An Inertial Model of Steady Coastal Upwelling

JOSEPH PEDLOSKY

Department of the Geophysical Sciences, The University of Chicago, Chicago, Ill. 60637

(Manuscript received 15 August 1977)

ABSTRACT

A nonlinear inertial model of a steady-state coastal upwelling circulation is presented. The model describes a longshore equatorward current and countercurrent structure which is independent of any parameterization of turbulent mixing. Solutions for flat and sloping bottoms are presented.

1. Introduction

Theoretical models of coastal upwelling generally fall into one of two categories. Numerical models which describe more fully the nonlinear aspects of the upwelling circulation are most often limited in the duration of the upwelling period they describe, while the vertical structure resolved is restricted by the number of layers used in the calculation (e.g., Hurlburt and Thompson, 1973). On the other hand, analytical models which attempt to model the long-term (i.e., seasonal) upwelling structure usually depend heavily on both a linearization of the momentum and thermal equations and very crude parameterization of the turbulent mixing of heat and momentum (e.g., Allen, 1973; Pedlosky, 1974). These assumptions lead to conceptually useful descriptions of the circulation; however, the strength and structure of the flow depend on the least certain aspects of such models, the parametric representations of turbulent mixing. In particular, in these models fluid elements can rise in the basic stratification field only by adjusting their density by mixing with their environment. By hypothesis the slope of the density surfaces must remain small. Furthermore, the strength of both the onshore and longshore flow depend directly on the forcing, i.e., the offshore Ekman flux in the upper mixed layer produced by a longshore wind stress, while observations indicate that the mean longshore flow, once set up, is relatively unaltered should the applied wind stress slacken.

In an attempt to construct a model which does not depend *explicitly* on turbulent mixing parameters, a simple though nonlinear inertial theory of upwelling is presented below. The model is based on three central hypotheses.

1) In the region below the mixed layer the motion is strictly inertial and the fluid conserves its density and potential vorticity as it rises.

2) For a basically uniform stress field the upwelling circulation enters the mixed layer in a mixing region sufficiently narrow in comparison with a Rossby deformation radius so that this region can be modelled as a delta function sink of fluid for the inertial region below.

3) In the steady state the fluid on the upper surface, outside the mixing region maintains the minimum density of the region, i.e., that the mixing zone insures that fluid flowing seaward of the zone is not heavier than the inertially controlled fluid below.

This last hypothesis is for the steady-state model a reasonable one, although no proof is given here that this condition will in fact be realized in the complete inertial *plus* dissipation physics that must obtain to describe the complete problem. As in the case of inertial theories of the Gulf Stream, the inertial theory I present here is capable of explicitly describing only the formation region of the boundary flow; the complete circulation problem must, for the steady state, involve dissipation and is beyond the scope of the present discussion.

2. The model and formulation

I consider a fluid whose density field $\rho(x,y,z)$ can be partitioned in a Boussinesq fashion, i.e.,

$$\rho = \rho_0 + \theta(x,y,z). \tag{2.1}$$

where ρ_0 is the mean density (constant), θ the density anomaly and $\theta \ll \rho_0$. The coordinates x, y, z measure distance eastward, northward and vertically in the usual way. The hypothesis that density is preserved implies that

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z} = 0, \tag{2.2}$$

where u, v and w are velocities in the x, y and z directions.

0022-3670/78/0171-0177$05.00
© 1978 American Meteorological Society

Since θ is a conservative field all fluid elements remain on the θ surface to which they are initially assigned and glide horizontally and vertically along that surface. Following Rossby (1938) it is useful to use θ instead of z as a vertical coordinate and consider z a dependent function of x, y and θ. This depends implicitly on the third modeling assumption, i.e., that $\partial\theta/\partial z < 0$ throughout the region described by this model.

