A local modification to monopole equations in 8-dimension

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Abstract

Monopole equations on 8-manifolds with spin(7) holonomy are given in [1]. In this work we write a local modification of the monopole equations on $\mathbb{R}^8$. We show that these new equations are admit non-trivial solutions including all the 4-dimensional solutions as a special case as in [1].

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1 Introduction

The Seiberg-Witten monopole equations are stated for 4-dimensional manifolds and these equations have great importance for the topology of smooth four-manifolds (See [6], [5]). There are various generalizations of these equations to higher dimensions (See [3], [6], [1]). In this work we propose a modification of the monopole equations on $\mathbb{R}^8$ which are stated in [1].

2 Preliminaries

The framework used hereafter is described in detail in [6].

Definition 1. *A spin$^c-$structure on a 2n-dimensional oriented real Hilbert space V is a pair (W, Γ) where W is a 2^n-dimensional complex Hermitian vector space and $\Gamma : V \longrightarrow End\,(W)$ is a linear map which satisfies*

$$\Gamma\,(v)^* + \Gamma\,(v) = 0, \qquad \Gamma\,(v)^*\,\Gamma\,(v) = |v|^2\,1$$

for every $v \in V$.

It is pointed out in [6] that such a map can be extended to an algebra isomorphism $\mathbb{C}l(V) \longrightarrow End\,(W)$ which satisfies $\Gamma\,(\widetilde{x}) = \Gamma\,(x)^*$, where $\mathbb{C}l(V) \cong Cl(V) \otimes \mathbb{C}$ is complex Clifford algebra over V, $\widetilde{x}$ is conjugate of x in $\mathbb{C}l(V)$ and $\Gamma\,(x)^*$ denotes hermitian-conjugate of $\Gamma\,(x)$.

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Let (W, Γ) be a spin^c structure on V. There is a natural splitting of W. We fix an orientation of V and denote by

$$\varepsilon = e_{2n}...e_2e_1 \in Cl(V)$$

the unique element of $Cl(V)$ which has degree $2n$ and is generated by a positively oriented orthonormal basis $e_1, ..., e_{2n}$. Then $\varepsilon^2 = (-1)^n$ and hence

$$W = W^+ \bigoplus W^-$$

where the $W^{\pm} = \{w \in W | \Gamma(\varepsilon) w = \pm i^n w\}$ are the eigen spaces of $\Gamma(\varepsilon)$. Note that $\Gamma(v) W^+ \subset W^-$ and $\Gamma(v) W^- \subset W^+$ for every $v \in V$. So the restrictions of $\Gamma(v)$ to W^+ for $v \in V$ determine a linear map $\gamma : V \longrightarrow Hom(W^-, W^+)$ which satisfies

$$\gamma(v)^*\gamma(v) = |v|^2 1$$

for every $v \in V$. The linear map $\Gamma : V \longrightarrow End(W)$ can be recovered from γ via $W = W^+ \bigoplus W^-$ and

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix}.$$

γ satisfies $\gamma(v)^*\gamma(v) = |v|^2 1$ if and only if Γ satisfies

$$\Gamma(v)^* + \Gamma(v) = 0, \Gamma(v)^* \Gamma(v) = |v|^2 1.$$

Let (W, Γ) be a spin^c structure on V. Such a structure gives an action of the space of 2−forms Λ^2V on W. This action is defined by the following:

Firstly, we identify Λ^2V with the space of second order elements of Clifford algebra $C_2(V)$ via the map

$$\Lambda^2V \longrightarrow C_2(V) \ ; \eta = \sum_{i<j} \eta_{ij} e_i \wedge e_j \longmapsto \sum_{i<j} \eta_{ij} e_i e_j.$$

