An improved economic growth model with endogenous fertility

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Abstract. This paper generalizes the model introduced by Cai in [5], where the Cass-Koopmans optimal growth model has been extended to allow for endogenous fertility choice. A sufficient condition for the existence of a nonzero steady state equilibrium is provided. If there is a unique nonzero steady state, this is a saddle point. If there are multiple nonzero steady states, the rightmost is a saddle point.

M.S.C. 2000: 91B62.
Key words: endogenous fertility, steady state.

1 Introduction

One of the key insights of Barro [2] was to think about utility maximizing individuals who care not only about their own consumption but also their children's consumption. This reasoning was extended by some authors to develop a number of growth models with endogenous fertility (see, e.g., [3], [4], [7], and [8]). In particular, Palivos introduced an endogenous fertility choice in the Cass-Koopmans optimal growth model. He showed that if the marginal product of capital minus the population growth rate is monotonically decreasing in capital, then the economy has only one steady state equilibrium, which has the saddle point property. If it does not decrease monotonically, multiple steady states and growth paths may emerge. In [5] Cai carried out a modified version of Palivos' model in which multiple steady states appear. In his model the maximized Hamiltonian is no longer concave with respect to capital, and so the uniqueness of the path satisfying the necessary condition for an optimum is no longer ensured. Thus, starting with an initial value of capital stock, there may exist several paths leading to different steady states. In this paper, we modify Cai's model by assuming a more general production function than the one he used. The corresponding model has a set of necessary conditions for optimality which can be reduced to a pair of differential equations. A sufficient condition for the existence of a steady state equilibrium is established for this dynamical system. Moreover, we prove that if there is only one nonzero steady state, then this is a saddle, and so the path that leads to this point is optimal and unique. If there is a finite number of nonzero steady states, then the rightmost nonzero steady state is a saddle point.

Applied Sciences, Vol.8, 2006, pp. 78-84.

2 The model

We assume a closed economy populated by a continuum of identical infinitely lived agents. In absence of immigration and mortality, there is a one-to-one correspondence between population growth rate and number of children. The agent's instantaneous utility is a separable function of consumption c and fertility n, i.e.

$$u(c,n) = u_1(c) + u_2(n). \tag{2.1}$$

The functions u_1 and u_2 in (2.1) are nonnegative, twice continuously differentiable, strictly increasing, i.e. $u_1'(c) > 0$, $u_2'(n) > 0$, strictly concave, i.e. $u_1''(c) < 0$, $u_2''(n) < 0$, and with the property that

$$\lim_{c\to 0} u_1'(c) = \lim_{n\to 0} u_2'(n) = \infty, \ \lim_{c\to\infty} u_1'(c) = \lim_{n\to b} u_2'(n) = 0.$$

The constant $b > 0$ represents the fertility limit which agents can reach. To simplify notation, time dependence of all variables is suppressed. Agents allocate their available unit of time between labor effort l and child-rearing $\phi(n)$, with ϕ a twice continuously differentiable function such that

$$\phi(0) = 0, \ \phi(b) = 1, \ \phi'(n) > 0, \ \phi''(n) \geq 0.$$

Each agent has access to a technology y described by a production function of the form $f(k,l) = g(k)l$, where g is a twice continuously differentiable function with

$$g(0) = 0, \ g'(k) > 0, \ g''(k) < 0, \ \lim_{k\to 0} g'(k) = \infty, \ \lim_{k\to\infty} g'(k) = 0.$$

The particular case of $g(k) = Ak^\alpha$, $0 < \alpha < 1$, $A > 0$, was studied in [5]. In this model, agents face two constraints. The first is a time allocation constraint,

$$l + \phi(n) = 1, \tag{2.2}$$

the second is the standard budget or resource constraint,

$$c + \dot{k} = f(k,l) - nk, \tag{2.3}$$

where a dot over a variable denotes time derivative.

A representative agent's optimization problem is given as

$$\max \int_0^\infty e^{-\rho t} u(c,n)\, dt \tag{2.4}$$

subject to equations (2.2), (2.3), and $k(0) = k_0 > 0$, with $\rho > 0$ the constant rate of time preference.