In this system of coordinates the horizontal momentum equations become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{\partial \pi}{\partial x}, \quad (2.3a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{\partial \pi}{\partial y}, \quad (2.3b)$$

where x and y derivatives are taken along surfaces of constant density and where

$$\pi = \frac{p}{\rho_0} + \frac{\theta}{\rho_0}gz. \quad (2.3c)$$

The hydrostatic relation [used to derive (2.3c)] takes the simple form

$$\frac{\partial \pi}{\partial \theta} = \frac{gz}{\rho_0}, \quad (2.3d)$$

while the condition of incompressibility is written

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \theta}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial z}{\partial \theta}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial z}{\partial \theta}\right) = 0. \quad (2.3e)$$

Nondimensional variables are introduced according to the following scheme:

$$\left.\begin{aligned} (u,v) &= U(u',v') \\ (x,y) &= L(x',y'), \quad t = \frac{L}{U}t' \\ \theta &= \Delta\theta\theta' \\ z &= Dz' \\ \pi &= UfL\pi' \end{aligned}\right\}. \quad (2.4)$$

In (2.4), D is the characteristic depth of the upwelling zone, $\Delta\theta$ the overall density difference between the surface and bottom, while L is chosen to be the Rossby deformation radius, *viz.*,

$$L \equiv (g\Delta\theta/\theta_0 D)^{1/2}/f, \quad (2.5)$$

so that all horizontal lengths are measured in deformation radii.

The velocity scale U will be chosen to characterize the magnitude of the onshore flow velocity. For the purposes of this study f is considered constant, although this is not essential.

The equation set (2.3) then becomes (dropping primes on dimensionless quantities)

$$\epsilon\frac{du}{dt} - v = -\frac{\partial \pi}{\partial x}, \quad (2.6a)$$

$$\epsilon\frac{dv}{dt} + u = -\frac{\partial \pi}{\partial y}, \quad (2.6b)$$

$$z = \epsilon\frac{\partial \pi}{\partial \theta}, \quad (2.6c)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \theta}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial z}{\partial \theta}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial z}{\partial \theta}\right) = 0, \quad (2.6d)$$

where

$$d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y, \quad (2.7a)$$

$$\epsilon = \frac{U}{fL} = \frac{U}{\left(g\frac{\Delta\theta}{\theta_0}D\right)^{1/2}}. \quad (2.7b)$$

The absence of the θ derivative in (2.7a) is due, of course, to the conservation of density, that is, to the fact that the "vertical" velocity ω in the x, y, θ system is identically zero by (2.2).

It follows from (2.6a), (2.6b) and (2.6d) that the conservation of potential vorticity is described by

$$\frac{d}{dt}\frac{(1+\epsilon\zeta)}{\partial z/\partial\theta} = 0, \quad (2.8)$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (2.9)$$

3. The inertial upwelling flow

We consider fluid in the region $x \leq 0$, $1 \leq \theta \leq 0$. At distances far offshore, i.e., as $x \to -\infty$, the flow is specified. That is, noncoastal dynamics will prescribe the starting conditions for the upwelling circulation. At an eastern boundary, for example, the β-effect, important on general oceanic scales, will determine the distribution of the onshore flow between the geostrophic interior and a lower Ekman layer. For the present calculation I assume that all the onshore geostrophic flow *locally* balances the offshore flux in the upper Ekman flow so that there is no significant flow in the lower Ekman layer, and that this onshore flow is *y independent*. It follows from this that (2.8) implies

$$\frac{1+\epsilon\zeta(x,\theta)}{\partial z/\partial\theta} = \frac{1}{\partial z_l/\partial\theta}, \quad (3.1)$$

where $\partial z_l/\partial\theta$ is the inverse of the static stability distribution far offshore. In deriving (3.1), I have also assumed that ζ is negligibly small far offshore. The relation (3.1) may be written

$$\epsilon\zeta = \epsilon \frac{\partial v}{\partial x} = \frac{\partial z/\partial\theta - \partial z_I/\partial\theta}{\partial z_I/\partial\theta} \qquad (3.2)$$

which is the form derived by Rossby (1938) in his study of the problem of geostrophic adjustment.