We compose this map with Γ to obtain a map $\rho : \Lambda^2V \longrightarrow End(W)$ given by

$$\rho\left(\sum_{i<j} \eta_{ij} e_i \wedge e_j\right) = \sum_{i<j} \eta_{ij} \Gamma(e_i) \Gamma(e_j)$$

for any orthonormal basis $e_1, ..., e_{2n}$ of V. This map is independent of the choice of the orthonormal basis $e_1, ..., e_{2n}$. The spaces $W^{\pm}$ are invariant under $\rho(\eta)$ for every 2−form $\eta \in \Lambda^2V$. So we can define

$$\rho^{\pm}(\eta) = \rho(\eta)|_{W^{\pm}}$$

for $\eta \in \Lambda^2V$. In 4-dimension $\rho^+(\eta) = \rho^+(\eta^+)$ for every 2−form $\eta \in \Lambda^2V$, where η^+ is the self-dual part of η. The map ρ extends to a map

$$\rho : \Lambda^2V \otimes \mathbb{C} \longrightarrow End(W)$$

on the space of complex valued 2−forms. If η is real valued 2−form then $\rho(\eta)$ is skew-Hermitian and if η is imaginary valued then $\rho(\eta)$ is Hermitian.

Globalizing Γ to $2n$-dimensional oriented manifolds X leads to a $spin^c$ structure $\Gamma : TX \longrightarrow End(W)$, W being a 2^n-dimensional complex Hermitian vector bundle on X. Such a structure exists iff $w_2(X)$ has an integral lift (See [4]). Γ extends to an isomorphism between the complex Clifford algebra bundle $\mathbb{C}l(TX)$ and $End(W)$. There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n}e_{2n-1}...e_1)$ where $e_1, e_2, ..., e_{2n}$ is any positively oriented local orthonormal frame of TX.

A Hermitian connection $\bigtriangledown$ on W is called a $spin^c$ connection (compatible with the Levi-Civita connection) if

$$\bigtriangledown_v(\Gamma(w)\Psi) = \Gamma(w)\bigtriangledown_v \Psi + \Gamma(\bigtriangledown_v w)\Psi$$

where Ψ is a spinor (section of W), v and w are vector fields on X and $\bigtriangledown_v w$ is the Levi-Civita connection on X. $\bigtriangledown$ preserves the sub bundles $W^{\pm}$.

There is a principal $Spin^c(2n)$-bundle P on X such that the bundle W of spinors, the tangent bundle TX, and the line bundle L_Γ can be recovered as the associated bundles

$$W = P \times_{Spin^c(2n)} \mathbb{C}^{2n}, \quad TX = P \times_{Ad} \mathbb{R}^{2n}$$

where Ad is being the adjoint action of

$$Spin^c(2n) = \left\{e^{i\theta}x : \theta \in \mathbb{R},\ x \in Spin(2n)\right\} \subset \mathbb{C}l_{2n}$$

on $\mathbb{R}^{2n}$. Then it can be obtain a complex line bundle $L_\Gamma = P \times_\delta \mathbb{C}$ where $\delta : Spin^c(2n) \to S^1$ by $\delta\left(e^{i\theta}x\right) = e^{2i\theta}$.

There is a one-to-one correspondence between $spin^c$ connections on W and $spin^c(2n) = Lie(Spin^c(2n)) = spin(2n) \oplus i\mathbb{R}$-valued connection 1-forms $\widehat{A} \in A(P) \subset \Omega^1(P, spin^c(2n))$ on P. Hence every $spin^c$ connection $\widehat{A}$ decomposes as

$$\widehat{A} = \widehat{A}_0 + \frac{1}{2^n} trace(\widehat{A})$$

where $\widehat{A}_0$ is the traceless part of $\widehat{A}$. Let $A = \frac{1}{2^n} trace(\widehat{A})$, this is an imaginary valued 1-form in $\Omega^1(P, i\mathbb{R})$ which satisfies

$$A_{pg}(vg) = A_p(v), \quad A_p(p.\xi) = \frac{1}{2^n} trace(\xi) \tag{2.1}$$

for $v \in T_pP$, $g \in Spin^c(2n)$, and $\xi \in spin^c(2n)$. Let

$$\mathcal{A}(\Gamma) = \left\{A \in \Omega^1(P, i\mathbb{R}) : A \text{ satisfies } (2.1)\right\}$$

