Seismic analysis of concrete gravity dams by decoupled modal approach in time domain

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ABSTRACT

A modal approach is proposed for dynamic analysis of general concrete gravity dam-reservoir systems in time domain. The method is explained initially, and the analysis of Pine Flat Dam is considered as a verification example. The proposed approach is proved to be a very effective technique. The main advantage being that it relies on eigen-vectors of decoupled system, which can be easily obtained by standard eigen-solution routines.

KEY WORDS

Concrete gravity dam, Pine Flat Dam, decoupled modal approach, Newmark's method

1 Introduction

The finite element method has been widely used in seismic analysis of concrete gravity dams. Although, there are different approaches available in this regard, the most natural method is based on the Lagrangian-Eulerian formulation, which employs nodal displacements and pressure degrees of freedom for the dam and reservoir region, respectively. Meanwhile, it is well known that in this formulation, the induced total mass and stiffness matrices of the coupled system are unsymmetrical due to interaction terms [1]. In direct method of analysis in time domain, it is possible to efficiently transform the direct integration algorithm in such a manner that allows one to work with symmetric matrices [1, 2]. However, in modal analysis, the symmetrization process requires introduction of additional variables in eigen-solution routines, which is not very efficient and creates complications in computer programming [3, 4].

In the present study, a modal analysis method is proposed which is dependent on mode shapes evaluated from the symmetric part of the original eigen-problem of the system. The formulation of this method is presented initially and the procedure is implemented in a special computer program "MAP-76" [5]. Subsequently, the analysis of Pine Flat Dam is considered as a numerical example. The proposed technique is applied for this dam and the results are compared against corresponding results related to direct method of analysis. Finally, the accuracy of the method is evaluated and its convergence is controlled.

2 Method of Analysis

Let us consider a general concrete gravity dam-reservoir system. In this study, the dam is discretized by plane solid finite elements, while plane fluid elements are utilized for the reservoir region. It can be easily shown that in this case, the coupled equations of the system may be written as [6]:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{p}} \end{bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{p}} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{B}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} -\mathbf{M}\,\mathbf{J}\,\mathbf{a}_{\mathrm{g}} \\ -\mathbf{B}\,\mathbf{J}\,\mathbf{a}_{\mathrm{g}} \end{bmatrix} \tag{1}$$

M, **C**, **K** in this relation represent the mass, damping and stiffness matrices of the dam body, while **G**, **L**, **H** are assembled matrices of fluid domain. The unknown vector is composed of **r,** which is the vector of nodal relative displacements and the vector **p** that includes nodal pressures. Meanwhile, **J** is a matrix with each two rows equal to a 2×2 identity matrix (its columns correspond to unit rigid body motion in horizontal and vertical directions) and $\mathbf{a}_{\mathrm{g}}$ denotes the vector of ground accelerations. Furthermore, **B** in the above relation is often referred to as interaction matrix.

The relation (1) can also be written alternatively in a more compact form as:

$$\overline{\mathbf{M}}\,\ddot{\bar{\mathbf{r}}} + \overline{\mathbf{C}}\,\dot{\bar{\mathbf{r}}} + \overline{\mathbf{K}}\,\bar{\mathbf{r}} = -\overline{\mathbf{M}}\,\overline{\mathbf{J}}\,\mathbf{a}_{\mathrm{g}} \tag{2}$$

Where $\bar{\mathbf{r}}$ and $\overline{\mathbf{J}}$ are defined as follows:

$$\bar{\mathbf{r}} = \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix} \tag{3}$$

$$\overline{\mathbf{J}} = \begin{bmatrix} \mathbf{J} \\ \mathbf{0} \end{bmatrix} \tag{4}$$

Meanwhile, the exact forms of $\overline{\mathbf{M}}, \overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$, are well apparent by matching relations (1) and (2) together. Obviously, these matrices can also be written as sum of the symmetric and unsymmetrical parts as below:

$$\overline{\mathbf{M}} = \mathbf{M}_{\mathrm{S}} + \mathbf{M}_{\mathrm{U}} \tag{5a}$$

$$\overline{\mathbf{C}} = \mathbf{C}_{\mathrm{S}} \tag{5b}$$

$$\overline{\mathbf{K}} = \mathbf{K}_{\mathrm{S}} + \mathbf{K}_{\mathrm{U}} \tag{5c}$$

It is noted from equation (1) that the damping matrix is totally symmetric, and the following relation also holds:

$$\mathbf{K}_{\mathrm{U}} = -\mathbf{M}_{\mathrm{U}}^{\mathrm{T}} \tag{6}$$

The coupled equation (1) can be integrated and solutions can be obtained through time by direct method as well as modal approach. The direct integration process is usually carried out by applying Newmark's algorithm. In the normal procedure, this is encountered with a non-symmetric system of equations to be solved at each time step that is not going to be efficient. However, this could be avoided by a Pseudo-Symmetric technique, which is discussed elsewhere, utterly [2]. In modal approach, which is the basis of the present study, the method relies on obtaining the natural frequencies and mode shapes of the system. Thereafter, the solution can be estimated based on the combination of these modes at different time steps.

2.1 Decoupled Modal Technique

The eigen value problem corresponding to relation (2) can be written as follows:

$$\overline{\mathbf{K}}\ \overline{\mathbf{X}}_j = \overline{\lambda}_j \overline{\mathbf{M}}\ \overline{\mathbf{X}}_j \tag{7}$$

Physically, it is clear that the eigen values of this system are real and free vibration modes exist. However, it is noted from the form of matrices $\overline{\mathbf{K}}, \overline{\mathbf{M}}$ (relation (5)) that the system is not symmetric and standard eigen value computation methods are not directly applicable. Although, there are techniques available to arrive at a symmetric form and reduce the problem to a standard eigen value one, it is computationally costly and additional variables are required to be introduced. Therefore, this path is not pursued in the present study. As a substitute, it was preferred to work with the eigen values and vectors extracted from the following eigen-problem:

$$\mathbf{K}_S\ \mathbf{X}_j = \lambda_j \mathbf{M}_S \mathbf{X}_j \tag{8}$$

Where $\mathbf{K}_S, \mathbf{M}_S$ are the symmetric parts of the $\overline{\mathbf{K}}, \overline{\mathbf{M}}$ matrices, as mentioned previously (relation (5)). Of course, the eigenvectors obtained through the above relation, are not the true mode shapes of the coupled system. However, these can be presumed as Ritz' vectors which can be similarly combined to estimate the true solution. The solution of this substitute eigen-problem are easily obtained by standard methods, since the involving matrices are symmetric and positive definite. Having the orthogonal condition and normalizing the modal matrix with respect to mass matrix, one would have:

$$\mathbf{X}^T \mathbf{M}_S \mathbf{X} = \mathbf{I} \tag{9a}$$

$$\mathbf{X}^T \mathbf{K}_S \mathbf{X} = \mathbf{\Lambda} \tag{9b}$$

Where $\mathbf{I}$ is the identity matrix and $\mathbf{\Lambda}$ is a diagonal matrix containing the eigen values of the symmetric substitute system. The following relations are also derived easily based on relations (5), (6) and (9):

$$\mathbf{X}^T \overline{\mathbf{M}}\ \mathbf{X} = \mathbf{I} + \mathbf{X}^T \mathbf{M}_U \mathbf{X} \tag{10a}$$

$$\mathbf{X}^T \overline{\mathbf{K}}\ \mathbf{X} = \mathbf{\Lambda} - \mathbf{X}^T \mathbf{M}_U^T\ \mathbf{X} \tag{10b}$$

