrum is

$$K(k) = c_1\epsilon_0^{2/3}[1 + (k/k_b)^{-4/3}]k^{-5/3}, \quad (18)$$

where $k_b = c_4(N^3/\epsilon_0)^{1/2}$. For $k \ll k_b$, Eq. (18) has a limiting form

$$K(k) \propto N^2k^{-3}, \quad (19)$$

although Lumley pointed out that on the subrange (19), the assumption (15) is violated, *viz.*,

$$|\partial \ln\epsilon/\partial \ln k| = 2. \quad (20)$$

Indeed, both the wavenumber dependence and the dependence on $N(z)$ in (19) are inconsistent with the GM model. On the other hand, the Bolgiano form (14) is consistent with a GM model in both wavenumber dependence as $k^{-2.2}$ and in its depth dependence as $[N(z)]^{4/5}$ times some unknown $[\overline{\rho' w'}(z)]^{2/5}$. This may not be far from WKBJ scaling. Neither is it clear to what extent Bolgiano's model implies "turbulence". Simply, the model implies that the spectrum is governed by the rate at which internal waves do work to raise the mass field. However, this is inconsistent with a random superposition of free waves since individual free waves do not provide vertical mass transport.

6. Conclusions

Recent efforts (Olbers, 1976; Müller and Olbers, 1975; Müller, 1976; McComas and Bretherton, 1977; McComas, 1977; Pomphrey *et al.*, 1980) to treat the oceanic internal wave field as an assemblage of weak, resonantly interacting waves have been widely discussed, for example, in reviews by Garrett and Munk (1979) or by Gregg and Briscoe (1980) and in texts by LeBlond and Mysak (1978) or by Lighthill (1978). Sometimes it is cautioned that the predicted evolution time of the internal wave spectrum is not longer than a wave period so that the theories are only marginally valid. In this paper it is seen that the spectral evolution time is irrelevant and that the relevant interaction time is much shorter. Indeed, for observed oceanic energy levels the interaction time is computed to be typically much shorter than the wave period, hence contradicting the weak wave premise. The same result can be obtained directly by comparing horizontal phase speeds with fluid velocities. The result seems clear: *oceanic internal waves are not weak waves* (in the sense of weak resonant interaction theories). It is further estimated that the ocean is too energetic by nearly two orders of magnitude to be treated as weak waves.

The nearly universal energy level of internal waves (as distinct from the shape of spectral distribution) suggests a possible saturation mechanism. It is seen that a transition from weakly to strongly interacting waves may provide such a mechanism, implying Richardson numbers of roughly Ri $\approx$ 4. On smaller scales, motion might be characterized as turbulent. With such a regime in mind, dimensional scaling arguments by Bolgiano (1959), Shur (1962) and Lumley (1964) are reviewed. It is seen that these scaling laws depend essentially on the average vertical mass flux—a quantity that vanishes identically in a field of weakly interacting waves.

7. Speculation

The foregoing picture is a gloomy one from the viewpoint of theoretical understanding. On the one hand, appeal to the careful calculations of weak wave theory seems hopeless given the observed energy levels. On the other hand, we avoid turbulence modeling both because we observe correlations that are wavelike and because turbulence modeling seems to offer only simple, dimensional scaling laws. In this section I will just sketch a third approach which might be classed a "renormalized perturbation method." The sketch is tentative inasmuch as this is work that I have not carried out but only mention for the possible interest of more active investigators in this field.

Progress on internal wave interactions would appear to depend on the following two steps:

1) Reformulation of the equations of motion

In previous discussions, we have criticized the statistical assumptions leading from an equation of motion (2) to an equation for average spectral evolution (7). Now we must worry as well whether (2) is valid. The concern is whether a Lagrangian-based derivation is suitable for describing finite-amplitude wave-wave interactions. Even among proponents of Lagrangian derivations there are important differences as between McComas (1975) and Olbers (1976). Both develop their Lagrangians in powers of the *small* fluid particle displacement ξ from a hypothetical rest position, where ξ draws contributions from all waves present. Then as we have seen above, near Ri $\approx$ 4 the displacement induced by the inertial shear is larger than the horizontal wavelength of internal waves during an internal wave period. Because both McComas and Olbers only retain terms up to $O(\xi^3)$, their forms for (2) are identical. However, inclusion of higher order terms in the method of McComas could lead to divergence at observed oceanic energy levels. In addition, one may ask whether the wave modes obtained by requiring the Lagrangian to be stationary at $O(\xi^2)$ are a complete basis for the fluid motion. These cautions prompt us to return to a direct formulation from the Boussinesq equations in Eulerian velocity and density fields. Then Fourier decomposition of velocity and density perturbation fields provides a natural basis for describing buoyant turbulence. However, linear wave modes are disguised because linear operators couple the density and velocity fields. To recover a description in interacting wave modes bases, one would rotate the density-velocity bases. Whether to rotate bases or not will turn out to be a matter of convenience or taste. Formally the result will resemble (2), *viz.*,

$$(\partial_t + i\mathbf{L})\hat{\boldsymbol{\phi}}_l^* = \int_\Delta dk_m dk_n \mathbf{M}_{lmn}\hat{\boldsymbol{\phi}}_m\hat{\boldsymbol{\phi}}_n. \quad (21)$$