There is a one-to-one correspondence between these 1-forms and $spin^c$ connections on W. Let $\bigtriangledown_A$ be the $spin^c$ connection corresponding to A. $\mathcal{A}(\Gamma)$ is an affine space with parallel vector space $\Omega^1(X, i\mathbb{R})$. Let $F_A \in \Omega^2(P, i\mathbb{R})$ be the curvature of the 1-form A and D_A denote the Dirac operator corresponding to $A \in \mathcal{A}(\Gamma)$,

$$D_A : C^\infty\left(X, W^+\right) \longrightarrow C^\infty\left(X, W^-\right)$$

defined by

$$D_A(\Psi) = \sum_{i=1}^{2n} \Gamma(e_i) \bigtriangledown_{A,e_i}(\Psi)$$

where $\Psi \in C^{\infty}(X, W^+)$ and $e_1, e_2, ..., e_{2n}$ is any local orthonormal frame.

The Seiberg-Witten equations can be expressed as follows:

Let $\Gamma : TX \longrightarrow End(W)$ be a fixed $spin^c$ structure on X and consider the pairs $(A, \Psi) \in \mathcal{A}(\Gamma) \times C^{\infty}(X, W^+)$. The Seiberg-Witten equations read

$$D_A(\Psi) = 0, \quad \rho^+(F_A) = (\Psi\Psi^*)_0$$

where $(\Psi\Psi^*)_0 \in C^{\infty}(X, End(W^+))$ is defined by $(\Psi\Psi^*)(\tau) = \langle \Psi, \tau \rangle \Psi$ for $\tau \in C^{\infty}(X, W^+)$ and $(\Psi\Psi^*)_0$ is the traceless part of $(\Psi\Psi^*)$.

3 Seiberg-Witten Equations on $\mathbb{R}^4$

We fix the constant spin^c structure $\Gamma : \mathbb{R}^4 \longrightarrow End(\mathbb{C}^4)$ given by

$$\Gamma(w) = \begin{pmatrix} 0 & \gamma(w) \\ -\gamma(w)^* & 0 \end{pmatrix}$$

where $\gamma : \mathbb{R}^4 \longrightarrow End(\mathbb{C}^2)$ is defined on generators e_1, e_2, e_3, e_4 by the following: $\gamma(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\gamma(e_2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\gamma(e_3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\gamma(e_4) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

The spin^c connection $\bigtriangledown = \bigtriangledown_A$ on $\mathbb{R}^4$ is given by

$$\bigtriangledown_j \Phi = \frac{\partial \Phi}{\partial x_j} + A_j \Phi$$

where $A_j : \mathbb{R}^4 \longrightarrow i\mathbb{R}$ and $\Phi : \mathbb{R}^4 \longrightarrow \mathbb{C}^2$. Then the associated connection on the line bundle $L_\Gamma = \mathbb{R}^4 \times \mathbb{C}$ is the connection $1-$form

$$A = \sum_{i=1}^{4} A_i dx_i \in \Omega^1(\mathbb{R}^4, i\mathbb{R})$$

and its curvature 2-form is given by

$$F_A = dA = \sum_{i=1}^{4} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbb{R}^4, i\mathbb{R})$$

where $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$ for $i, j = 1, ..., 4$.

According to the above data Seiberg-Witten equations on $\mathbb{R}^4$ can be written in the following form:

$D_A \Phi = 0$ can be expressed as

$$-\bigtriangledown_1 \Phi + i\sigma_1 \bigtriangledown_2 \Phi + i\sigma_2 \bigtriangledown_3 \Phi + i\sigma_3 \bigtriangledown_4 \Phi = 0$$

and $\rho^+(F_A) = (\Phi\Phi^*)_0$ rewrite

$$\begin{aligned} F_{12} + F_{34} &= -\tfrac{i}{2}\left(\phi_1\overline{\phi}_1 - \phi_2\overline{\phi}_2\right) \\ F_{13} - F_{24} &= \tfrac{1}{2}\left(\phi_1\overline{\phi}_2 - \phi_2\overline{\phi}_1\right) \\ F_{14} + F_{23} &= -\tfrac{i}{2}\left(\phi_1\overline{\phi}_2 + \phi_2\overline{\phi}_1\right) \end{aligned}$$

4 Monopole Equations in 8-dimension

The Seiberg-Witten equations make sense on a manifold X of any even dimension. However, they have interesting consequences only in dimension 4 . For different from 4 dimensions, the second equation $\rho^+(F_A) = (\Phi\Phi^*)_0$ yields an over-determined set of equations having no non-trivial solutions even locally (see [2]).

