Over the evolution of fundamental ideas of fracture mechanics

Razvan Raducanu

Abstract

This paper presents three major directions of the evolution of fundamental ideas of fracture mechanics: experimental methods, numerical methods and analytical methods, starting with the observation of Leonardo da Vinci until nowadays.

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Nothing lasts forever. Failure of structures often occurs because of the cracks, which expand over a certain "safe" distance. There are cracks in all structures as a result of manufacture defects or different types of loading. In case the loading is applied cyclically, some cracks could appear or propagate as a result of the material fatigue property. In the final stages of crack propagation, the crack speed is growing very fast and failure of structure can appear.

According to some reports, e.g. [38], [115], in the United States of America in 1982 the cost of damages caused by cracks was estimated to 119 billion dollars per year, i.e. almost 4% from the net intern income. It was also underlined the idea that 35 billion dollars will be spared if the modern technology of studying cracks is used. A similar report of the European Union, [43] shows that, in the same way, in Europe can be spared almost 80 billion dollars per year.

From the early ages people noticed that the stone blocks used in construction present cracks. The propagation of those cracks was dangerous because it was frequently leading to structure failures. A characteristic property of these stone blocks - which were easy to model - is the propagation of cracks with a low energy of dissipation. The first time that man noticed the existence of fracture energy was when he cracked different type of stones to manufacture weapons and different kind of tools.

Leonardo da Vinci [148] was the first who tried to study the fracture phenomena around the year 1500. He noticed the inverse proportionality between the length of a cord and its strength. He also was the first to notice and study the size effect, i.e. the structure size / its inner strength relationship. One hundred years later, Galileo Galilei [55](1638) was the first to investigate from a mathematical point of view the problem of cracks. He also introduced the notion of tension in mechanics. He noticed that the workers paid more attention to the building of big ships than to the building of the small ones. Thus, continuing the da Vinci ideas, he studied the size effect in

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fracture mechanics. His theories concerned the relationship between the dimension of animals and the inner strength of their bones. His experiments were further developed by Mariotte [90](1686), who emphasized the relationship between the strength of a body and its dimensions. A century later, in 1776, Coulomb [34](1776) has studied crack propagation in stones under compression.

In 1898 G. Kirsch has obtained an analytical solution, capable to yield the tension around a circular hole in an infinite plate under uniform tensions at a remote point. Kirsch can be considered as a pioneer in the study of cracks by analytical methods. His ideas were further developed in 1907 by the Russian mathematician Iury Vasilievich Kolosov. Kolosov proposed an analytical method for the calculation of the tensions around a hole in an infinite plate. Inglis [68](1913) continued the work of Kolosov and in 1913 published a paper in which he made an analysis of the state of tension in the neighbourhood of a crack tip.

The ideas from Inglis paper were further developed in 1920 by a young engineer, Alan Arnold Griffith, who analysed the phenomena from an energetic point of view in [60]. Griffith was the head of the Department of Physics at the Royal Aircraft Establishment from Farnborough. By the time he studied the cracks that occur in brittle materials; in particular he made some experiments on glass. Griffith noticed that if in a brittle material there are some cracks, then the tension will concentrate and grow very fast around the tips of these cracks. Thus, he obtained his famous relation for an elliptical crack with traction-free crack surfaces

$$\sigma^2 = \frac{2\gamma E}{\pi a}, \tag{0.1}$$

where a is the crack length, 2γ is the applied tension and E is the Young modulus. The main idea of Griffith's work was the fact that crack propagation is determined by the relation between the strain energy and the necessary surface energy for the new surface area as the cracks advances. Griffith asserted that when a crack propagates the decreasing of the strain energy is compensated by the increasing of the potential energy caused by the tension in the newly created crack surfaces ([59]). This idea was applied in 1939 by Wetergaard [151] to the case of a two-dimensional solid, and later in 1946 by Sneddon [131] in the case of a three-dimensional solid. All these theories concerned brittle materials.