It is convenient to take the θ derivative of (3.2) and use the fact that to O(ϵ), which is a small parameter in all oceanographically relevant upwelling situations, the longshore velocity satisfies

$$\epsilon \frac{\partial v}{\partial \theta} = \frac{\partial z}{\partial x}. \qquad (3.3)$$

This leads to a simple equation for z, *viz.*,

$$\boxed{\frac{\partial^2 z}{\partial x^2} - \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial\theta}\bigg/\frac{\partial z_I}{\partial\theta}\right) = 0} \qquad (3.4)$$

The independence of the longshore coordinate is not a necessary feature of the analysis but leads to a considerable simplification of the analysis. The results of linear calculations (Pedlosky, 1974) imply that this should be a good model for that part of the wind forcing with longshore scales larger than some critical value which in the linear theory was O(500 km).

The boundary conditions for (3.4) are

$$z = z_I(\theta), \qquad x \to -\infty, \qquad (3.5a)$$

$$\frac{\partial z}{\partial\theta} = -\delta(\theta - 1), \quad x = 0, \qquad (3.5b)$$

$$z = 0, \qquad \theta = 0, \qquad (3.5c)$$

$$z = z_B(x), \qquad \theta = 1. \qquad (3.5d)$$

The first condition merely states that far from the coast the distribution of density with depth approaches its "interior" value. The second condition forces each density surface to enter the sink in the upwelling corner at the coast. That is, each density lamina participates in the upwelling or else by continuity the onshore velocity on that density surface would vanish from the condition for two dimensional steady flow:

$$\frac{\partial}{\partial x}\left(u\frac{\partial z}{\partial\theta}\right) = 0. \qquad (3.6)$$

The last two conditions [(3.5c) and (3.5d)] specify that the least dense fluid remain on the upper surface and the most dense remain on the lower surface $z = z_B(x)$, although in the θ coordinate frame it is the lower surface which "stays" at $\theta = 1$.

As an example, consider the case where

$$\frac{\partial z_I}{\partial\theta} = -1, \qquad (3.7)$$

i.e., a case of linear stratification and where $z_B = 1$, i.e., a flat bottom. The solution of (3.4) subject to (3.5a)–(3.5d) is

$$z = \frac{2}{\pi}\tan^{-1}\left(\frac{\sinh\pi x}{1+\cosh\pi x}\tan\pi\theta/2\right). \qquad (3.8)$$

Surfaces of constant θ are plotted in the x-z plane in Fig. 1. The longshore velocity v is determined, up to a constant, by (3.2) and (3.3) and is

$$v = -\epsilon^{-1}[x + \pi^{-1}\log(\cosh\pi x + \cos\pi\theta)]. \qquad (3.9)$$

The profile of the longshore velocity at three offshore positions is plotted in Fig. 2. Where v is positive the flow is poleward, where it is negative (i.e., near the surface) it is equatorward. Note that the poleward flow occupies a larger fraction of the total depth in the inshore profiles. To obtain the *dimensional* value v_* of the longshore current, it is necessary to multiply by U. It follows from (3.9), therefore, that

$$v_* = fLv(x,z) \qquad (3.10)$$

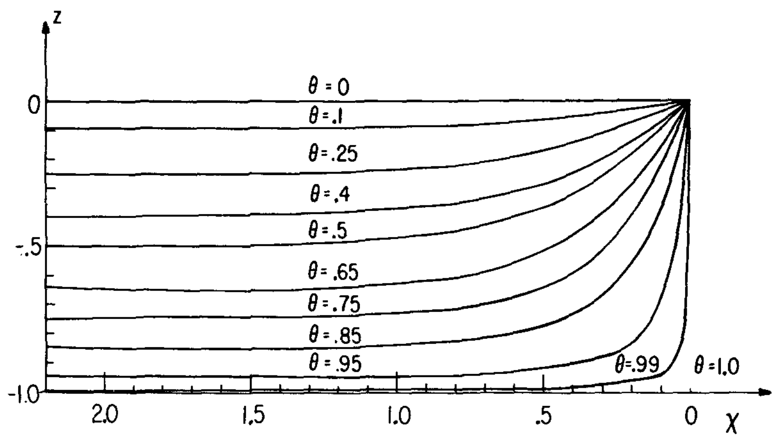

FIG. 1. Equilibrium density surfaces for a flat bottom.