As usual in modal techniques, the solution is written as a combination of different modes:

$$\overline{\mathbf{r}} = \mathbf{X}\ \mathbf{Y} \tag{11}$$

The vector $\mathbf{Y}$ contains the participation factors of the modes. Substituting this relation into (2) and multiplying both sides of that equation by $\mathbf{X}^T$, it yields:

$$(\mathbf{X}^T \overline{\mathbf{M}}\ \mathbf{X})\ddot{\mathbf{Y}} + \mathbf{C}^* \dot{\mathbf{Y}} + (\mathbf{X}^T \overline{\mathbf{K}} \mathbf{X})\mathbf{Y} = \mathbf{F}^*(t) \tag{12}$$

In this relation, additional matrix definitions are utilized as below:

$$\mathbf{C}^* = \mathbf{X}^T \overline{\mathbf{C}} \mathbf{X} \tag{13a}$$

$$\mathbf{F}^*(t) = -\mathbf{X}^T \overline{\mathbf{M}}\ \overline{\mathbf{J}}\ \mathbf{a}_g(t) \tag{13b}$$

Or alternatively, the following relation is obtained by employing (10):

$$(\mathbf{I} + \mathbf{X}^T \mathbf{M}_U \mathbf{X})\ \ddot{\mathbf{Y}} + \mathbf{C}^* \dot{\mathbf{Y}} + (\mathbf{\Lambda} - \mathbf{X}^T \mathbf{M}_U^T \mathbf{X})\mathbf{Y} = \mathbf{F}^*(t) \tag{14}$$

Applying the Newmark's technique for integration of this equation, would yield the following equation at each new time step:

$$\hat{\mathbf{K}}\ \mathbf{Y}_{n+1} = \hat{\mathbf{F}}_{n+1} \tag{15}$$

$\hat{\mathbf{K}}$ and $\hat{\mathbf{F}}_{n+1}$ are denoted as the generalized effective stiffness matrix and the generalized effective force vector of the system at time step n+1, respectively. They are defined as below:

$$\hat{\mathbf{K}} = a_0(\mathbf{I}+\mathbf{X}^T\mathbf{M}_U\mathbf{X}) + a_1\mathbf{C}^* + (\mathbf{\Lambda}-\mathbf{X}^T\mathbf{M}_U^T\mathbf{X}) \tag{16}$$

$$\hat{\mathbf{F}}_{n+1} = \mathbf{F}^*_{n+1} + (\mathbf{I}+\mathbf{X}^T\mathbf{M}_U\mathbf{X})(a_0\mathbf{Y}_n + a_2\dot{\mathbf{Y}}_n + a_3\ddot{\mathbf{Y}}_n) + \mathbf{C}^*(a_1\mathbf{Y}_n + a_4\dot{\mathbf{Y}}_n + a_5\ddot{\mathbf{Y}}_n) \tag{17}$$

In general, the vector of participation factors can be solved through relation (15). Thereafter, the unknown vector is obtained by equation (11) as usual in the modal procedure. It must be also mentioned that the generalized effective stiffness matrix employed in relation (15) is inherently unsymmetrical. However, it may be easily transformed to a symmetric matrix by multiplying certain rows of the matrix relation (15) by an appropriate factor. This can be shown as follows:

It is noticed through (1) and (5) that the symmetric matrices utilized in the substitute eigen-problem, corresponds to the decoupled dam-reservoir system. Therefore, the natural frequencies and eigenvectors are actually related to this decoupled system. It should be noted that in actual programming, one can modify the usual subspace iteration routines to converge to the desired lowest modes of the dam first, and similarly for the finite reservoir region afterwards by appropriate selection of initial vectors. Meanwhile, they can also be obtained as two separate problems. Let us now assume for simplicity, without loss of generality that the mode shapes for the dam are ordered first and the ones for the finite reservoir considered subsequently in the modal matrix,

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 \end{bmatrix} \tag{18}$$

and similar arrangements for the eigen values in the diagonal matrix $\mathbf{\Lambda}$.

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{bmatrix} \tag{19}$$

Then, it is clear that the following relations hold:

$$\mathbf{X}^T\mathbf{M}_U\mathbf{X} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{X}_2^T\mathbf{B}\mathbf{X}_1 & \mathbf{0} \end{bmatrix} \tag{20a}$$

$$\mathbf{X}^T\mathbf{M}_U^T\mathbf{X} = \begin{bmatrix} \mathbf{0} & \mathbf{X}_1^T\mathbf{B}^T\mathbf{X}_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{20b}$$

$$\mathbf{C}^* = \begin{bmatrix} \mathbf{C}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2^T\mathbf{L}\mathbf{X}_2 \end{bmatrix} \tag{20c}$$

Of course, it must be also mentioned that, it is usually assumed that the damping matrix of the dam is of viscous type for the analysis carried out by modal approach in time domain. Therefore, this would lead to:

$$\mathbf{C}_1^* = 2\beta_d\mathbf{\Lambda}_1^{1/2} \tag{21}$$

where β_d is the equivalent damping factor, which is assumed constant in this study for all of the modes. Substituting relations (20) into (16, 17), the expanded form of equation (15) is now concluded as:

$$\begin{bmatrix} a_0 \mathbf{I}_1 + a_1 \mathbf{C}_1^* + \boldsymbol{\Lambda}_1 & -\mathbf{X}_1^T \mathbf{B}^T \mathbf{X}_2 \\ a_0 \mathbf{X}_2^T \mathbf{B} \mathbf{X}_1 & a_0 \mathbf{I}_2 + a_1 \mathbf{X}_2^T \mathbf{L} \mathbf{X}_2 + \boldsymbol{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix}_{n+1} = \begin{bmatrix} \hat{\mathbf{F}}_1 \\ \hat{\mathbf{F}}_2 \end{bmatrix}_{n+1} \quad (22)$$

It is noticed that, the vector of participation factors is also assumed to be partitioned into two parts in this relation, and as before the indices 1, 2 correspond to dam and reservoir modes, respectively. Relation (22) is equivalent to (15) and as mentioned previously, this is initially an unsymmetrical system of equation. However, it is now revealing that this matrix relation could become symmetric by multiplying the lower matrix equation by a factor of $-a_0^{-1}$, which yields the following relation:

$$\begin{bmatrix} a_0 \mathbf{I}_1 + a_1 \mathbf{C}_1^* + \boldsymbol{\Lambda}_1 & -\mathbf{X}_1^T \mathbf{B}^T \mathbf{X}_2 \\ -\mathbf{X}_2^T \mathbf{B} \mathbf{X}_1 & -a_0^{-1} (a_0 \mathbf{I}_2 + a_1 \mathbf{X}_2^T \mathbf{L} \mathbf{X}_2 + \boldsymbol{\Lambda}_2) \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix}_{n+1} = \begin{bmatrix} \hat{\mathbf{F}}_1 \\ -a_0^{-1} \hat{\mathbf{F}}_2 \end{bmatrix}_{n+1} \quad (23)$$

This can now be readily solved for the vector of participation factors at each time step, and the original unknown vector is obtained accordingly by the help of relation (11).