In [1] the authors construct a consistent set of monopole equations on eight-manifolds with $Spin(7)-$ holonomy, using the generalized self-duality notion of 2-forms.

BDK stated monopole equations for 8-manifolds with $Spin(7)$ holonomy as follows:

Let X be an 8-manifold with $Spin(7)$ holonomy, so that the structure group is reducible to $Spin(7)$) and gives rise to the global subbundles $S_1,\ S_2 \subset \wedge^2(T^*X)$ which are 7 and 21 dimensional respectively. They concentrate on the 7 dimensional subbundle S_1, which is called bundle of self-dual 2-forms and consider its complexification $V^+ = S_1 \otimes \mathbb{C}$. Let F be an imaginary valued 2-form and its projection onto V^+ denoted by F^+ and the projection of the map $\Psi\Psi^*$ onto the sub bundle $\rho^+(V^+) \subset End(W^+)$ by $(\Psi\Psi^*)^+$. Then the monopole equations of BDK are as follows:

$$
\begin{array}{rcl}
D_A\Psi & = & 0 \\
\rho^+\left(F_A^+\right) & = & (\Psi\Psi^*)^+
\end{array}
\tag{4.2}
$$

They exemplify this on $\mathbb{R}^8$ as follows:

Firstly they choose the following basis $f_1, f_2, ..., f_7 \in \wedge^2\mathbb{R}^8$ for S_1 by using the generalized self-duality notions of 2-forms.

$$
\begin{array}{rcl}
f_1 & = & dx_1 \wedge dx_5 + dx_2 \wedge dx_6 + dx_3 \wedge dx_7 + dx_4 \wedge dx_8, \\
f_2 & = & dx_1 \wedge dx_2 + dx_3 \wedge dx_4 - dx_5 \wedge dx_6 - dx_7 \wedge dx_8, \\
f_3 & = & dx_1 \wedge dx_6 - dx_2 \wedge dx_5 - dx_3 \wedge dx_8 + dx_4 \wedge dx_7, \\
f_4 & = & dx_1 \wedge dx_3 - dx_2 \wedge dx_4 - dx_5 \wedge dx_7 + dx_6 \wedge dx_8, \\
f_5 & = & dx_1 \wedge dx_7 + dx_2 \wedge dx_8 - dx_3 \wedge dx_5 - dx_4 \wedge dx_6, \\
f_6 & = & dx_1 \wedge dx_4 + dx_2 \wedge dx_3 - dx_5 \wedge dx_8 - dx_6 \wedge dx_7, \\
f_7 & = & dx_1 \wedge dx_8 - dx_2 \wedge dx_7 + dx_3 \wedge dx_6 - dx_4 \wedge dx_5.
\end{array}
$$

They also associate each 2-form a skew-symmetric matrices in an obvious way, and obtain a constant $spin^c$ structure $\Gamma : \mathbb{R}^8 \longrightarrow End(\mathbb{C}^{16})$ given by

$$
\Gamma(e_i) = \begin{pmatrix} 0 & \gamma(e_i) \\ -\gamma(e_i)^* & 0 \end{pmatrix}
$$

(e_i, $i = 1, 2, ..., 8$ being the standard basis for $\mathbb{R}^8$), where $\gamma(e_1) = Id$, $\gamma(e_i) = f_{i-1}$ for $i = 2, ..., 8$. Then, according to the above form of f_i and Γ, we get the explicit monopole equations on $\mathbb{R}^8$.