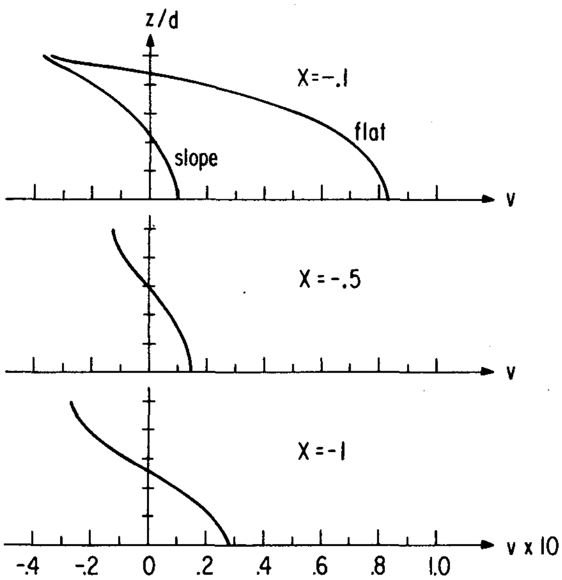

FIG. 2. The longshore velocity at three offshore stations. Note that at $x = -1$ the scale of v has been multiplied by 10. Positive velocities are northward. Dimensional velocity is obtained by multiplication by fR, where f is the Coriolis parameter and R the Rossby deformation radius. The range of z/d is from 0 to 1.0.

and is *independent* of the strength of the onshore flow. With $f = 10^{-4}$ s^{-1} and $L = O(10$ km), this yields a $v_* = O(10–20$ cm s$^{-1})$ at $x = 0.5$ (i.e., about half a deformation radius offshore). The longshore velocity is logarithmically singular at $x = 0$, $z = 0$, an unavoidable consequence of the delta function representation of the sink flow. The independence of the longshore speed, in the steady-state inertial theory, of the upwelling circulation is due to the following simple fact. Once the density surface has deformed upward, following the initial transient phase of upwelling, the resulting longshore flow is a steady equilibrium solution of the full inviscid equations. It will decay only on the long dissipative time scale by processes ignored in the present theory.

The density surfaces, once set up, act as fixed guides for the onshore flow. Fluid elements glide along the density surfaces into the upper layer sink with velocity [from (3.6)]

$$u = \frac{U_I \partial z/\partial\theta}{\partial z_I/\partial\theta} = -U_I \frac{(\cosh\pi x + \cos\pi\theta)}{\sinh\pi x}, \quad (3.11a)$$

$$w = -U_I \frac{\sin\pi\theta}{\sinh\pi x}, \quad (3.11b)$$

the relation for w following from

$$w = u\partial z/\partial x. \quad (3.12)$$

Fig. 3 shows the profiles with depth of the onshore flow at three offshore positions. The flow at infinity is uniform with depth, but as the coast is approached the effect of the boundary decelerates the flow at depth and in compensation for this, accelerates the onshore flow near the surface. This effect becomes more marked closer to the coast. This in turn, in the *transient* problem, produces the longshore structure shown in Fig. 2. Concurrently, the variations of u with x produce a vertical velocity in the x-z frame. This velocity at mid-depth, $z = -0.5$, is shown in Fig. 4. The vertical velocity falls to one-tenth its maximum value at $x = -0.9$, i.e., very nearly one deformation radius offshore. The onshore and vertical velocities do depend linearly on the upwelling circulation strength which determines the rate at which fluid elements pass through the density structure which itself is independent of U_I.