3 Numerical Example

In this section, the analysis of Pine Flat Dam is considered as a verification example. The dam is 121.92 m high, with the crest length of 560.83 m and it is located on the King's River near Fresno, California.

A special computer program "MAP-76" [5] is used as the basis of this study. The program was already capable of analyzing a general dam-reservoir system by direct approach in the time domain [2]. In this study, the modal analysis option is also included in the program based on the formulation presented in the previous section.

3.1 Modeling and Basic Data

The Pine Flat dam-reservoir system is considered over a rigid foundation. The two-dimensional finite element model is displayed in Figure 1. The dam section relates to the tallest monolith (121.92 m), and it is assumed in a state of plane stress. The water level is considered at the height of 116.19 m above the base, similar to the previous study [2]. Meanwhile, a length of 200 m is included in the model for the reservoir domain.

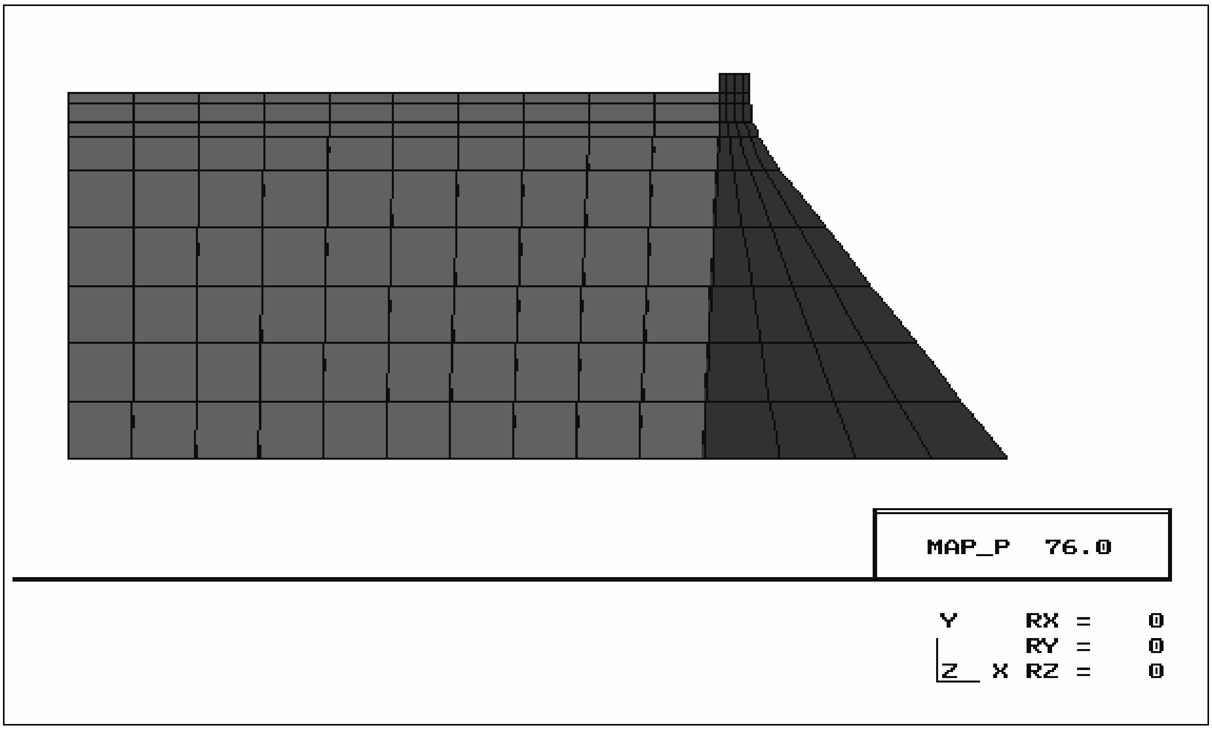


Figure - 1 Dam-reservoir system discretization

The dam body is discretized with 8-node plane solid elements, while 8-node fluid elements are used for the reservoir region. The model consists of a total of 439 nodes and 568 degrees of freedom and it includes 40, 90 plane solid and fluid elements, respectively.

3.1.1 Basic Parameters

The concrete is assumed to be homogeneous and isotropic with the following basic properties:

- Elastic modulus $E_c = 22.75$ GPa
- Poisson's ratio $\nu_c = 0.20$
- Unit weight $\gamma_c = 24.8$ kN/m^3

The water is taken as compressible, inviscid fluid, with weight density of 9.81 kN/m^3 and pressure wave velocity of 1440.0 m/s. Meanwhile, the sommerfeld boundary condition is imposed at the upstream boundary of the impounded water domain, and the reservoir bottom condition is assumed completely reflective.

The main analysis carried out, is based on modal analysis, and viscous damping is assumed for this case. The viscous damping coefficients are considered constant for all the modes ($\beta_d = 0.05$). However, a second case is also analyzed based on the direct approach, where the Rayleigh damping matrix is applied and the corresponding coefficients are determined such that equivalent damping for frequencies close to the first and third modes of vibration would be 5% of critical damping.

3.1.2 Loading

It should be mentioned that static loads (weight, hydrostatic pressures) are each visualized as being applied in one separate increment of time. Therefore, the same time step of 0.01 second, which is chosen in dynamic analysis, is also considered as time increment of static loads application. It is noted that time for static analysis is just a convenient tool for applying the load sequentially, but it is obvious that inertia and damping effects are disregarded in the process. In this respect, the dead load is applied in one increment and hydrostatic pressures thereafter in another increment at negative range of time. At time zero, the actual dynamic analysis begins with the static displacements and stresses being applied as initial conditions.

The dynamic excitation considered, is the S69E component of Taft earthquake records, which is applied in the horizontal x-direction. The time duration utilized, is 13 seconds in each case.

3.2 Analysis Results

As mentioned, the main analysis is performed by modal approach. As a first step of this case, the eigen-problem is solved based on the symmetric parts of the total mass and stiffness matrices. This is actually a decoupled system, and the natural frequencies obtained correspond to either the dam or the reservoir (finite region considered). The first five natural frequencies of each domain are listed in Table 1.

Table 1: Natural frequencies of the dam and reservoir.