The explicit form of $D_A\Psi = 0$ is

$$
\begin{array}{c}
\partial_1\psi_1 - \partial_2\psi_5 - \partial_3\psi_2 - \partial_4\psi_6 - \partial_5\psi_3 - \partial_6\psi_7 - \partial_7\psi_4 - \partial_8\psi_8 = \\
-A_1\psi_1 + A_2\psi_5 + A_3\psi_2 + A_4\psi_6 + A_5\psi_3 + A_6\psi_7 + A_7\psi_4 + A_8\psi_8
\end{array}
$$

$$\partial_1\psi_2 - \partial_2\psi_6 + \partial_3\psi_1 + \partial_4\psi_5 + \partial_5\psi_4 - \partial_6\psi_8 - \partial_7\psi_3 + \partial_8\psi_7 = \\ -A_1\psi_2 + A_2\psi_6 - A_3\psi_1 - A_4\psi_5 - A_5\psi_4 + A_6\psi_8 + A_7\psi_3 - A_8\psi_7$$

$$\partial_1\psi_3 - \partial_2\psi_7 - \partial_3\psi_4 + \partial_4\psi_8 + \partial_5\psi_1 + \partial_6\psi_5 + \partial_7\psi_2 - \partial_8\psi_6 = \\ -A_1\psi_3 + A_2\psi_7 + A_3\psi_4 - A_4\psi_8 - A_5\psi_1 - A_6\psi_5 - A_7\psi_2 + A_8\psi_6$$

$$\partial_1\psi_4 - \partial_2\psi_8 + \partial_3\psi_3 - \partial_4\psi_7 - \partial_5\psi_2 + \partial_6\psi_6 + \partial_7\psi_1 + \partial_8\psi_5 = \\ -A_1\psi_4 + A_2\psi_8 - A_3\psi_3 + A_4\psi_7 + A_5\psi_2 - A_6\psi_6 - A_7\psi_1 - A_8\psi_5$$

$$\partial_1\psi_5 + \partial_2\psi_1 + \partial_3\psi_6 - \partial_4\psi_2 + \partial_5\psi_7 - \partial_6\psi_3 + \partial_7\psi_8 - \partial_8\psi_4 = \\ -A_1\psi_5 - A_2\psi_1 - A_3\psi_6 + A_4\psi_2 - A_5\psi_7 + A_6\psi_3 - A_7\psi_8 + A_8\psi_4$$

$$\partial_1\psi_6 + \partial_2\psi_2 - \partial_3\psi_5 + \partial_4\psi_1 - \partial_5\psi_8 - \partial_6\psi_4 + \partial_7\psi_7 + \partial_8\psi_3 = \\ -A_1\psi_6 - A_2\psi_2 + A_3\psi_5 - A_4\psi_1 + A_5\psi_8 + A_6\psi_4 - A_7\psi_7 - A_8\psi_3$$

$$\partial_1\psi_7 + \partial_2\psi_3 + \partial_3\psi_8 + \partial_4\psi_4 - \partial_5\psi_5 + \partial_6\psi_1 - \partial_7\psi_6 - \partial_8\psi_2 = \\ -A_1\psi_7 - A_2\psi_3 - A_3\psi_8 - A_4\psi_4 + A_5\psi_5 - A_6\psi_1 + A_7\psi_6 + A_8\psi_2$$

$$\partial_1\psi_8 + \partial_2\psi_4 - \partial_3\psi_7 - \partial_4\psi_3 + \partial_5\psi_6 + \partial_6\psi_2 - \partial_7\psi_5 + \partial_8\psi_1 = \\ -A_1\psi_8 - A_2\psi_4 + A_3\psi_7 + A_4\psi_3 - A_5\psi_6 - A_6\psi_2 + A_7\psi_5 - A_8\psi_1$$

where $\partial_i\psi_j = \frac{\partial\psi_j}{\partial x_i}$. The explicit form of $\rho^+\left(F_A^+\right) = (\psi\psi^*)^+$ is

$$F_{15} + F_{26} + F_{37} + F_{48} = \\ \frac{1}{4}\left(\psi_1\overline{\psi}_3 - \psi_3\overline{\psi}_1 - \psi_2\overline{\psi}_4 + \psi_4\overline{\psi}_2 - \psi_5\overline{\psi}_7 + \psi_7\overline{\psi}_5 - \psi_6\overline{\psi}_8 + \psi_8\overline{\psi}_6\right)$$