4. The effect of topography

If the bottom of the upwelling zone is no longer flat, certain changes occur in the upwelling circula-

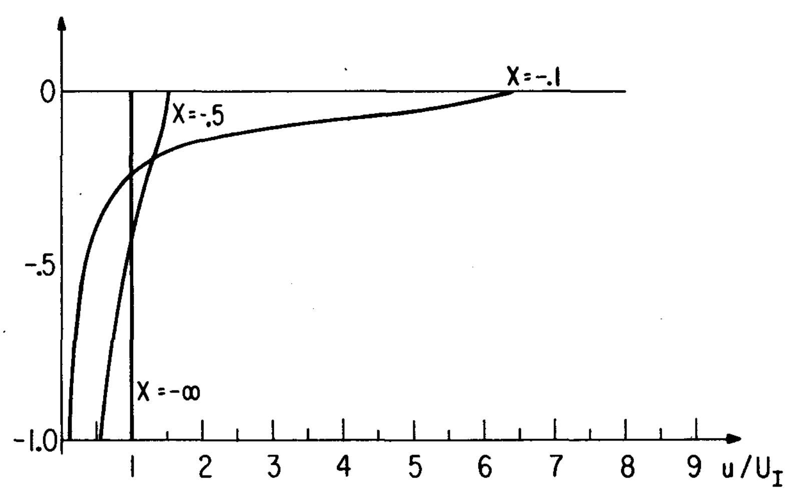

FIG. 3. Velocity profiles for the onshore flow. At $x = -\infty$, $u = U_I$.

tion predicted by potential vorticity conservation. Consider the circulation in the region $-L \leqslant x \leqslant 0$, $z_B(x) \leqslant z \leqslant 0$, where $L \gg 1$, i.e., where the domain of the problem in x is considerably greater than a deformation radius. Furthermore, let

$$\left.\begin{aligned} z_B(0) &= -d \\ z_B(-L) &= -1 \end{aligned}\right\} . \tag{4.2}$$

Although the geometry of the region in the x-z frame is a complicated wedge-shaped region, the x-θ coordinate frame remains rectangular and the only change in the problem posed by (3.4) and (3.5) is that (3.5a) and (3.5b) become

$$z_I(\theta) = -\theta, \qquad x = -L, \tag{4.3a}$$

$$\frac{\partial z}{\partial \theta} = -d\delta(\theta - 1), \quad x = 0. \tag{4.3b}$$

As an example, consider the case where

$$z_B(x) = (1 - d)\frac{x}{L} - d \tag{4.4}$$

in the region $-L \leqslant x \leqslant 0$. The solution to (3.4) subject to (4.3a,b) and (3.5c), (3.5d) is

$$z(x,\theta) = (1 - d)(x/L) + (2d/\pi)\tan^{-1} \times \left(\frac{\sinh\pi x}{1 + \cosh\pi x}\tan\pi\theta/2\right). \tag{4.5}$$

Fig. 5 shows the equilibrium density surfaces in the case $L = 2.5$, $d = 0.25$. Again, the onshore flow glides along the density surfaces into the upwelling sink. The longshore velocity, determined by (3.2) and (3.3), is

$$\epsilon v = -[x - (d/\pi)\log 2(\cosh\pi x + \cos\pi\theta)] - \frac{(1-d)}{L}\frac{(x^2 - \theta^2)}{2} + V_0, \tag{4.6}$$

where V_0 is an undetermined constant. In the flat-bottom case the constant V_0 was chosen such that as $x \to -\infty$, the longshore velocity vanished, which while reasonable, is not necessary. In the case with a sloping bottom, the sloping isopycnal surfaces at $x = -L$ require nonzero longshore velocities at $x = -L$, and there seems no "natural" choice for V_0. In Fig. 2, however, I have shown the derived profile (4.6) at $x = -0.1$ with the arbitrary choice for V_0 that the surface velocity at this inshore station remains unaffected by topography. In general, however, the effect of topographic slope is to reduce the the longshore velocity induced by upwelling [as can be seen from the alteration of the coefficient of the logarithmic term in (4.6)], as a consequence of the reduced slope of the isopycnal surfaces required to bring them to the surface.