Mode number (i)	Natural frequencies f_i (Hz)	
	Dam	Reservoir
1	3.146408	3.115126
2	6.475173	4.749112
3	8.738600	7.795491
4	11.248678	9.300412
5	16.989656	9.958278

It is noticed that the first natural frequency of the reservoir is actually slightly lower than the one corresponding to the dam. Meanwhile, the natural frequencies of the dam are wider spread in comparison to the ones related to the reservoir domain. This means that a much higher number of modes are required for the reservoir in comparison with the dam for an accurate solution.

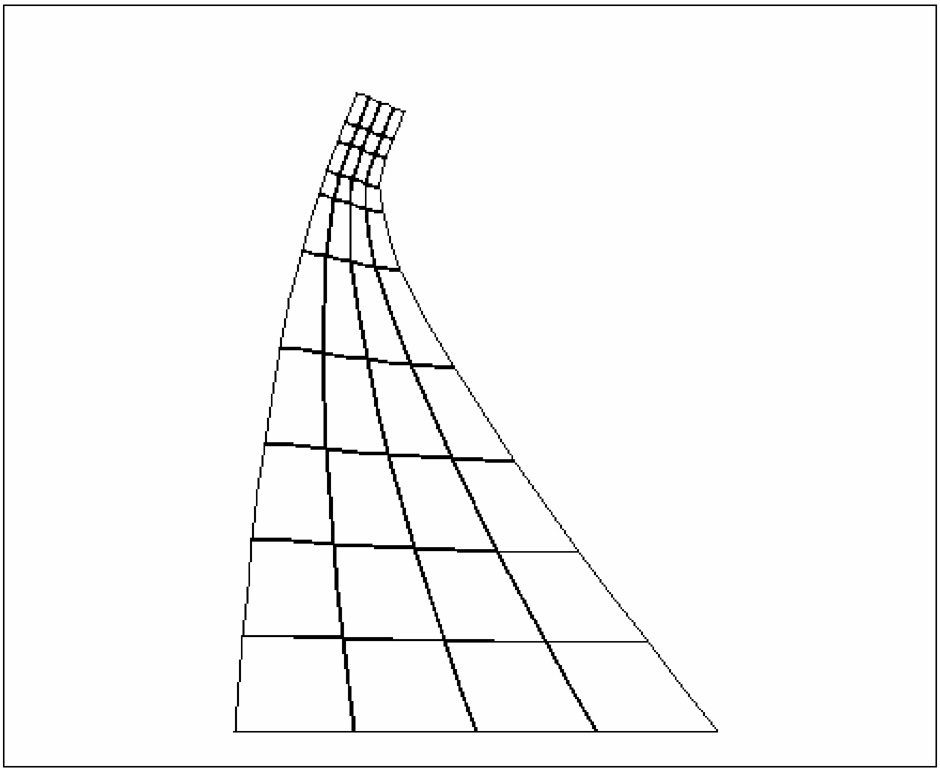
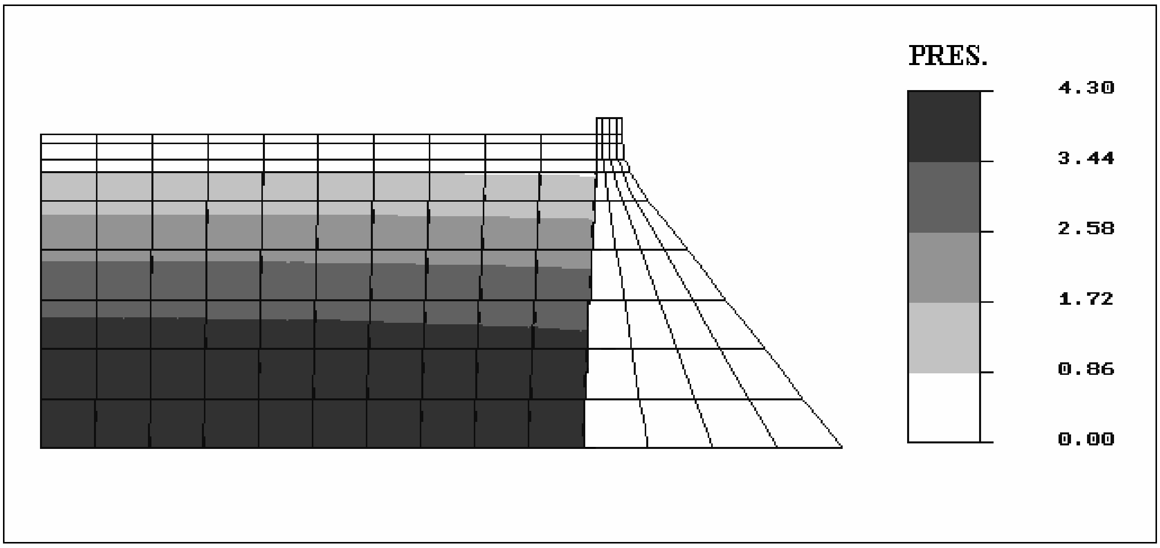
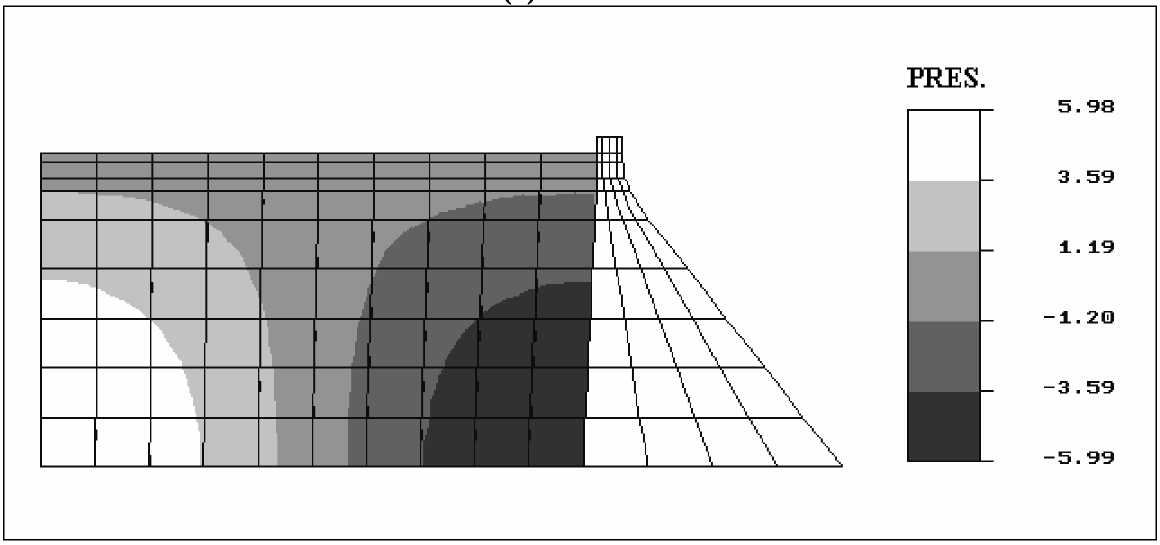


Figure 2. The first mode shape of the dam

The first mode shape of the dam is displayed in Fig. 2. Meanwhile, the first two mode shapes of the reservoir region are depicted in Fig. 3. It is noticed that the first mode of the fluid domain corresponds to a nearly symmetric case which pressures are approximately constant in the horizontal direction, while the second mode is very close to a perfect anti-symmetric case.