To perform the maximization in (2.4), we consider the current value Hamiltonian

$$H(k,c,n,\lambda) = u_1(c) + u_2(n) + \lambda[g(k)(1-\phi(n)) - c - nk],$$

where λ represents the co-state variable associated to (2.3). Applying the Pontryagin maximum principle yields the following first-order necessary conditions for the optimization problem

$$\dot{k} = H_\lambda \quad \Rightarrow \quad \dot{k} = g(k)(1-\phi(n)) - c - nk, \tag{2.5}$$

$$H_c = 0 \quad \Rightarrow \quad u_1'(c) = \lambda, \tag{2.6}$$

$$H_n = 0 \quad \Rightarrow \quad u_2'(n) = \lambda[g(k)\phi'(n) + k], \tag{2.7}$$

$$\dot{\lambda} = \rho\lambda - H_k \quad \Rightarrow \quad \dot{\lambda} = \lambda[\rho + n - g'(k)(1-\phi(n))], \tag{2.8}$$

where the subindex denotes the variable with respect to which the partial derivative is taken, plus the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda k = 0. \tag{2.9}$$

3 Existence of steady states

To investigate the dynamic of our model, we use equations (2.6) and (2.7) to express the co-state variable λ and the fertility rate n as a function of the state variable k and the consumption c.

Proposition 1.

i) $\lambda = \lambda(k, c)$; $\lambda_k = 0$ *and* $\lambda_c < 0$.

ii) $n = n(k, c)$; $n_k < 0$ *and* $n_c > 0$. *Moreover*

$$[g(k)\phi'(n(k,c)) + k]u_1'(c) = u_2'(n(k,c)). \tag{3.1}$$

Proof. i) Immediate from (2.6). ii) Set

$$F(k, c, n) = [g(k)\phi'(n) + k]u_1'(c) - u_2'(n).$$

Evidently (2.7) gives $F(k, c, n) = 0$. Since for any $(k_0, c_0) \in \mathbb{R}_+^2$,

$$F_n(k_0, c_0, n) = g(k_0)\phi''(n)u_1'(c_0) - u_2''(n) > 0,$$

$$\lim_{n\to 0} F(k_0, c_0, n) = -\infty, \ \lim_{n\to b} F(k_0, c_0, n) = [g(k_0)\phi'(b) + k_0]u_1'(c_0) > 0,$$

we have that the Implicit function theorem [6] implies that there exists a unique differentiable function $n = n(k, c)$ in the positive orthant of the (k, c)-plane such that $F(k, c, n(k, c)) = 0$. In particular, this gives (3.1). The chain rule applied to $F(k, c, n(k, c)) = 0$ yields $n_k = -F_k/F_n$ and $n_c = -F_c/F_n$. The result now follows from being

$$F_k = [g'(k)\phi'(n) + 1]u_1'(c) > 0, \ F_c = [g(k)\phi'(n) + k]u_1''(c) < 0,$$

$$F_n = g(k)\phi''(n)u_1'(c) - u_2''(n) > 0.$$

□

Corollary 1. *Let* $(u_2')^{-1}$ *be the inverse function of the map* $n \to u_2'(n)$. *Then*

$$n(k,c) = (u_2')^{-1}\left\{[g(k)\phi'(n(k,c)) + k]u_1'(c)\right\}. \tag{3.2}$$

Moreover, for any $k_0 \in \mathbb{R}_+$, $\lim_{c\to 0} n(k_0,c) = 0$ *and* $\lim_{c\to\infty} n(k_0,c) = b$.

Proof. There exists $(u_2')^{-1}$ since $n \to u_2'(n)$ is monotone decreasing. So (3.2) is immediate from (3.1). Let $k_0 \in \mathbb{R}_+$. Then $\lim_{c\to 0} n(k_0,c) = (u_2')^{-1}(\infty) = 0$, and $\lim_{c\to\infty} n(k_0,c) = (u_2')^{-1}(0) = b$. □

Proposition 2. *The following system*

$$\begin{cases} \dot{k} &= g(k)(1-\phi(n(k,c))) - c - n(k,c)k, \\ \dot{c} &= [\rho + n(k,c) - g'(k)(1-\phi(n(k,c)))]\, u_1'(c)/u_1''(c), \end{cases} \tag{3.3}$$

describes the dynamic behaviour in the (k,c)*-plane of the system of necessary conditions for an optimal program together with* $k(0) = k_0 > 0$ *and the transversality condition* (2.9).

Proof. Differentiating (2.6) with respect to time yields $u_1''\dot{c} = \dot{\lambda}$. Hence the statement follows from Proposition 1. □

Steady states of (3.3) are reached when $\dot{k} = \dot{c} = 0$. If we set

$$F_1(k,c) = g(k)(1-\phi(n(k,c))) - c - n(k,c)k,$$

$$F_2(k,c) = \rho + n(k,c) - g'(k)(1-\phi(n(k,c))),$$

then this is equivalent to write $F_1(k,c) = 0 = F_2(k,c)$.

Lemma 1. $F_1(k,c) = 0$ *determines a unique differentiable curve* $c = c_1(k)$ *on* $(0,\infty)$ *such that* $\lim_{k\to 0} c_1(k) = 0$ *and* $c_1(k) > 0$ *for all* $k > 0$.