In 1948 George Rankine Irwin, professor at the Leigh University, showed that the Griffith's relation should include the work done in the plastic region, i.e. the crack will propagate if the strain energy is bigger than the total necessary energy (work done to create new crack surfaces and the work done in the plastic region). In 1957 Irwin advanced the idea of a straight segment crack, which has singularities at the two ends; he also proposed a generalization of the Griffith theory to an arbitrary crack. At the same time, he introduced in fracture mechanics a new crack propagation principle which states that a crack propagates if the rate of the strain energy G is bigger than the critical work necessary to create new crack surfaces. One of the most important results belonging to Irwin was the demonstration of the fact that the state of tension in the neighbourgood of the crack tip is completely determined by the stress intensity factors K_I and K_{II}. His argument relays on the work of Westergaard [151](1939). Irwin also stressed out that any fracture can be described in terms of the three fundamental cases of fracture: I. the symmetrical opening case, II. the sliding case and III. the tearing case (fig.1). The letter K for the stress intensity factors seems to be after the name of Kies (1952-1954), a colleague of Irwin as it is mentioned in [30]. For relatively simple geometries and loadings there are tables containing certain values and relationships for the stress intensity factors cases. In fracture mechanics, the determination of the parameters describing the tensions and displacements in the neighbourhood of the crack tip is of great importance. Thus, if the stress intensity factor exceeds a certain critical value, specific to the material, the crack will propagate and the material structure will fail.

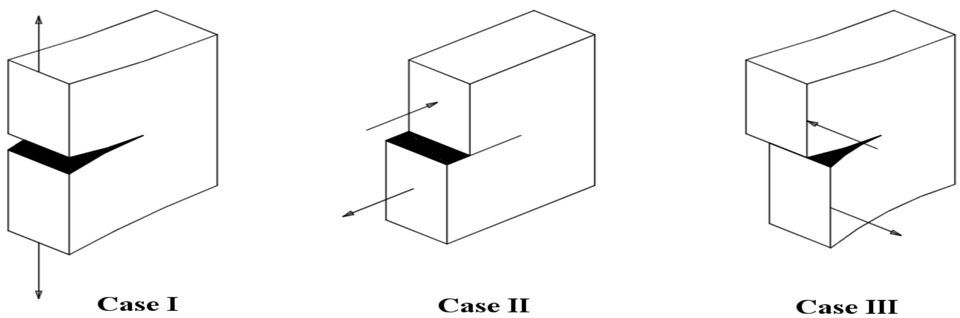

Fig.1 The fundamental cases of fracture.

One of the most notorious methods for the construction of approximate solutions for the differential equations is the Finite Element Method (FEM). It is applied to study the cracks in different kinds of structures, the thermal loading of electronic microchips etc. The first attempts to study plates by methods of Galerkin type belonged to MacNeil [87](1951) and Levy [82](1953). In the '50, R. Courant [35](1943) used Ritz type methods within a variational framework to study the vibrations of certain systems. He is the first who initiated a finite element type analysis. The bases of this method were established in the '60 by Turner, Clough, Martin and Topp [142](1956). Then, it was Clough [33](1960) that established the method in the form known and used nowadays.

Then error analysis methods that characterize FEM accuracy were developed. One of the first attempts in this respect belongs to MacNeil [87](1951).

In 1957 Irwin proposed [75], a first experimental method for the study of cracks, called electric resistive tensometry. In 1958 the first papers on the experimental analysis of the stress intensity factors based on photoelasticity appeared ([150]). Starting with the ideas of Wells and Post [150](1958), Irwin [76](1958) makes an analysis of the stress intensity factors based on the data of an izocromatic line.

The method of cohesive interfaces, proposed by Barenblatt [10](1959), can also be located within the general framework of the Galerkin type methods. In 1959, Barenblatt [10] was the first to take into account cohesive forces in the neighbourhood of the crack tip in the linear elasticity theory. His theory was based on an original tractions-displacements relation, which considered interfaces displacements. Within the linear elasticity framework, Barenblatt assumes the existence of cohesive forces around the crack tip, which are able to totally eliminate the friction between the crack surfaces. In the general model developed in [10], the distribution of these cohesive forces was not specified. In 1960, Dugdale [39] assumes that these forces are known and constant. These ideas were later implemented in standard finite element algorithms, in commercial codes, incorporating the interfaces in the structures of the finite elements.