(a) Mode 1

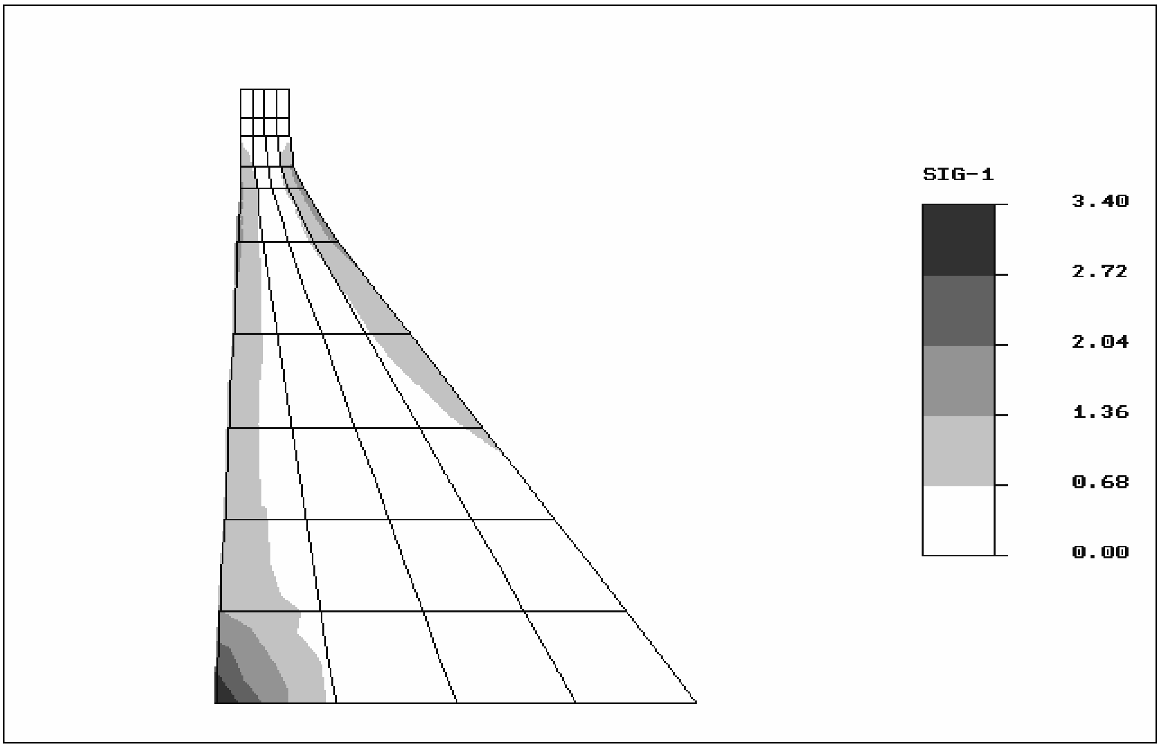


(b) Mode 2

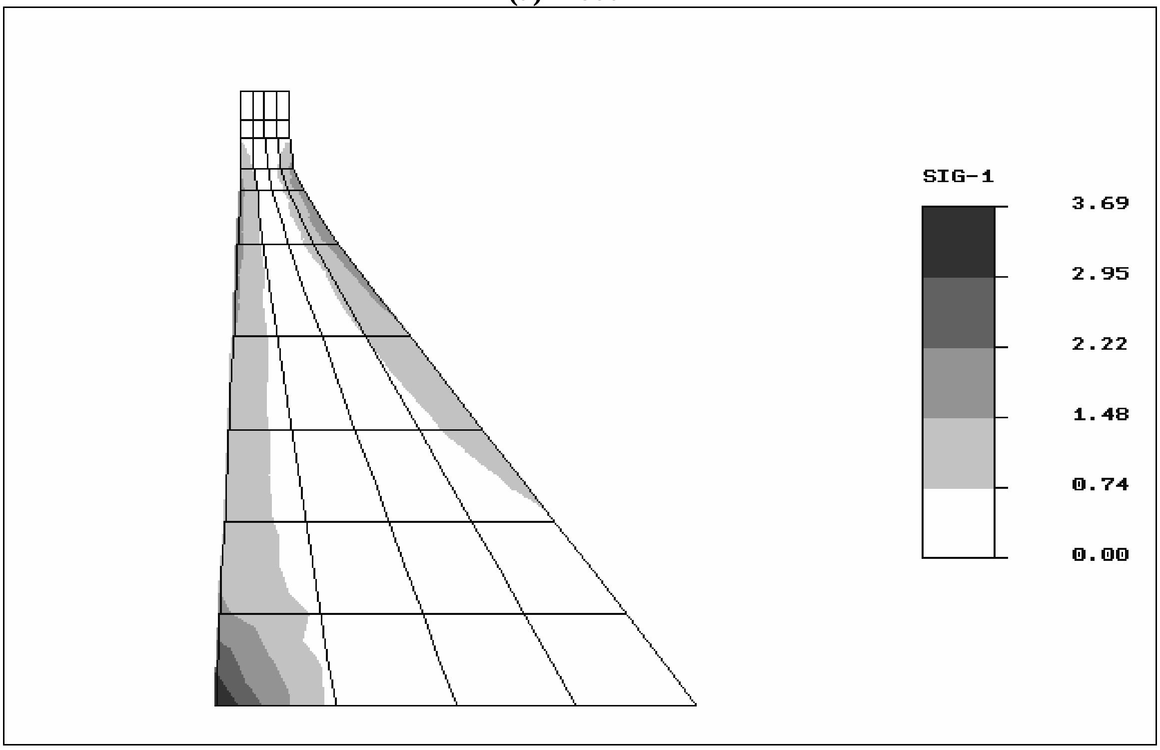
Figure 3. The first two modes of the reservoir

In the next step, the modal analysis is carried out by utilizing 25, 75 modes for the dam and the reservoir domain, respectively.

For comparison purposes, the same model is also analyzed based on the direct approach. All the basic data in this case, are similar to the original case, except for the type of damping. That is Rayleigh damping in comparison with viscous damping. Although, this could cause slight differences in result, it should not be very significant. Neglecting this minor source of difference, it is well known that the direct method results can be considered as exact for the discretization employed, since it is actually equivalent to considering all of the modes of both discrete domains.