Proof. For any $k_0 > 0$, Corollary 1 implies

$$\lim_{c\to 0} F_1(k_0,c) = g(k_0) > 0, \quad \lim_{c\to\infty} F_1(k_0,c) = -\infty.$$

Moreover, for all $c \in (0,\infty)$ it is

$$(F_1)_c = -[g(k)\phi'(n(k,c)) + k]n_c - 1 < 0.$$

Thus the Implicit function theorem states that $F_1(k,c) = 0$ determines a unique differentiable function $c = c_1(k)$ for all $k \in (0,\infty)$ and $F_1(k,c_1(k)) = 0$. As $n(0,c_1(0)) = (u_2')^{-1}(0) = b$, by taking $k \to 0$ in $F_1(k,c_1(k)) = 0$, it follows that $\lim_{k\to 0} c_1(k) = 0$. □

Lemma 2.

$F_2(k,c) = 0$ *determines a unique differentiable curve* $c = c_2(k)$ *on* $(0,\bar{k})$ *such that* $\lim_{k\to\bar{k}} c_2(k) = 0$, *with* $\bar{k} > 0$ *denoting the unique solution of the equation* $g'(k) = \rho$. *Furthermore, if* $(u_1')^{-1}$ *denotes the inverse function of the map* $c \to u_1'(c)$, *then*

$$c_2(k) = (u_1')^{-1}\{u_2'[g'(k)(1-\phi(n(k,c_2(k)))) - \rho]/[g(k)\phi'(n(k,c_2(k))) + k]\}.$$

Proof. Existence and uniqueness of $\bar{k}$ follow from being $k \to g'(k)$ a map monotonically decreasing to zero. Since for any $k_0 \in (0, \bar{k})$,

$$\lim_{c\to 0} F_2(k_0, c) = \rho - g'(k_0) < 0, \ \lim_{c\to\infty} F_2(k_0, c) = \rho + b > 0,$$

and

$$(F_2)_c = [1 + g'(k)\phi'(n(k, c))]n_c > 0$$

for all $c \in (0, \infty)$ and $k \in (0, \bar{k})$, the Implicit function theorem yields that the equation $F_2(k, c) = 0$ determines a unique differentiable function $c = c_2(k)$ for all $k \in (0, \bar{k})$, and $F_2(k, c_2(k)) = 0$. As $g'(\bar{k}) = \rho$, for $k \to \bar{k}$ in $F_2(k, c_2(k)) = 0$, we get

$$\lim_{k\to\bar{k}} n(k, c_2(k)) = -\rho \lim_{k\to\bar{k}} \phi(n(k, c_2(k))).$$

From this we must have $\lim_{k\to\bar{k}} n(k, c_2(k)) = 0$. Therefore, by Corollary 1, it follows $\lim_{k\to\bar{k}} c_2(k) = 0$. Finally, the map $c \to u_1'(c)$ is monotone decreasing and so it has an inverse. The statement comes from (3.2) and $F_2(k, c_2(k)) = 0$. □

The previous Lemmas provide an information on how the curves $F_1(k, c) = 0$ and $F_2(k, c) = 0$ divide the positive orthant of the (k, c)-plane. On the curve $c = c_1(k)$, it is $\dot{k} = 0$. Hence, the first quadrant of the (k, c)-plane is separated into two parts: $\dot{k} < 0$ above the curve $c = c_1(k)$ and $\dot{k} > 0$ below the curve $c = c_1(k)$. On the curve $c = c_2(k)$, we have $\dot{c} = 0$. So, the first quadrant of the (k, c)-plane is separated into two parts: $\dot{c} < 0$ above the curve $c = c_2(k)$ and $\dot{c} > 0$ below the curve $c = c_2(k)$. It is now immediate the next result which establishes a sufficient condition for the existence of nonzero steady states equilibrium.

Proposition 3. *Under the following limit condition*

$$\limsup_{k\to 0} c_2(k) > 0, \tag{3.4}$$

there exists at least a nonzero steady state for the dynamical system (3.3).

Corollary 2. *Condition* (3.4) *is equivalent to*

$$\liminf_{n\to b} u_2'(n)/g(k(n)) < \infty,$$

where $k(n) = (g')^{-1}\{(\rho + n)/(1 - \phi(n))\}$.