In 1952, in the framework of linear elasticity, Williams has developed the formulas for displacements and tensions in the neighbourhood of a corner [153]. Later on, in 1957 he studied the limit case of a crack in [154]. More details on the study of the state of tension in the neighbourhood of a corner of a two-dimensional body can be found in V. G. Blinova et al. [23](1995).

Along with the development of the FEM, one can notice the imposing of the Direct Stiffness Method (of Galerkin type) proposed by Turner et al. [143](1959), [144](1964), over the Force Method - the idea is discussed in detail in Felipa [46](1995), [49](2000). In 1960, Argyris and Kelsey [5] described the classical mechanics in a matricial form. Starting with this paper, the FEM occured in a natural way.

In 1960 the first experiments on stones and concrete took place, considering the Griffith model. In this direction, Kaplan [78](1961) has studied the implementation of the Griffith's theory of fracture mechanics within the framework of linear elasticity of concrete. MacClintock and Walsh [89](1962) have considered cracks with friction between the crack surfaces. In 1967 the crush of the mines was a really important problem. In order to solve it, Bienawski [22](1965) presented the results of his experiments on rocks.

In 1961 was the first time that the problem of crack propagation due to the material's fatigue was considered. Thus, Paris [104] stated his famous law on crack propagation within the frame of linear elasticity

$$\frac{da}{dN} = C(\Delta K)^n, \tag{0.2}$$

where $a$represents the length with which the crack moves forward during a loading cycle, and ΔK is the variation of the stress intensity factors during the same loading cycle; the parameter C depends on the loading and geometry of the model. A generalization of Paris law, characteristic to the study of crack propagation due to the property of fatigue is

$$\frac{da}{dN} = f(\Delta K), \tag{0.3}$$

where $f(\Delta K)$is a function, which depends on the variation of the stress intensity factors. Integrating the equation (3), one can obtain the number of cycles necessary for a crack to propagate from an initial length to the critical length, where the structure fails

$$N_{cr} = \int_{l_0}^{l_{cr}} \frac{1}{f(\Delta K)} da. \tag{0.4}$$

In the following years, numerous researchers have developed different particular forms of the function $f(\Delta K)$. Another well-known law in this direction was proposed by Forman in 1967.

The paper of Melosh [93](1963) belongs to the line of the methods dealing with the accuracy of the FEM; he proposed a test based on the property of invariance of rigid bodies. In the paper of Irons et al. [71](1964) a test based on the condition of constant strain is discussed. One of the most famous and important methods on the validation of the FEM is the patch test, proposed in 1966 by Irons [72]. Subsequently this method has been improved by many researchers and continues to be a powerful tool in the analysis of the FEM in present.

The interface element method was another important method of Galerkin type in the evolution of fracture mechanics. The pioneers of this method were Ngo et al. [98](1967), who studied a two-dimensional theory and Goodman et al. [58](1968). In the interface element method, the main role is played by an interface element located at the interface between two structural finite elements. This element can characterize the crack locally, i.e. it can transmit or not the loading between the two neighbouring elements. A comparative analysis of the interface element method and the cohesive interface method was made by Foulk et al. [54](2000).

A first book on the applications of the FEM was written by Zienkiewicz and Cheung [156](1967). In 1968, Rice noticed that the accurate evaluation of the stress intensity factors is very important because they characterize totally the behavior of the crack. The determination of these factors is the main objective of the computational fracture mechanics, because they are essential in the study of stability and crack propagation [116].

Forman et. al [53](1967) continues the work of Paris concerning the crack propagation due to the fatigue of the material. Thus, in order to characterize the behavior of the rate of the crack propagation when the stress intensity factor approaches its critical value, Forman proposed his famous law

$$\frac{da}{dN} = \frac{C(\Delta K)^n}{(1-R)K_{cr} - \Delta K}, \tag{0.5}$$

where K_{cr} is the critical value of the stress intensity factor and $R$is the ratio between the maximum and the minimum values of the stress intensity factor.