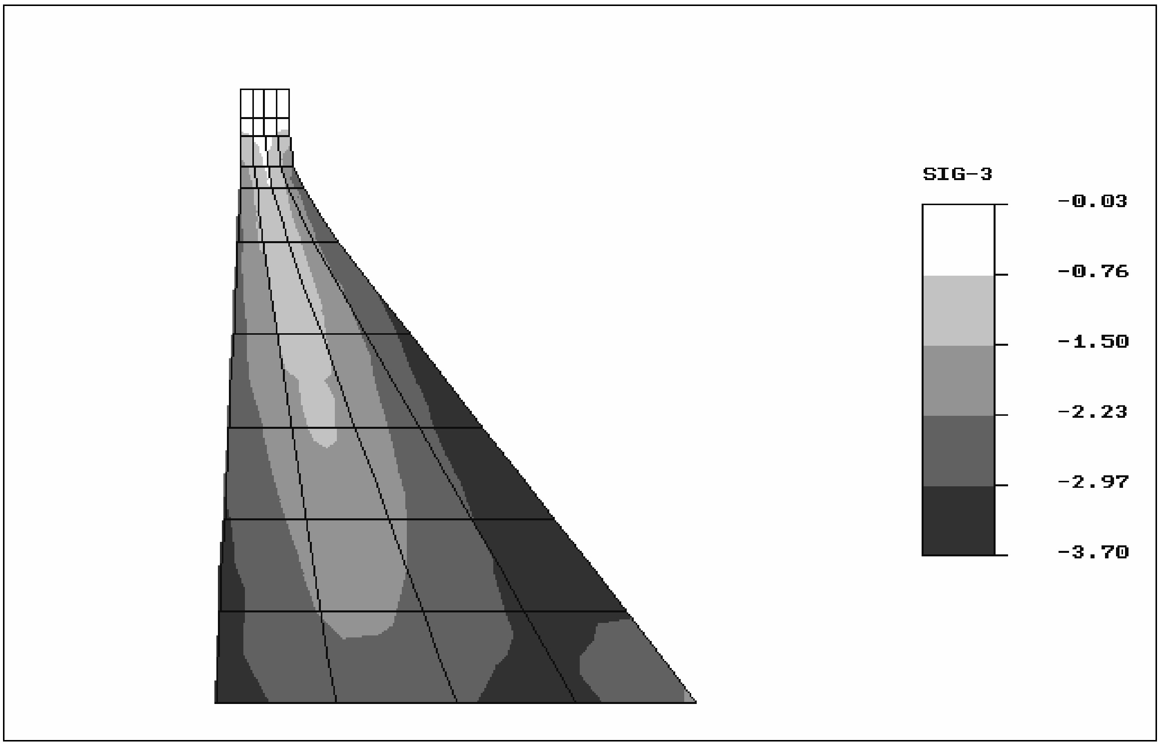
(a) Modal



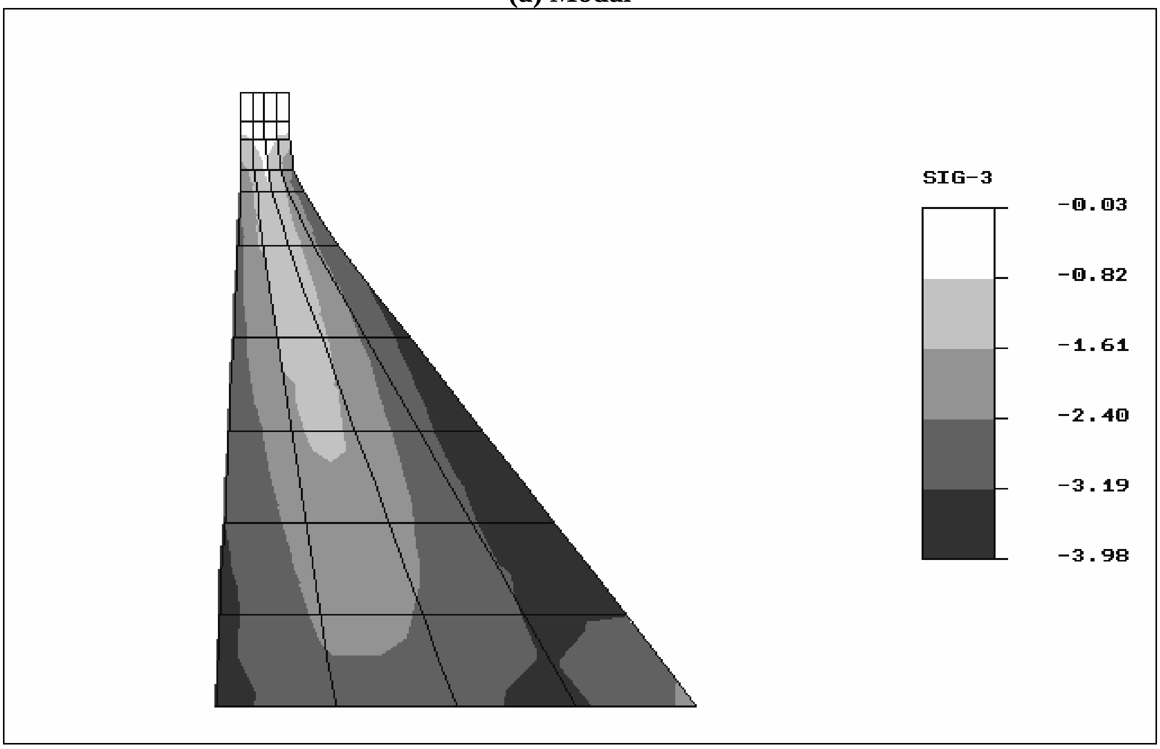
(b) Direct

Figure 4. Envelopes of maximum tensile principal stresses (MPa)

Both cases are analyzed, and the results corresponding to envelope of maximum tensile and compressive stresses are illustrated in Figures 4 and 5. It is observed that the distributions of maximum stresses are very similar for the modal and direct approaches. However, it is noticed that maximum values of tensile and compressive stresses, are about 7.9 and 7.0 percents lower for the modal approach in comparison to the direct method.



(a) Modal



(b) Direct

Figure 5. Envelopes of maximum compressive principal stresses (MPa)