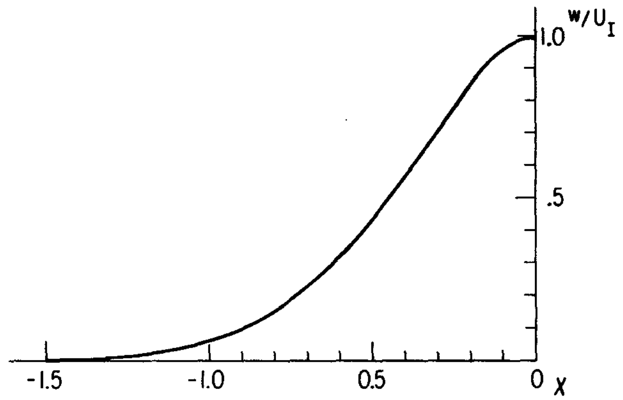

FIG. 4. The vertical velocity at mid-depth as a function of offshore distance.

The general solution for the isopycnal surfaces in the case of arbitrary $z_B(x)$, such that

$$z_B(x) \to -1, \quad \text{as } x \to -\infty,$$

can be found in the form

$$z = \tfrac{1}{2}\int_{-\infty}^{0} d\xi[1 + z_B(\xi)]\sin\pi\theta \times \left(\frac{1}{\cosh\pi(x + \xi) + \cos\pi\theta}\right.$$

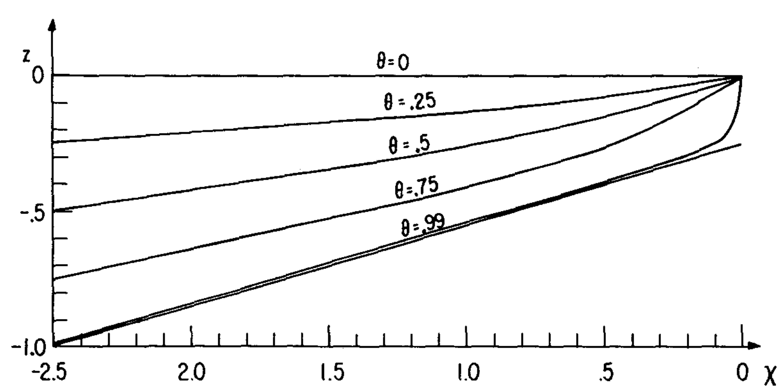

FIG. 5. Equilibrium density surfaces for a sloping bottom.

$$
\left. - \frac{1}{\cosh\pi(x-\xi) + \cos\pi\theta} \right) + (2d/\pi)\tan^{-1}
$$

$$
\times \left(\frac{\sinh\pi x}{1+\cosh\pi x} \tan\pi\theta/2 \right) - (1-d)\theta, \quad (4.7)
$$

where again $z_B(0) = -d$. This solution is sufficiently complex, however, that I prefer the simple example (4.5) for illustrative purposes.

5. Conclusions

The simple model presented in this paper demonstrates the equilibrium structure for the dynamic variables in an upwelling zone determined solely by potential vorticity and density conservation. The basic assumption is that after upwelling has been initiated each density surface rises to allow fluid to enter the upper mixed layer in a narrow surface region near the coast. The reasonableness of this assumption is subjectively strengthened by comparison with observation. Fig. 6 shows the observed isopycnal surfaces off the Oregon coast in late summer (29 August 1973), late in the upwelling season. The qualitative similarity to Figs. 1 and 5 is apparent. Nevertheless, as is the case for all recirculating fluid systems dissipation must be important over a portion of the fluid path not governed by the inertial theory. This is implicitly part of the present model which does not describe the offshore flux in the thin upper Ekman layer, nor does the theory describe the viscous boundary layer in which the longshore velocity is brought to rest at the coast. The most important deficiency of the inertial model, though, is its inability to describe the mixing process, implicitly assumed in the theory, whereby dense fluid flows into the corner region where in-