Proof. Lemma 2 yields $\limsup_{k\to 0} c_2(k) > 0$ if and only if $\liminf_{k\to 0} u_1'(c_2(k)) < \infty$. From $F_2(k, c) = 0$, we see that $k \equiv k(n) = (g')^{-1}((\rho+n)/(1 - \phi(n)))$. So $k \to 0$ if and only if $n \to b$. From (3.2) we get that $u_1'(c) = u_2'(n)/(g(k)\phi'(n) + k)$. As $\lim_{k\to 0} g(k)/k = \infty$ and $\phi'(b) > 0$, the statement follows immediately. □

From Arrow's theorem [1], we know that if the maximized current-value Hamiltonian, defined as

$$\begin{aligned} H^0(k,c) &= \max_{n,\lambda} H(k,c,n,\lambda) \\ &= u_1(c) + u_2(n(k,c)) + \lambda(k,c)[g(k)(1-\phi(n(k,c)) - c - n(k,c)k] \end{aligned}$$

is concave in the state variable k, given the co-state variable λ, then the first order conditions, together with a transversality condition, are necessary and sufficient to characterize the maximum. As

$$H^0_k(k,c) = \lambda(k,c)[g'(k)(1-\phi(n(k,c))) - n],$$

for this to be true it must hold the condition that the marginal product of capital minus the population growth rate is monotonically decreasing in capital. However, in this model, this condition does not hold and so multiple steady states equilibrium may emerge.

Proposition 4. *If the dynamical system* (3.3) *has a unique nonzero steady state, then this is a saddle point. If* (3.3) *has a finite number of nonzero steady states, then the rightmost nonzero steady state is a saddle point.*

Proof. Let $k^* \in (0,\bar{k})$ be the unique point of intersection of the curves $c = c_1(k)$ and $c = c_2(k)$. Since this intersection corresponds to the nonzero steady state of (3.3), if we set $c^* = c_1(k^*) = c_2(k^*)$, then (k^*, c^*) is the unique steady state of (3.3). As $c_1(k)$ is over $c_2(k)$ when k approximates $\bar{k}$, it follows that $c_1(k) < c_2(k)$ when $0 < k < k^*$, and $c_1(k) > c_2(k)$ when $k^* < k < \bar{k}$. Therefore the first quadrant of the (k,c)-plane is separated into four regions and we see that (k^*, c^*) is a saddle point by phase portrait analysis. Now let assume that the dynamical system (3.3) has a finite number of nonzero steady states. Let (k_1^*, c_1^*) be the steady state on the extreme right of the (k,c)-plane. The curve $c = c_1(k)$ is above the curve $c = c_2(k)$ on the interval $(k_1^*, \bar{k})$, and below the curve $c = c_2(k)$ on the interval $(k_1^* - \delta, k_1^*)$, for $\delta > 0$ chosen sufficiently small. So proceeding as before we get the statement. □

If the dynamical system (3.3) has a unique nonzero steady state, then the unique saddle path is the optimal growth path. If it has multiple nonzero steady states, then the steady state on the extreme right is a saddle point, and the saddle path is the optimal growth path for the per capita capital. Moreover, as the next result confirms, the steady state with higher per capita capital has higher per capita consumption and lower fertility.

Proposition 5. *Let* (k_1^*, c_1^*) *and* (k_2^*, c_2^*) *be two nonzero steady states of the dynamical system* (3.3)*. Let* $n_i^* = n_i(k_i^*, c_i^*)$, $i = 1, 2$*. If* $k_1^* < k_2^*$*, then* $c_1^* < c_2^*$ *and* $n_1^* > n_2^*$*.*

Proof. Since the function $k = k(n)$ is monotonically decreasing, it has an inverse. Hence $k_1^* = k(n_1^*) < k(n_2^*) = k_2^*$ yields $n_1^* > n_2^*$. From $F_1(k,c) = 0$ we have

$$c_2^* - c_1^* = [g(k_2^*) - g(k_1^*)](1-\phi(n_2^*)) + g(k_1^*)(\phi(n_1^*) - \phi(n_2^*)) - n_2^* k_2^* + n_1^* k_1^*$$

$$= [g(k_2^*) - g(k_1^*)](1-\phi(n_2^*)) + g(k_1^*)(\phi(n_1^*) - \phi(n_2^*)) - n_2^*(k_2^* - k_1^*) + k_1^*(n_1^* - n_2^*)$$

$$= [g'(\xi)(1 - \phi(n_2^*)) - n_2^*](k_2^* - k_1^*) + [g(k_1^*)\phi'(\eta) + k_1^*](n_1^* - n_2^*) > 0,$$

where $\xi \in (k_1^*, k_2^*)$ and $\eta \in (n_2^*, n_1^*)$. Note that $F_2(k, n) = 0$ and $k \to g'(k)$ monotone decreasing imply $g'(\xi)(1 - \phi(n_2^*)) > g'(k_2^*)(1 - \phi(n_2^*)) = \rho + n_2^*$. □

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