One of the first attempts to develop an analytical method for the determination of the stress intensity factors and the state of stress in the neighbourhood of the crack tip in shells belongs to Folias [51], [52](1969). At the same time, Pian and Tong [110](1969) have written a paper, which proposed very interesting and original methods in the approaching of crack problems, i.e. mixt formulations and hybrid variational principles.

When a crack occurs in a body, it appears implicitly a plasticity region in the neighbourhood of the crack tip. Depending on the size of this plasticity region, the concept of the rate of the strain energy, proposed by Griffith, cannot be applied directly. In the energy balance one should take into account the plasticity region. In order to take into account these plasticity regions, Rice introduced a new parameter in the study of cracks: the J integral (path independent around the crack tip) Rice [116](1968). He also proposed a new propagation criterion. According to this criterion, within the framework of linear elastic fracture mechanics, there is an equivalence between the J integral and the rate of the strain energy G.

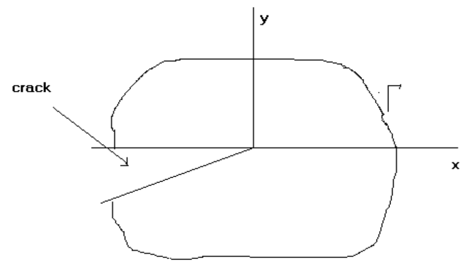

Fig. 2. An instance of a crack

The most important feature of the J integral is that the characterization of the displacement and tension discontinuities uses some integrals on paths that do not pass through the discontinuities [91]. Hillerborg [66](1983) underlines that the Rice's propagation criterion does not produce good results on brittle materials.

In the '70 the application of the FEM was possible only on powerful computers, which were the property of some public institutions. In Chan et al. [31](1970) the FEM in fracture mechanics is very well discussed. He uses linear finite elements and proposed a refinement of the mesh to characterize better the singularity. At the same time, Byskov [27](1970) proposed a new kind of triangular finite element, which contained a crack; this element was meant to describe the crack better than all the other triangular finite elements considered before. His approach is based on an analytical solution in a "principal element" for the whole crack field, making use of the Muskhelisvilli's [96](1963) complex functions. A method for the computation of the stress intensity factors is also proposed.

The weight function method, proposed by Bueckner [26](1970), is also in the line of the analytical methods to study cracks. This method makes use of complex functions; it assumes the apriori knowledge of a complete solution for the problem. Bueckner noticed that the weight functions are independent of the loading but dependent on the model geometry.

Irwin's ideas from 1958 of experimental crack analysis based on photoelasticity were continued in 1970 by Bradley and Kobayashi [24]. They made an analysis of stress intensity factors based on the data from two izocromatics.

The work of Elber [41](1970) [42](1971) should also be mentioned here. He studies the relationship between the rate of crack propagation due to the fatigue of the material and the ratio R of the maximum and minimum value of the stress intensity factors.

In 1971 Tracey [140] had a new brilliant idea which has generated results for crack modelling much more realistic compared to the results obtained up to that moment. He used triangular finite elements and he considered a polynomial representation of the displacement, as it is shown by Owens and Fawkes in [103](1983).

An original approach of crack modelling with finite elements belongs to Rao et al. [113](1971). In their analytical approach, Rao makes use of a coupling technique between two kinds of finite elements: ones which contain singularities - to model the crack, and others which do not - to model the rest of the body. The consistency is not accomplished.

In [7](1971) Babuska proposed an extension of the patch test for mixed formulations. He proposed the control of the stability conditions through an elipticity condition of infsup type. Later on, this type of approach was discussed by him in [8](1973) and by Brezzi in [25](1974). In 1972, Irons and Rozzaque [73] discussed some problems of the convergence analysis of the FEM from the patch test point of view.