Here we suppose $\hat{\boldsymbol{\phi}}$ is a vector whose components may consist of density and velocity coefficients while **L** and **M** are tensors replacing Ω and A (scalars) in (2). By rotating bases into wave modes, one may diagonalize **L**. For the case of internal waves on a vertical plane, such a diagonalization with a discussion of symmetries and invariants has been given explicitly by Ripa (1980). The main point here is that in any particular representation we may choose for (21), we are assured of an exactly quadratically nonlinear equation of motion on a complete basis set regardless of wave amplitude.

2) Renormalization of the spectral evolution equation

The problem of strongly interacting waves has been an urgent one for some time in the field of plasma turbulence. Kadomtsev (1965) has described a closure theory for such strongly interacting waves (i.e., interacting in times as short or shorter than their wave periods) by extension of the "direct interaction" (DI) theory of turbulence as proposed by Kraichnan (1959). These theories may be classed as renormalization methods because of their close resemblance to line or mass renormalizations in quantum field theory. Such renormalizations are not unique. DI is only the simplest second-order line renormalization. For the problem of fluid turbulence a choice among possible renormalizations cannot be made analytically in contrast with an analytical solution for the problem of electromagnetic fields. Thus, the basis for strong wave and turbulence theories is not deductively established. Moreover, attempts to employ the very difficult formalism of Kadomtsev in quantitative calculations have not shown very close agreement with data on plasma turbulence.

A problem which resembles the internal wave-turbulence problem but is rather easier is that of interacting Rossby waves and two-dimensional, quasi-geostrophic turbulence. Rhines (1975) has described this motion is which long waves propagate rapidly with (presumably) weak interactions among themselves whereas shorter Rossby waves propagate very slowly and are completely dominated by advection. For Rossby waves, the interaction equations are simpler since the barotropic wave is defined by a single scalar field, the streamfunction. Fourier transformed, the equations of motion are naturally of the form (2). We can then concentrate on the statistical closure question, hoping that the more difficult problem for internal waves and turbulence may be solved similarly.

A closure theory for strongly interacting Rossby waves has been developed and compared favorably with data from numerical flow simulations by Holloway and Hendershott (1977). Here we just sketch those points on which Holloway and Hendershott differ from the weak wave theories leading to (7). First, we do not suppose the interaction time is long compared to wave periods. The contrary may be the case. Thus we omit (3) which would decompose a wave into an amplitude and a fast varying phase. Then we are not concerned with the rather delicate limit-taking in (5). Second, we do not suppose strictly the random phase or quasi-gaussian assumption (6). Instead we expect that nonlinear interactions will build up higher order, non-gaussian correlations as must be the case since it is the triple correlation $\langle\hat{\phi}_l\hat{\phi}_m\hat{\phi}_n\rangle$ not equal to zero which is responsible for any energy transfer at all. We suppose that a sum over quadruple correlations will draw some contribution from triple correlations, *viz.*,

$$\tfrac{1}{2}\sum_{p,q}(\langle\hat{\phi}_l\hat{\phi}_m\hat{\phi}_p\hat{\phi}_q\rangle + \langle\hat{\phi}_l\hat{\phi}_n\hat{\phi}_p\hat{\phi}_q\rangle + \langle\hat{\phi}_m\hat{\phi}_n\hat{\phi}_p\hat{\phi}_q\rangle$$
$$= \Phi_l\Phi_m + \Phi_l\Phi_n + \Phi_m\Phi_n - \mu_{lmn}\langle\hat{\phi}_l\hat{\phi}_m\hat{\phi}_n\rangle, \quad (22)$$

where μ_{lmn} is a coupling coefficient to be determined in a self-consistent way. Substitution of (22) in place of (6) results in an equation much as (7), *viz.*,

$$\frac{\partial\Phi_l}{\partial\tau} = T_l = \int_\Delta dk_m dk_n B_{lmn}(\Phi_m\Phi_n - \Phi_l\Phi_m)\theta_{lmn}, \quad (23)$$

where

$$\theta_{lmn} = \mathrm{Real}[\mu_{lmn} + i(\Omega_l + \Omega_m + \Omega_n)]^{-1} \quad (24)$$

and it remains to determine μ_{lmn}.