$$F_{12} + F_{34} - F_{56} - F_{78} = \\ \frac{1}{4}\left(\psi_1\overline{\psi}_5 - \psi_5\overline{\psi}_1 - \psi_2\overline{\psi}_6 + \psi_6\overline{\psi}_2 + \psi_3\overline{\psi}_7 - \psi_7\overline{\psi}_3 + \psi_4\overline{\psi}_8 - \psi_8\overline{\psi}_4\right)$$

$$F_{16} - F_{25} - F_{38} + F_{47} = \\ \frac{1}{4}\left(\psi_1\overline{\psi}_7 - \psi_7\overline{\psi}_1 + \psi_2\overline{\psi}_8 - \psi_8\overline{\psi}_2 - \psi_3\overline{\psi}_5 + \psi_5\overline{\psi}_3 + \psi_4\overline{\psi}_6 - \psi_6\overline{\psi}_4\right)$$

$$F_{13} - F_{24} - F_{57} + F_{68} = \\ \frac{1}{4}\left(\psi_1\overline{\psi}_2 - \psi_2\overline{\psi}_1 + \psi_3\overline{\psi}_4 - \psi_4\overline{\psi}_3 + \psi_5\overline{\psi}_6 - \psi_6\overline{\psi}_5 - \psi_7\overline{\psi}_8 + \psi_8\overline{\psi}_7\right)$$

$$F_{17} + F_{28} - F_{35} - F_{46} = \\ \frac{1}{4}\left(\psi_1\overline{\psi}_4 - \psi_4\overline{\psi}_1 + \psi_2\overline{\psi}_3 - \psi_3\overline{\psi}_2 - \psi_5\overline{\psi}_8 + \psi_8\overline{\psi}_5 + \psi_6\overline{\psi}_7 - \psi_7\overline{\psi}_6\right)$$

$$F_{14}+F_{23}-F_{58}-F_{67}=\frac{1}{4}\left(\psi_6\overline{\psi}_1-\psi_1\overline{\psi}_6-\psi_2\overline{\psi}_5+\psi_5\overline{\psi}_2-\psi_3\overline{\psi}_8+\psi_8\overline{\psi}_3+\psi_4\overline{\psi}_7-\psi_7\overline{\psi}_4\right)$$

$$F_{18}-F_{27}+F_{36}-F_{45}=\frac{1}{4}\left(\psi_1\overline{\psi}_8-\psi_8\overline{\psi}_1-\psi_2\overline{\psi}_7+\psi_7\overline{\psi}_2-\psi_3\overline{\psi}_6+\psi_6\overline{\psi}_3-\psi_4\overline{\psi}_5+\psi_5\overline{\psi}_4\right)$$

In [1] it is pointed out that if the pair (A,Φ) with

$$A=\sum_{i=1}^{4}A_i\left(x_1,x_2,x_3,x_4\right)dx_i \text{ and } \Phi=\left(\phi_1\left(x_1,x_2,x_3,x_4\right),\phi_2\left(x_1,x_2,x_3,x_4\right)\right)$$

is a solution of the 4-dimensional Seiberg-Witten equations, then the pair (B,Ψ) with

$$B=\sum_{i=1}^{4}A_i\left(x_1,x_2,x_3,x_4\right)dx_i$$

(i. e. the first four components B_i of B coincide with A_i, thus not depending on x_5,x_6,x_7,x_8 and the last four components of B vanish) and

$$\Psi=(0,0,\phi_1,\phi_2,0,0,i\phi_1,-i\phi_2)$$

is a solution of the equations $D_A(\Psi)=0$, $\rho^+\left(F_A^+\right)=\left(\Psi\Psi^*\right)^+$ on $\mathbb{R}^8$.