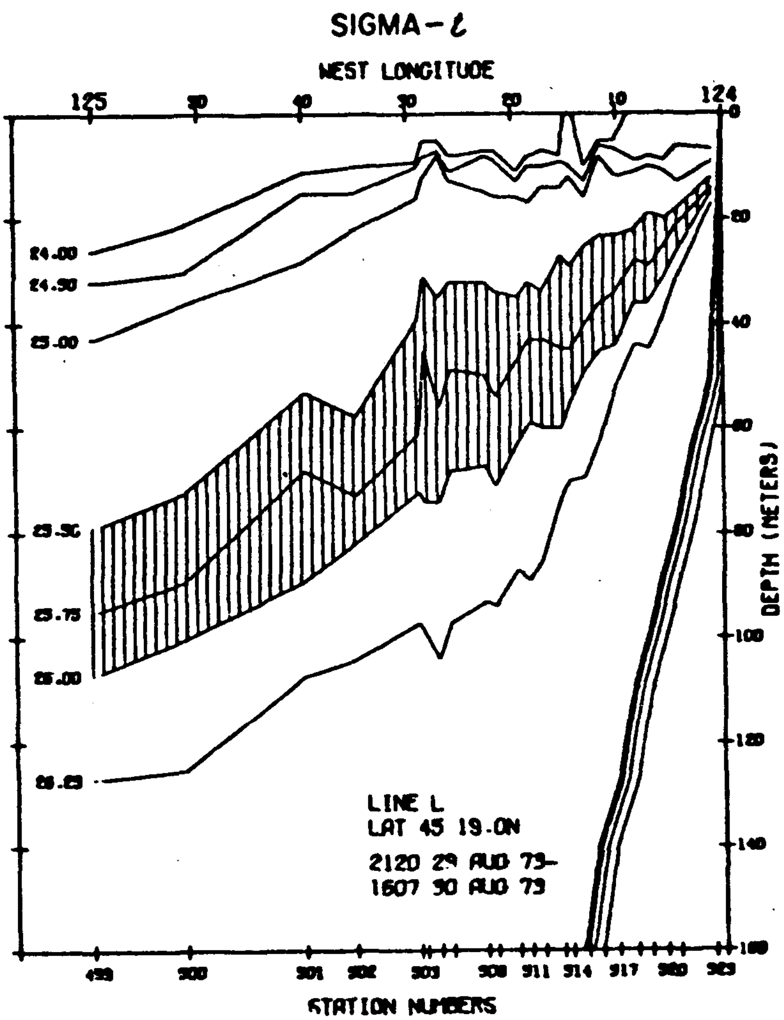

FIG. 6. Observed isopycnal surfaces off the coast of Oregon, August 1973 (courtesy of Holbrook and Halpern, 1974).

tense mixing with the surface waters allows the water to flow seaward with the surface density. Nevertheless, the *advantage* of the inertial theory is obvious. It yields predictions for the upwelling scale, the velocity field and the density field independent of any turbulent mixing coefficient and independent of any linearization hypothesis.

Acknowledgment. This research was supported in part by a Grant from the Atmospheric Research Section, National Science Foundation.

REFERENCES

Allen, J. S., 1973: Upwelling and coastal jets in a continuously stratified ocean. *J. Phys. Oceanogr.*, **3**, 245–257.

Holbrook, J. E., and Halpern, D., 1974: CUEA DATA Report 12, STD Measurements off the Oregon Coast (July/August 1973).

Hurlburt, H. E., and J. D. Thompson, 1973: Coastal upwelling on a β-plane. *J. Phys. Oceanogr.*, **3**, 16–32.

Pedlosky, J., 1974: Longshore currents, upwelling and bottom topography. *J. Phys. Oceanogr.*, **4**, 214–226.

Rossby, C. G., 1938: On the mutual adjustment of pressure and velocity distributions in certain simple current systems II. *J. Mar. Res.*, **1**, 239–263.