In 1972 the first papers on the study of circumferential cracks in cylindrical shells issued. In this respect, Duncan Fama et al. [40](1972) could be considered pioneers. In the same year, Rice [119](1972) proposed a new, very efficient numerical method for the study of cracks in plates: the line spring method. This method uses a projection of the spatial problem in plane. Rice [118](1972) developed a new point of view in the weight function method. He approached the problem from an energetic point of view, proposing some possibilities of application of the theory to a three-dimensional space.

Shah and Kobayashi [126] proposed in 1972 a new numerical method for the study of cracks (the alternate method), combining analytical and numerical results from the finite element analysis to obtain formulas for the stress intensity factors.

The experimental methods have also been developed in 1972 in the paper of Schroedel et al. [125], in which stress intensity factors were studied through the "frozen tensions" method. In the same line is situated the work of Theocaris and Gdoutos [138](1972) in which it is introduced the "caustics method", characteristic to the fields with singularities.

In 1973 Strang and Fix published their book [132] which presented the mathematical background of the FEM. They have mentioned the evolution of this method and its major impact on the computational mechanics. In the same year, it should be mentioned the paper of Tong [139] which studied the problem of crack propagation by means of the FEM. This work continues the ideas of Rao et al. [113](1971) and solves the consistency problem of the coupling method from an energetic point of view, introducing Lagrange multipliers. Moreover, besides Pian, Tong could be also considered as a pioneer of the study of cracks with hybrid FEM. In [81] Leicester continues the ideas of Galilei form 1638 concerning the influence of the size of a body versus its state of tension. Later on, Hillerborg will continue and develop this work proposing new elements, which can characterize better the crack behaviour.

Veubeke [147](1974) brings new ideas to the patch test. He uses a variational formulation to obtain the equations of the linear two-dimensional elasticity. Thus he has interpreted the patch test as a hybrid finite element model, taking into consideration non-conformal FEM.

The virtual extension method is another method widely used in fracture mechanics. Its pioneers were Parks [107](1974) and Hellen [62](1975). This method appeared as a natural continuation of the extension force method.

In 1974 Cartwright and Rooke[29](1974) proposed a new method, namely the compound method. To study a model with a crack within this method, the model is decomposed in some "elementary" simpler models; the solution of the initial crack problem is obtained by composing the solutions of the elementary models.

The papers of Benzley [19](1974) and Akin [3](1976) used special forms for the representation of displacement field around the crack tip. Henskell and Shaw [64](1975) and Barsoum [12](1976), have independently developed methods for the study of singularities based on izoparametric shell and plate elements to describe in an accurate way the state of tension in the neighbourhood of the crack tip. More specifically, they have tried to model the singularity of the $r^{-1/2}$ order from the neighbourhood of the crack tip. They proposed a new kind of singular finite element, namely the quarter point element. The name comes from the fact that for a square finite element one will displace the nodes belonging to the sides adjacent to the crack tip to a position located at a distance equal to a quarter of the square finite element side from the crack tip. This new kind of finite element produces a singularity in tractions of $r^{-1/2}$ order along the sides of the element adjacent to the crack tip. The problem of these singularities along the elements's sides was solved by Barsoum [12](1976). He has modified the element considered by Henskell and Shaw in [64], considering a square degenerated element, i.e. a triangular element that presented singularities of order $r^{-1/2}$ along the sides adjacent to the crack tip.

In 1976 Paris et al. [105] has suggested a method for the computation of the two-dimensional weight functions based on the FEM. His method brings a lot of clarifying simplifications to the algorithm used until that moment due to the implementation of the FEM. The work of Labbens et al. [79](1976) is in the same spirit, calculating the weight function in nodes by means of the compliances matrix.

In 1976, Hillerborg [65] was a professor of Material Science at the Univ. of Lund. He proposed a model, which generalizes the preceding theories of Barenblatt [10](1959) and Dugdale [39](1960) in such a way that these theories are limit cases of his theory. Thus, the crack can be interpreted as a reunion of two regions: the first region of the real crack where there is no tension transfer and the second one, the "affected" region, where we encounter tension transfer. After 1975 the papers of Bazant [14](1976) and Bazant and Cedolin [15](1979) appeared. They continued Rice's ideas from 1972 and described another energetic model of crack propagation more suited to the finite element analysis.

