Weak Interactions of Equatorial Waves in a One-Layer Model. Part II: Applications

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(Manuscript received 19 July 1982, in final form 16 March 1983)

ABSTRACT

There are pairs of resonant triads with two common components. Analytic solutions describing the evolution of a system with such a *double resonant triad* are presented and compared with the resonant three-wave problem. Both solutions for constant energies (and shifted frequencies) and for maximum energy exchange (and unshifted frequencies) are discussed. The latter problem is integrable; a subclass of solutions can be written in terms of those of the one-triad system.

Unlike problems of mid-latitude quasi-geostrophic flow and internal gravity waves in a vertical plane, there are resonant triads of equatorial waves with the same speed which have a finite interaction coefficient. This includes the case of second-harmonic resonance or, more generally, a chain of resonant harmonics (a finite number of them in the case of Rossby waves, but an infinite number for inertia-gravity modes). Some analytic and numerical solutions describing the evolution of different chains of resonant harmonics are presented and compared with the (resonant) three-wave problem. Both solutions for constant energies (and shifted frequencies) and for maximum energy exchange (and unshifted frequencies) are presented. The evolution of a chain of resonant harmonics with more than five components is aperiodic, chaotic and unstable.

The derivation of the equations of long-short wave resonances and Korteweg-deVries is straightforward from the evolution equations in phase-space, i.e., there is no need of the usual and cumbersome perturbation expansion in physical space. These equations govern the interaction of a packet of Rossby and inertia-gravity waves with a long Rossby mode of the same group velocity and the self-interaction of long Rossby waves, respectively.

1. Introduction

In the first part of this paper (Ripa, 1983), we posed the problem of weak interactions of equatorial waves in a one-layer model of the ocean or the atmosphere. A classification of all possible resonant triads was presented, particular cases are triads of waves with the same speed and non-local triads (in frequency space). This paper is devoted to applications of those results.

The simplest nonlinear wave system is probably that of one isolated resonant triad for which there is an explicit general solution [see Ripa (1981), hereafter R81]. The next step in complexity may be that of two resonant triads with two common components, a total of four waves. Here we study the possibility of such systems in the equatorial waveguide and some of their dynamics.

In the systems of quasi-geostrophic flow (QGF) and internal gravity waves on a vertical plane (IGW) waves with the same speed do not interact; as a consequence, one wave is found to be an exact nonlinear solution. That is not true for equatorial waves [see Ripa (1982), hereafter R82], and thus an interesting problem is to study the evolution of the system when originally there is only one wave. This is addressed here for the particular case in which the initial wave resonates with harmonics.

Finally, there are modes which effectively resonate with a continuum of waves instead of a discrete set, e.g., when the group velocities of two members of a resonant triad coincide. These cases must be studied with the evolution equation for wave packets which is usually obtained by a perturbation expansion appropriate to each particular problem. We present here an alternative way of deriving that equation which we believe to be simpler and more general.

This paper is developed as follows: Resonant double triads are studied in Section 2, and the resonant interactions among harmonics are discussed in Section 3. Section 4 deals with the derivation of the evolution equation for wave-packets, and a summary is provided in Section 5.

Sections, equations and tables from the first part of this paper are quoted using the prefix I-. The reader is referred to the Appendix in that publication for notation used throughout.

2. Resonant double triads

The simplest truncated system with nonlinear behavior is the three-wave problem (resonance with the harmonic, discussed in the following section, is but a limiting case of that one). Here we investigate the existence of resonant four-wave systems, made up of a couple of resonant triads with two common components in the equatorial wave-guide and study some of their dynamics.