In (23), just as in (7), the fundamental interaction rate is

$$\eta_l = -\frac{\delta T_l}{\delta \Phi_l} = \int_\Delta dk_m dk_n B_{lmn} \Phi_m \theta_{lmn}, \quad (25)$$

here modified by the replacement of $\pi\delta(\Omega_l + \Omega_m + \Omega_n)$ by θ_{lmn}. We may understand η_l as a frequency uncertainty in mode l due to wave interactions. We may also observe in (24) that μ_{lmn} appears as the frequency uncertainty among three interacting waves, i.e., as a broadening of the resonance condition (4). This motivates us to identify the frequency uncertainty among three waves with the individual interaction rates η_l, η_m and η_n, simply

$$\mu_{lmn} = \eta_l + \eta_m + \eta_n. \quad (26)$$

Now μ_{lmn} has been specified in a way that is consistent with the interaction rate computed in (25). It is on this point that weak wave theories have been inconsistent, *viz.*, supposing $\mu_{lmn} \to 0$ while computing finite η_l, η_m and η_n.

Together (23), (24), (25) and (26) are a closed set of equations guiding the evolution of the variance spectrum Φ_l. In the limit of small amplitude, $\eta_l^2/\Omega_l^2 \to 0$ from (25), then $\theta_{lmn} \to \pi\delta(\Omega_l + \Omega_m + \Omega_n)$ so that one recovers a weak wave theory in the limit that such a theory would be valid. The limit of large amplitudes is such that the natural frequency Ω_l becomes insignificant, i.e., $\Omega_l^2/\eta_l^2 \to 0$, then $\theta_{lmn} \to (\eta_l + \eta_m + \eta_n)^{-1}$. This limit corresponds to a theory of strong turbulence as proposed by Edwards (1964). One simple, unifying step, namely reintroducing the interaction rate into the calculation of the interaction rate, takes one from a theory of weak waves to a theory of strong turbulence. However, the turbulence theory of Edwards (1964), like that of Kraichnan (1959), is known to be defective in its treatment of the interaction between very long and very short scales. The problem is basically that of random Doppler shifting as discussed previously so that one cannot sum frequency uncertainties as in (26). Rather, the effect of advection by very long waves which contributes substantially to η_l, η_m and η_n should not significantly affect μl_{mn}. Kadomtsev (1965) has suggested limiting the integration in (25) to cut off long-wave contributions. Kraichnan (1971) has suggested a reweighting of the integrand in (25). With such modifications it has been shown that (23) is consistent with Kolmogorov inertial scaling at large Reynolds number and without stratification.

Another possible defect has been suggested in various forms by Legras (1978), Carnevale (1979) and Holloway (1979). The question concerns η_l and μ_{lmn} which are given above as real quantities because θ_{lmn} is real in (24). In fact, it would be possible to drop the real part operator in (24) allowing T_l, θ_{lmn}, η_l and μ_{lmn} to be complex. Then $\partial\Phi_l/\partial\tau$ would be RealT_l, while Imagμ_{lmn} would shift the resonance curve to $\Omega_l + \Omega_m + \Omega_n +$ Imag$\mu_{lmn} = 0$ and Realμ_{lmn} would be the resonance broadening. This and other modifications may be argued in a number of ways but have not been demonstrated in any specific example.

This outline is only a sketch that no doubt will be shown to be wrong in parts. However, certain *prima facie* features could be stressed:

- There is no inherent limit to wave amplitudes. For waves of very small amplitude, a valid weak wave theory would be recovered. For waves of extremely large amplitude, stratification would be negligible and the theory would predict inertial turbulence including a Kolmogorov subrange. Hopefully, such a theory may yield sensible results at intermediate amplitudes.
- There may be a saturation mechanism in (23) which is absent in (7). This is due to the coupling among (24), (25) and (26). For small amplitudes, θ_{lmn} is close to $\pi\delta(\Omega_l + \Omega_m + \Omega_n)$ so that η_l draws contributions in (25) only from nearly resonant triads. With increasing wave amplitude, μ_{lmn} increases as $\eta_l + \eta_m + \eta_n$, hence the rates η_l, η_m and η_n in turn draw contributions from the richer collection of off-resonant triads.
- If in (21) we deal with ϕ_l in primitive bases, i.e., velocity and density components, then we seek in (23) the evolution of the tensor variance spectrum $\mathbf{\Phi}_l = \langle \boldsymbol{\phi}_l \boldsymbol{\phi}_{-l} \rangle$ which has among its components the density covariance and the vertical and horizontal components of mass flux as well as the velocity covariance tensor. Thus we may consider the generation, and interaction with, density finestructure as part of a unified treatment including the rate of working against gravity.

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