The next step in the evolution of the study of cracks was made in 1976 by Hillerborg who proposed the idea of the characteristic length of a crack for brittle materials. This step can be considered as a continuation of the work of Galilei [55](1638) and Leicester [81](1973) on the influence of the size on the robustness of a structure. Thus, the strength of structural elements depends on their size. In this direction, Varpentieri [28](1986) proposed a measure of the structural strength: "Carpentieri's number" or "the brittleness number" of a structure.

The ideas from the '60 of Barenblatt [10](1959) and Dugdale [39](1960) on the cohesive interfaces method and the interface element method were further developed by Ingraffea [69](1977) who proposed the continuous interface method. The method was later developed by Liaw [83](1984). This method is the most notorious because the interface between two elements remains continuous until the crack propagates. A further development of the method belongs to Gerken [56](1996); the method was implemented in Gerken [57](2001) and in the comercial finite element code ABAQUS [1](1997).

Some of the pioneers of the non-conformal FEM, besides Veubeke [147](1974), were Taylor, Beresford and Wilson [136](1976). They have implemented the first ideas on non-conformal finite elements, which are of great interest in crack analysis and in the analysis of domains with corners. They considered an element of Q6 type and they added other specific nodes, thus obtaining a new non-conformal finite element QM6.

Oliveira [101](1977) has set the mathematical background on the patch test. He also made some remarks on the patch test as a tool for the validation of the new non-conformal finite elements QM6. He noticed the patch test together with the completion property imply the convergence of the method. Veubeke's ideas from [147](1974) were further developed by Sander and Beckers. They worked on the equivalence between the patch test and the hybrid element method.

Tracey and Cook [141](1977) worked on the ideas of Tracey [140](1971), considering rectangular elements and a potential function of order p in order to characterize the singularities of order r^{-p}.

Although the FEM is well founded, it is not best suited to the study of discontinuities. The modelling of a space with discontinuities through the classical FEM assumes a relationship between the element's topology and the geometry of the discontinuity. This implies the remeshing as the crack propagates, which leads to errors and a large number of computations. Thus appeared the meshless methods, which are suited to the study of domains with discontinuities. These methods construct the approximations using the data at the nodes and some weight functions with compact support in the domain. One of the pioneers of this method was Lucy [86](1977) who has proposed the Smooth Particle Hydrodynamics method. He used a weight function to construct local discrete approximations. Is was noticed that for an arbitrary configuration of nodes the approximation consistency depends upon the choice of the weight function. In Randles et al. [112](1996) some variants of the method, which can be applied to large deformations and to the fracture of brittle materials, were developed.

In 1977, Rybicki and Kanninen [121](1977) proposed a method to study cracks based on the J integral method proposed by Rice in 1968. This method implies a finer discretization in the neighbourhood of the crack tip; thus the results are in a completely agreement with the analytical results whenever available.

Pian [111](1978) describes the implementation of hybrid elements in the study of cracks through the FEM from a historical point of view. MacNeil [88] proposed in the same year a very original Galerkin type approach: the assumed strain method.

Starting with the weight function method of Rice [117](1972), Vanderglass [145](1978) and, later, Parks and Kamenetzki [108](1979) applied the virtual extension method on two meshes of finite elements (a reference one and a virtual displaced one) to compute the stress intensity factors and the weight functions. Petroski and Achenbach [109](1978) proposed a method for the study of cracks based on the analytical expressions of the displacement in the neighbourhood of the crack tip; their ideas were further developed by Fett et al. [50](1987).

In 1980, Tan and Fenner [135](1980) have implemented the boundary element method (BEM) in fracture mechanics to compute the stress intensity factors for cylindrical shells. The BEM is rarely used in the study of cracks, because of the great number of computation implied and the difficulties in the implementation at the boundary (i.e. the conditions imposed to the Green functions). The advantage of the BEM consists in the fact that it avoids remeshing. In the same year, Sanford et al. [124] developed an experimental method based on notions of photoelasticity, using the least squares method to compute the stress intensity factors. Their paper is a continuation of the ideas from Sanford et al. [123](1979).