An N-wave system with only resonant interactions has N degrees of freedom and (at least) three integrals of motion: quadratic energy $E^{(2)}$, quadratic pseudo-momentum $P^{(2)}$, and the value of the hamiltonian H [see (I-3.13) and (I-3.14)]. The three-wave problem is then integrable: its general solution, from an arbitrary initial condition, is written in R81. We briefly discuss its characteristics now, for the sake of comparison with larger systems. Two nondimensional parameters, m' and m, fully describe the properties of each solution. These may take any value on the triangle ($-1 \leqslant m' \leqslant 1, 0 \leqslant m \leqslant 1 - |m'|$); m' is related to the value of $P^{(2)}/E^{(2)}$ and m to that of $H^2/[E^{(2)}]^3$. The energies $|Z_i|^2$, and instantaneous frequencies, $-d_t \arg(Z_i)$, are periodic functions of time [with the sole exception of the ($m' = 0$, $m = 1$) solution].

The cases $m' = -1$ or $m' = +1$ correspond to all energy in one of the stable components (extreme slowness and thus smaller frequencies).

For ($m = 0$, $-1 < m' < 1$), there is no energy exchange and the instantaneous frequencies are constant and shifted from their free values. These cases correspond to $\mathrm{Re}(Z_1Z_2Z_3) = 0$ (e.g., $X_a = iY_a$ with Y_a real, in the notation of Section I-3.b).

At the other sides of the (m, m') triangle, ($m = 1 - |m'|$, $-1 < m' < 1$), there is the maximum energy exchange compatible with $E^{(2)}$ and $P^{(2)}$ conservation and the frequencies are fixed at their free values. The X_a are real, aside from unimportant constant phase factors, restricted by $\mathrm{Im}(Z_1Z_2Z_3)= 0$. [Thus $H \equiv 0$, and (I-3.14) does not represent a constraint independent from (I-3.13a).] If $|m'| \approx 1$ there is a catalytic energy exchange between one of the stable components and the unstable one in a time scale determined by the energy of the other stable mode, which is quite unaffected by the process. If $m' = 0$, on the other hand, the solution represents the complete decay of the unstable component into the stable ones followed by generation of the former by the latter, just once.

Finally, for ($0 < m < 1 - |m'|$, $-1 < m' < 1$), the energy exchange is partially inhibited by the conservation of H and the instantaneous frequencies are both shifted and modulated.

a. Kinematics

Given a resonant triad with components (*1, 2, 3*) and in which s_1 is between s_2 and s_3, we investigate the possible existence of a component *4* such that the interaction in the triad (*4, 1*, 2*) is also resonant. [Component *1**, with frequency and wavenumber equal to $-\omega_1$ and $-k_1$, is physically the same as *1*, because its amplitude is equal to $-Z_1^*$; see (I-2.19).] The interaction and resonance conditions, for both triads, take the form

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad s_1\omega_1 + s_2\omega_2 + s_3\omega_3 = 0, \qquad (2.1a)$$

$$\omega_4 - \omega_1 + \omega_2 = 0, \quad s_4\omega_4 - s_1\omega_1 + s_2\omega_2 = 0, \qquad (2.1b)$$

$$n_1 + n_2 + n_3 = \text{odd}, \quad n_4 + n_3 = \text{even}. \qquad (2.1c)$$

All cases of a pair of RT with two common components can be written in this way, via an appropriate renumbering. Components *1* and *4* have the maximum $|\omega|$ of triads (*1, 2, 3*) and (*4, 1*, 2*) respectively, i.e., the four slownesses are ordered, in increasing *or* decreasing values, as (s_2, s_4, s_1, s_3); this implies $\mathrm{sgn}(\omega_4) = \mathrm{sgn}(\omega_1) = -\mathrm{sgn}(\omega_2) = -\mathrm{sgn}(\omega_3)$ [cf., Eqs. (I-3.4) and (I-3.5)].