The aim of this work is to modify the equations above on $\mathbb{R}^8$. To do this modification we also consider the subspace S_2 which is the orthogonal complement of S_1 , the spaces of self-dual $2-$forms, in $\wedge^2\mathbb{R}^8$. S_2 is $21-$dimensional subspace of $\wedge^2\mathbb{R}^8$. Let $G=\sum_{i<j}G_{ij}dx_i\wedge dx_j$ be any element of S_2 then it is orthogonal to each element of S_1, in particular it is orthogonal to the basis elements $f_1,f_2,...,f_7$ of S_1, that is $\langle G,f_i\rangle=0$ for $i=1,...,7$. Then we get following equations which determine S_2:

$$\begin{aligned}
G_{15}+G_{26}+G_{37}+G_{48} &= 0\\
G_{12}+G_{34}-G_{56}-G_{78} &= 0\\
G_{16}-G_{25}-G_{38}+G_{47} &= 0\\
G_{13}-G_{24}-G_{57}+G_{68} &= 0\\
G_{17}+G_{28}-G_{35}-G_{46} &= 0\\
G_{14}+G_{23}-G_{58}-G_{67} &= 0\\
G_{18}-G_{27}+G_{36}-G_{45} &= 0
\end{aligned}$$

From these equations we obtain following basis element for S_2

$$\begin{array}{rcl}
g_1 &=& dx_1\wedge dx_2 - dx_3\wedge dx_4\\
g_2 &=& -dx_5\wedge dx_6 + dx_7\wedge dx_8\\
g_3 &=& dx_1\wedge dx_2 + dx_3\wedge dx_4 + dx_5\wedge dx_6 + dx_7\wedge dx_8\\
g_4 &=& dx_1\wedge dx_3 + dx_2\wedge dx_4\\
g_5 &=& -dx_5\wedge dx_7 - dx_6\wedge dx_8\\
g_6 &=& dx_1\wedge dx_3 - dx_2\wedge dx_4 + dx_5\wedge dx_7 - dx_6\wedge dx_8\\
g_7 &=& dx_1\wedge dx_4 - dx_2\wedge dx_3\\
g_8 &=& -dx_5\wedge dx_8 + dx_6\wedge dx_7\\
g_9 &=& dx_1\wedge dx_4 + dx_2\wedge dx_3 + dx_5\wedge dx_8 + dx_6\wedge dx_7\\
g_{10} &=& dx_1\wedge dx_5 - dx_2\wedge dx_6\\
g_{11} &=& dx_3\wedge dx_7 - dx_4\wedge dx_8\\
g_{12} &=& dx_1\wedge dx_5 + dx_2\wedge dx_6 - dx_3\wedge dx_7 - dx_4\wedge dx_8\\
g_{13} &=& dx_1\wedge dx_6 + dx_2\wedge dx_5\\
g_{14} &=& -dx_3\wedge dx_8 - dx_4\wedge dx_7\\
g_{15} &=& dx_1\wedge dx_6 - dx_2\wedge dx_5 + dx_3\wedge dx_8 - dx_4\wedge dx_7\\
g_{16} &=& dx_1\wedge dx_7 - dx_2\wedge dx_8\\
g_{17} &=& -dx_3\wedge dx_5 + dx_4\wedge dx_6\\
g_{18} &=& dx_1\wedge dx_7 + dx_2\wedge dx_8 + dx_3\wedge dx_5 + dx_4\wedge dx_6
\end{array}$$

$$\begin{array}{rcl}
g_{19} &=& dx_1\wedge dx_8 + dx_2\wedge dx_7\\
g_{20} &=& dx_3\wedge dx_6 + dx_4\wedge dx_5\\
g_{21} &=& dx_1\wedge dx_8 - dx_2\wedge dx_7 - dx_3\wedge dx_6 + dx_4\wedge dx_5
\end{array}$$

Among the above basis elements $g_3, g_6, g_9, g_{12}, g_{15}, g_{18}, g_{21}$ are each non-degenerate and moreover the associated skew-symmetric matrices of these $2-$forms constitute a Radon-Hurwitz family of matrices.