Another valuable paper on the patch test belongs to Stummel [134](1980). Here were pointed out some non-conformal finite elements which passed the patch test proposed by Irons in 1966, but those still were not convergent. The conclusion is that the patch test is just a necessary condition for convergence.

In the '80 a new Galerkin type method for the numerical modelling of cracks appeared in the papers of Bergan [21](1980) and Bergan and Nygard [20](1984). It was called the free formulation method.

In 1983, Owens and Fawkes [103](1983) published a book in which they presented the methods for the stress intensity factors computation using the FEM. Theirin the principal methods: the J integral method (Rice [116](1968)), the virtual extension method (Parks [107](1974)) and the displacement extrapolation method are described.

The results of Shah and Kobayashi from 1972 were developed by Nyshioka et al. [100] (1983) in the study of elliptical cracks in bounded bodies. It should be also mentioned the paper of Irons and Loikkanen [74](1983) which established the mathematical background of a convergence criterion for the FEM.

In 1985, Hull et al. [67](1985) used a displacement extrapolation technique in the study of circumferential cracks in cylindrical shells; formulas for the stress intensity factors are mentioned. Haber and Koh [61](1985) proposed an explicit formulation for the variation of the energy during the virtual extension of the crack, using quarter point finite elements in the neighbourhood of the crack tip and regular izoparametric elements in the rest.

Bathe and Dvorkin [13](1985) have developed MacNeil's assumed strain method from 1978; in the same line are the works of Park and Stanley [106](1985) and Simo and Hughes [129](1986).

The papers of Taylor et al. [137](1986) and Rozzaque [120](1986) belong to the papers, which develop and emphasize the importance of the patch test. The patch test is presented here as a guide to construct new finite elements; it is stressed the importance of the convergence of the patch test in the study of stability.

Nishikov and Atluri [99](1987) have proposed in 1987 a generalization of the virtual extension principle to a three-dimensional case. They have developed some original expressions for the J integral in a three-dimensional space. Starting with Hellen's [62](1975) formulas for the stress intensity factors, they developed formulas for the three-dimensional case. In 1987 there can be mentioned new contributions to the development of the weight function method. Thus, Sham [127] has implemented a variational formulation on a finite element mesh to determine the weight functions.

In 1987, Verma and Melosh [146](1987) proposed a new definition of convergence within the patch test framework.

One year later, Shivakumar et al. [128], have implemented brick elements with eight and twenty nodes in the theory of Rybicki and Kanninen from 1977.

A new meshless method very useful in the study of domains with discontinuities belonged to Belytschko et al. [16](1988). This method was designed to characterize cracks as arbitrary discontinuities. These ideas were further developed by Simo et al. [130](1993) and later by Armero et al. [6](1996) and Oliver [102](1996).

An original method for the study of cracks was proposed in 1989 by Miazaky and Kaneko [95](1989); this method consists in a combination of the spring line method of Rice et al. [119](1972) and the BEM. In the same year, Hellen [63] applied the virtual extension method to nonlinear materials, using a theory based on incrementations. It should be also mentioned in the paper of White and Abel [152](1989) on the accuracy of the shell finite element.

In 1991, Amnipour and Holsapple [4] proposed a new method for the study of cracks on the interface between two materials based on hybrid finite elements.

Brueckner's ideas from 1970 and Rice's ideas from 1972 were successfully combined by Akimkin and Nishikov [2](1991) who used the spring line method in the study of weight functions. This method is of the same type as the method of Myazaki and Kaneko from 1989 concerning the coupling technique.

A method, which implemented high performance finite elements in the study of cracks, was proposed by Militelo and Felipa [94](1991) and Felipa and Militello [44](1992). They have developed the assumed deviatoric natural strain method, based on the free formulation method of Bergan and Nygard [20](1984) and the assumed strain method of MacNeil [88](1978).