The value of s_4 may be calculated, from (2.1a, b), as

$$s_4 = \frac{[2s_1s_2 - s_3(s_1 + s_2)]}{[s_1 + s_2 - 2s_3]}. \qquad (2.2)$$

In order to evaluate n_4, we use (I-4.12) with (a, b, c) = (*1, 2, 3*) and (*1*, 2, 4*), which yields

$$2(1 - s_1s_2c^2)\omega_1\omega_2/\beta c = N_3 - N_1 - N_2$$
$$= N_1 + N_2 - N_4,$$

and finally, from the last two terms,

$$n_4 = 2n_1 + 2n_2 - n_3 + 1 + [s_1 + s_2 - (s_3 + s_4)/2]c. \qquad (2.3)$$

Using the fourteen classes of RT from Table I-3 (including the permutation of b and c) we search for values of s_4 and n_4, from (2.2) and (2.3), that correspond to an equatorial wave, namely such that n_4 is an integer larger than -2 and with s_4 within the appropriate bounds from Table I-1. It is found that there are isolated solutions (depending on the values of n_1, n_2 and n_3), which fall into one of the following classes: 1) *R'RRR*, 2) *R'RRK*, 3) *GGG'R'*, 4) *GGR'G'*, 5) *GGR'K*, and 6) *GGKR'*, where *R'*: *R* or *M*($sc \leqslant -1$), and *G'*: *G* or *M*($sc > -1$). Thus, there are a total of 16 types of double-RT; classes (2), (3) and (6) have an infinite number of solutions in each branch of (s_1, s_2, s_3).

b. Dynamics

Let γ and γ' denote interaction coefficients of the two triads, namely,

$$\gamma = \gamma(1, 2, 3), \quad \gamma' = \gamma(4, 1^*, 2). \qquad (2.4)$$

Truncating the expansion of the system to only these four components, its evolution is controlled by [see (I-3.10)]

$$[d_t + i\omega_4]Z_4 = -\gamma'\omega_4 Z_1 Z_2^*,$$

$$[d_t + i\omega_1]Z_1 = \gamma'\omega_1 Z_2 Z_4 + \gamma\omega_1 Z_2^* Z_3^*,$$
$$[d_t + i\omega_2]Z_2 = -\gamma'\omega_2 Z_4^* Z_1 + \gamma\omega_2 Z_3^* Z_1^*,$$
$$[d_t + i\omega_3]Z_3 = \gamma\omega_3 Z_1^* Z_2^*. \quad (2.5)$$

A complete discussion of the types of solutions of (2.5) is beyond the scope of this paper; there does not seem to exist a general (analytic) solution, unlike for the three-wave problem. A few types of solutions are discussed here.

1) STABILITY PROBLEM

First we consider the stability of each of the four components; solutions of the form

$$Z_j \propto \exp[-i(\omega_j + \delta\omega_j)t], \quad j = 1, 2, 3, 4, \quad (2.6)$$

valid as long as $|Z_a| \gg |Z_b|$ for $b \neq a$. Solving for the $\delta\omega_j$ and neglecting nonlinear terms that do not involve Z_a (for each value of a), we get

$$\delta\omega_j = (0, -\mu^*, \mu, \mu), \quad \mu^2 = \omega_2\omega_3|\gamma Z_1|^2(\alpha^2 - 1), \quad \text{for } a = 1, \quad (2.7a)$$

$$\delta\omega_j = (\mu, 0, -\mu, \mu), \quad \mu^2 = -\omega_1\omega_3|\gamma Z_2|^2(1 + \alpha^2) > 0, \quad \text{for } a = 2, \quad (2.7b)$$

$$\delta\omega_j = (\mu, -\mu, 0, 0), \quad \mu^2 = -\omega_1\omega_2|\gamma Z_3|^2 > 0, \quad \text{for } a = 3, \quad (2.7c)$$

$$\delta\omega_j = (\mu, \mu, 0, 0), \quad \mu^2 = \omega_1\omega_2|\gamma' Z_4|^2 < 0, \quad \text{for } a = 4, \quad (2.7d)$$

where

$$\alpha^2 = -(\gamma'/\gamma)^2(\omega_4/\omega_3) \geq 0. \quad (2.8)$$

Therefore, component *1* is *unstable* if $\alpha^2 < 1$ [dominance of the (*1*, *2*, *3*) RT] or *stable* if $\alpha^2 > 1$ [dominance of the (*4*, *1**, *2*) RT], component *2* is *stable,* component *3* is *stable* and component *4* is *unstable.*