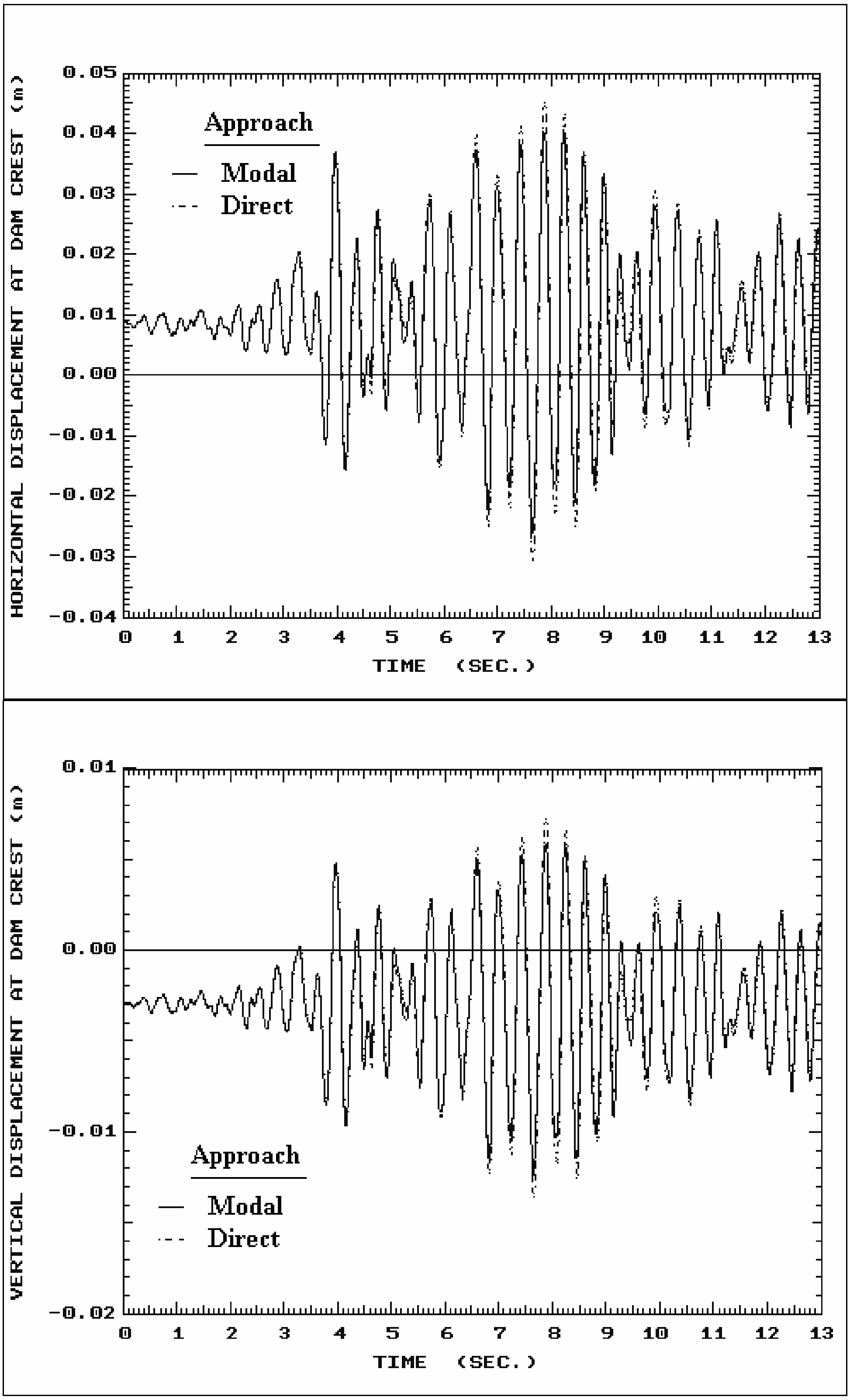


Figure 6. Comparison of displacements histories at dam crest between the two approaches.

For a better evaluation of the results, time histories of some important quantities are depicted in Figures 6 and 7. These are the horizontal, vertical components of displacement at dam crest and the maximum tensile, compressive principal stresses at dam heel. In each graph, the result corresponding to direct method is also shown for comparison purposes.

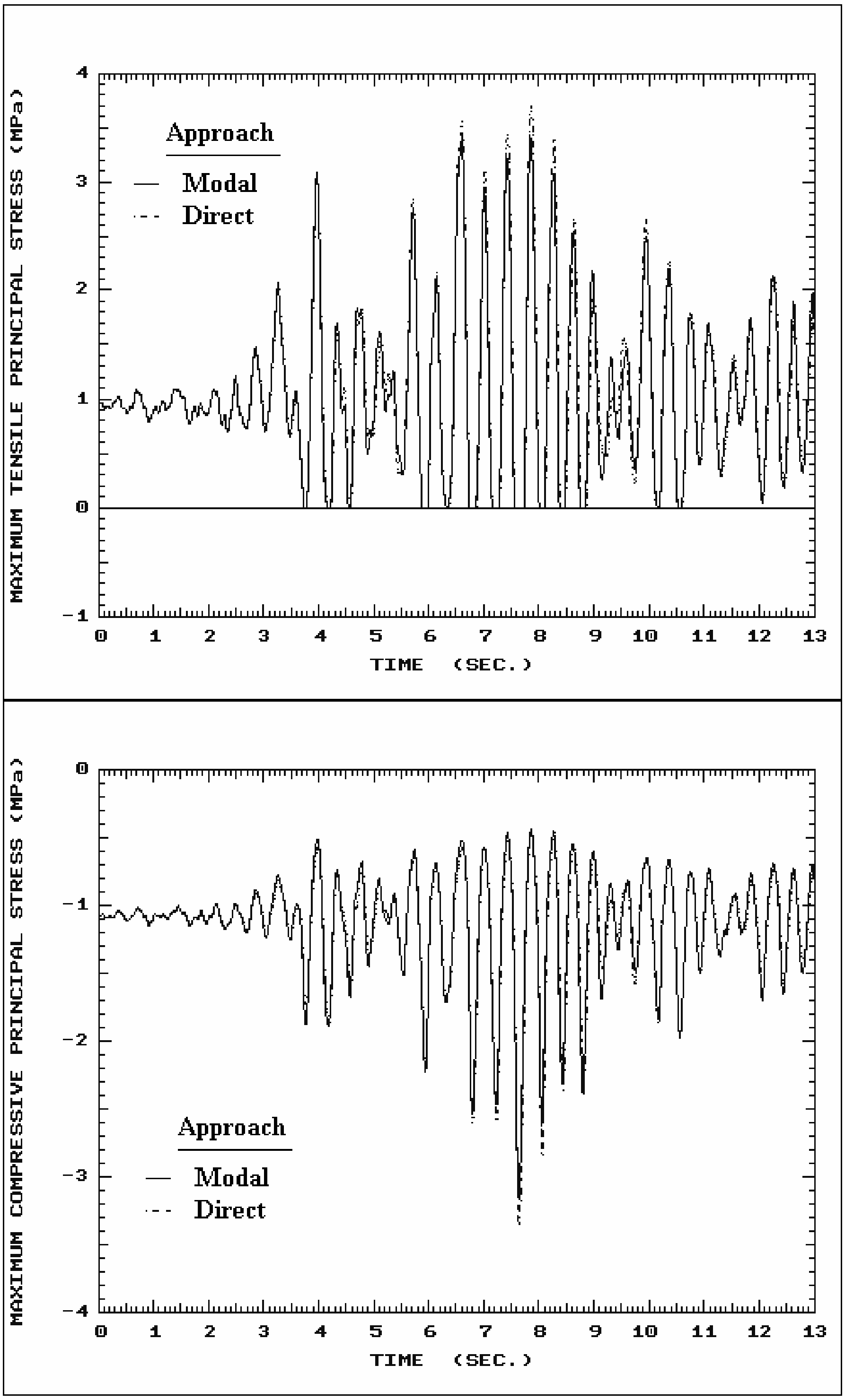
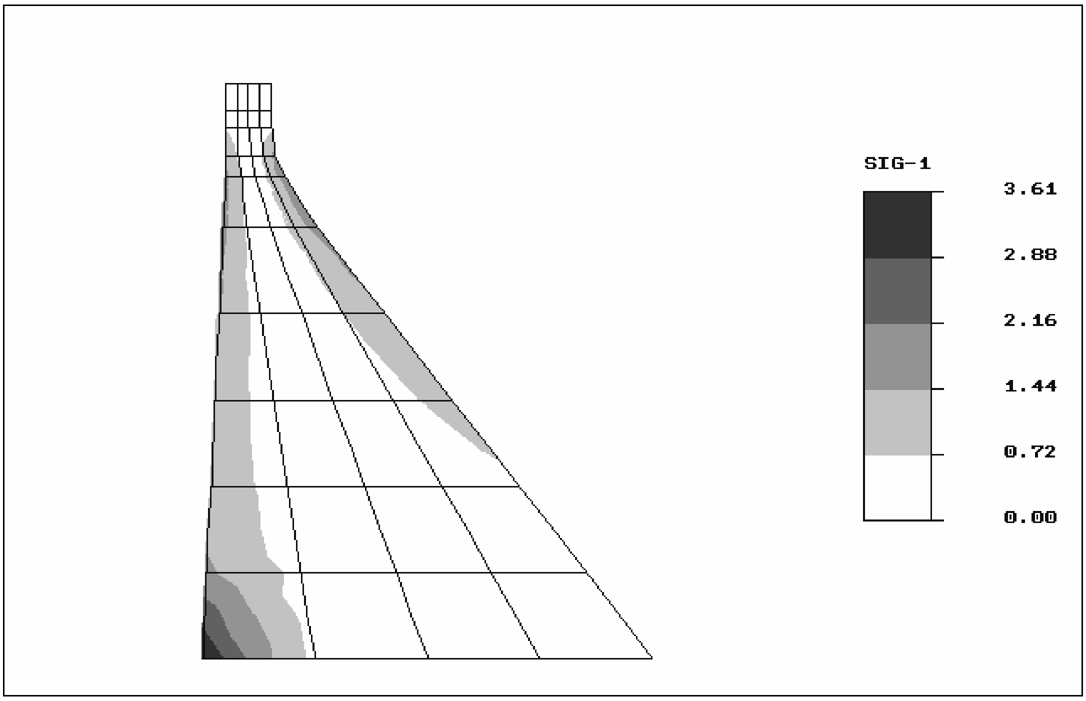