A year later, Barbero and Reddy [11](1992) have developed a variant of the virtual extension method called the method of Jacobian derivative. This method is based on a model, which describes the potential energy in terms of displacement and nodal coordinates, using izoparametric elements. The object of their work was to study cracks, which appear in laminate composite materials. In the same year, Weissberg and Arcan [149] used the method proposed by Rybicki and Kaninen [121](1977) to analyze the singularities of the stress intensity factors for a crack with adhesive faces.

Nayroles et al. [97](1992) proposed a meshless method: the diffuse element method. This method uses weight functions to characterize the discontinuities in bodies.

The paper of Chapelle and Bathe [32](1993) proposed a generalization of the patch test, called the infsup test, and discussed some two-dimensional examples which underline the accuracy of the proposed test.

Badary Narayana et al. [9](1994) used the method proposed by Rybicki and Kanninen [121](1977) to study crack problems with brick elements with eight nodes, in a three-dimensional space. A new concept of integration is proposed as well. The crack front is divided in segments, thus the remeshing is no longer needed.

Belytschko et al. [17](1994) proposed the element free Galerkin method. This method uses a moving least squares technique to construct an approximation. This technique was proposed earlier by Lancaster et al. [80](1981). It is pointed out that it is "almost impossible" to remesh for arbitrary cracks.

In 1995, Davidson et al. [36] developed a mixed numerical-analytical method for the study of cracks. They managed to produce formulas for the energy released during the crack propagation.

It should be mentioned the papers: Felipa [45](1994) and Felipa [46](1996) which present some parameterized variational principles which can describe almost every classical variational formulation of elasticity by means of some special functionals. This paper is of the same generality as the one of Veubeke [147](1974). Some other authors have developed these ideas and created template finite elements that are easier to use and that produce excellent results. In 1995, Felipa and Haugen [47] proposed finite elements templates based on parameterized variational principles and on the patch test.

Duarte et al. [37](1996) showed that for a certain nodes configuration and a certain choice of weight functions, the basis functions of the element free Galerkin method proposed by Belytschko are equivalent to the functions of the RKPM ("Reproducing

Kernel Particle Method") proposed by W.K.Liu et al. [84](1995), [85](1995). RKPM formulation shows that the kernel (weight function) from the smooth particle hydrodynamic method can be used to construct a basis in an approximation space, with a certain correction. This correction imposed the solving of a system of equations at each node of the domain. Thus, the convergence is naturally obtained. In Melenk [92](1996) it is shown that the element free Galerkin method and the reproducing Kernel particle method can be generalized to applications of the partition of unity method. Meshless methods are applied to mechanical problems in which the traditional FEM is difficult to implement. These problems concern dynamical and statical fracture mechanics, the study of large deformations in solids etc. In these cases appropriate methods are smooth particle hydrodynamics, element free Galerkin type methods, and the RKPM.

In 1997, Zhang and Chen [155](1997) have studied hybrid finite elements starting with a variational principle of Hu-Washitzu type. This article can be a guide for the construction of new finite elements. Another important paper that should be mentioned here is the one of Zienkiewicz and Taylor [157](1997), which made an analysis of the patch test with simple examples for plates.

The ideas of Chapelle and Bathe in [32](1993) were further developed by Iosilevich et al. [70](1998) which studied some applications of the infsup criterion for Mindlin-Reissner plates using variational approaches. In Rashid [114](1998), it is developed a study of cracks based on finite element of special type.

The advantages of meshless methods in the study of cracks can be seen as well in the methods based on the partition of unity in which the local functions of completion are included in the approximation. The method was developed by Strouboulis [133](1998) for the scalar problem of Laplace in domains with corners. Jirousek [77](1996) was the first who implemented the idea of incorporating local functions in approximations.

In [18] Belytschko (1999) used a partition of unity method enriching locally the finite element approximation; he introduced discontinuous functions in the approximation to characterize arbitrary discontinuities. This method is not suited for large cracks, and this completion with discontinuous functions is not recommended to represent the displacement away from the crack tip. Thus, the mesh needs to be remeshed in a certain way in the neighbourhood of the crack tip.

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Author's address:

Razvan Raducanu
Department of Applied Mathematics, "Al.I.Cuza" University, Iaşi, Romania
e-mail: rrazvan@uaic.ro