Let V^- be the complexification of the subspace of S_2 which is spanned by $\{g_3, g_6, g_9, g_{12}, g_{15},\ g_{18}, g_{21}\}$ and F_A^- be the projection of F_A on to the subspace V^-. The image $\rho^+(V^-)$ is a subspace of $End(W^+)$. We know that for any $\Psi \in C^\infty(\mathbb{R}^8, W^+)$ associates the endomorphism $\Psi\Psi^* \in End\,(W^+)$. We project $\Psi\Psi^*$ on to the subspace $\rho^+(V^-)$ and denote by $(\Psi\Psi^*)^-$. Hence we consider the (new) equation $\rho^+\left(F_A^-\right) = (\Psi\Psi^*)^-$. With the last modification, the monopole equations on $\mathbb{R}^8$ take the following form:

$$\begin{array}{rcl}
D_A\Psi &=& 0\\
\rho^+\left(F_A^+\right) &=& (\Psi\Psi^*)^+\\
\rho^+\left(F_A^-\right) &=& (\Psi\Psi^*)^-
\end{array} \tag{4.3}$$

The explicit form of the third equation with respect to above constant *spin*^{c}-structure Γ is:

$$F_{15} + F_{26} - F_{37} - F_{48} = \frac{1}{4}\left(\psi_1\overline{\psi}_3 - \psi_3\overline{\psi}_1 + \psi_2\overline{\psi}_4 - \psi_4\overline{\psi}_2 - \psi_5\overline{\psi}_7 + \psi_7\overline{\psi}_5 + \psi_6\overline{\psi}_8 - \psi_8\overline{\psi}_6\right)$$

$$F_{12}+F_{34}+F_{56}+F_{78}=$$
$$\frac{1}{4}\left(\psi_5\overline{\psi}_1-\psi_1\overline{\psi}_5+\psi_2\overline{\psi}_6-\psi_6\overline{\psi}_2+\psi_3\overline{\psi}_7-\psi_7\overline{\psi}_3+\psi_4\overline{\psi}_8-\psi_8\overline{\psi}_4\right)$$

$$F_{16}-F_{25}+F_{38}-F_{47}=$$
$$\frac{1}{4}\left(\psi_1\overline{\psi}_7-\psi_7\overline{\psi}_1-\psi_2\overline{\psi}_8+\psi_8\overline{\psi}_2-\psi_3\overline{\psi}_5+\psi_5\overline{\psi}_3-\psi_4\overline{\psi}_6+\psi_6\overline{\psi}_4\right)$$

$$F_{13}-F_{24}+F_{57}-F_{68}=$$
$$\frac{1}{4}\left(\psi_2\overline{\psi}_1-\psi_1\overline{\psi}_2+\psi_3\overline{\psi}_4-\psi_4\overline{\psi}_3-\psi_5\overline{\psi}_6+\psi_6\overline{\psi}_5-\psi_7\overline{\psi}_8+\psi_8\overline{\psi}_7\right)$$

$$F_{17}+F_{28}+F_{35}+F_{46}=$$
$$\frac{1}{4}\left(\psi_1\overline{\psi}_4-\psi_4\overline{\psi}_1-\psi_2\overline{\psi}_3+\psi_3\overline{\psi}_2-\psi_5\overline{\psi}_8+\psi_8\overline{\psi}_5-\psi_6\overline{\psi}_7+\psi_7\overline{\psi}_6\right)$$

$$F_{14}+F_{23}+F_{58}+F_{67}=$$
$$\frac{1}{4}\left(\psi_1\overline{\psi}_6-\psi_6\overline{\psi}_1+\psi_2\overline{\psi}_5-\psi_5\overline{\psi}_2-\psi_3\overline{\psi}_8+\psi_8\overline{\psi}_3+\psi_4\overline{\psi}_7-\psi_7\overline{\psi}_4\right)$$

$$F_{18}-F_{27}-F_{36}+F_{45}=$$
$$\frac{1}{4}\left(\psi_1\overline{\psi}_8-\psi_8\overline{\psi}_1+\psi_2\overline{\psi}_7-\psi_7\overline{\psi}_2+\psi_3\overline{\psi}_6-\psi_6\overline{\psi}_3-\psi_4\overline{\psi}_5+\psi_5\overline{\psi}_4\right)$$

It can be easily checked that the pair (B,Ψ) which is solution to (4.2) is also a solution to(4.3).

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