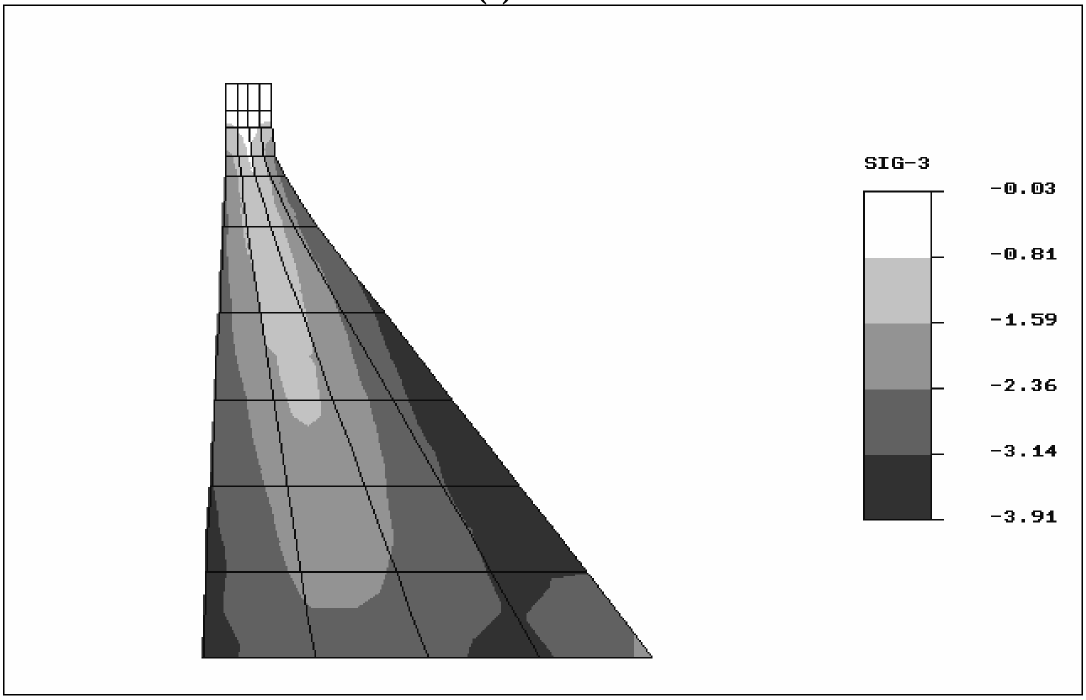
Figure 7. Comparison of maximum principal stresses histories at dam heel between the two approaches.

It is noticed that trends of all quantities monitored, are very similar for the modal and direct approaches. However, the modal technique predicts slightly lower maximum values.

Finally, to evaluate the modal technique more closely, it was decided to double the number of modes utilized over the original case. The envelopes of this case are shown in Figures 8a, and 8b for the maximum tensile and compressive stresses, respectively. These Figures could be compared with corresponding results of direct method (Figures 4b, and 5b).



(a) Tensile



(b) Compressive

Figure 8. Envelopes of maximum principal stresses (MPa) when the number of modes utilized are doubled over the original case

It is noticed that distributions become almost precisely the same. Meanwhile, the absolute maximum tensile and compressive stresses are now merely 2.2 and 1.8 percents lower than the direct approach results. Although, this degree of accuracy is seldom required for practical cases, this illustrates vividly the convergence of the proposed technique.

4 Conclusions

A technique is proposed for earthquake analysis of concrete gravity dams, which is referred to as decoupled modal approach. The method is described initially, and the procedure is implemented in a special computer program "MAP-76". Meanwhile, the analysis of Pine Flat Dam is considered as a numerical example and for verification purposes. The original case analyzed, is based on the proposed technique, and it is carried out by utilizing 25, 75 modes for the dam and the reservoir domain, respectively. Meanwhile, the direct method of analysis is used for comparison purposes. Overall, the main conclusions obtained can be listed as follows:

- The results obtained based on the decoupled modal approach, compare very well with the corresponding results of direct method. More specifically, by comparing the envelopes of maximum tensile and compressive stresses, it is shown that distributions of maximum stresses are very similar for both approaches. Meanwhile, the maximum values of tensile and compressive stresses are only about 7.9 and 7.0 percents lower for the modal approach in comparison to the direct method. Furthermore, trends of all quantities monitored, are very similar and in good agreement for both methods throughout the execution time.
- By doubling the number of modes utilized over the original case, the convergence of the technique is also controlled. For this case, it is noticed that absolute maximum tensile and compressive stresses are merely 2.2 and 1.8 percents lower than the corresponding direct method results.
- The proposed decoupled modal approach is proved to be an effective technique for seismic analysis of concrete gravity dams. The main advantage of this modal technique is that it employs eigen-vectors of the decoupled system, which can be easily obtained by standard eigen-solution routines.

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