In summary, *2* and *3* are stable, as required by E and P conservation (I-3.13) because these components have the extreme values of slowness; *4* is unstable, as the analysis of triad (*4*, *1**, *2*) alone determines; finally, the stability of *1*, which is the unstable component (intermediate slowness) of (*1*, *2*, *3*) and one of the stable components (extreme slowness) of (*4*, *1**, *2*) depends upon the parameters of which triad dominates.

2) EQUILIBRIUM SOLUTIONS

As a second example, we seek solutions of (2.5) with constant-energies and shifted-frequencies, *viz.,* of the form

$$Z_a = iY_a \exp[-i\omega_a(1 + \delta)t], \quad (2.9)$$

where Y_a and δ are real and constant. Substitution of (2.9) in (2.5) poses a nonlinear eigenvalue problem, whose solution is given by the eigenvector

$$Y_2 = \pm Y_1, \quad \delta Y_4 \pm \gamma' Y_1^2 = 0, \quad \delta Y_3 \pm \gamma Y_1^2 = 0, \quad (2.10)$$

and the eigenvalue

$$\delta^2 = (\gamma^2 + \gamma'^2)Y_1^2 \equiv \tfrac{1}{3}(\gamma^2 + \gamma'^2)E^{(2)}. \quad (2.11)$$

This is not the only equilibrium solution; those of (2.7b, c) are other examples of solutions without energy exchange. In comparison with the one-dimensional continuum of equilibrium solutions from the three-wave problem, solutions (2.7a, b) (with all the energy in an extreme-slowness component) are equivalent to the $m' = \pm 1$ cases, whereas solutions (2.9) through (2.11) (with equal relative frequencies shift) correspond to the $m' = 0$ case (see R81).

The $m' = 0$ case of the three-wave problem, Eqs. (I-4.5) and (I-4.6), may be recovered by making either $\gamma = 0$ or $\gamma' = 0$ in (2.9)–(2.11). Notice that the relative frequency-shift δ is larger for the double-RT than for the three-wave problem (for the same total energy). Also, there is no equi-partition of energy in (2.10), unlike in its three-component counterpart.

3) MAXIMUM-ENERGY-EXCHANGE

As in the three-wave problem, we expect these solutions to correspond to unshifted frequencies, e.g. (I-3.9) with X_a real. For this class of solutions $H \equiv 0$, and thus (I-3.13a) and (I-3.14) do not represent independent constraints on the evolution of the system. However, the solutions of (2.5) *with real* X_a have an extra integral of motion,

$$I = \gamma\omega_3 X_4 - \gamma'\omega_4 X_3 = \text{constant}. \quad (2.12)$$

Together with (I-3.13) this allows for a complete integration of the model equations. We then expect all possible solutions of $X_a(t)$ to be regular and periodic, just as for the one-triad case [which has, for real X_a, three components and two integrals of motion, $E^{(2)}$ and $P^{(2)}$].

As an example, a vanishing value of I reduces the evolution equations in the form

$$I = 0 \Rightarrow \begin{cases} dX_1/dt = \gamma(1 + \alpha^2)\omega_1 X_2 X_3, & (2.13a) \\ dX_2/dt = \gamma(1 - \alpha^2)\omega_2 X_3 X_1, & (2.13b) \\ dX_3/dt = \gamma\omega_3 X_1 X_2, & (2.13c) \end{cases}$$

which are, via an appropriate renormalization of the X_a, the equations for the maximum energy-exchange solutions of the three-wave problem [e.g., see Eq. (5.23) in R81 for the expression of the $X_a(t)$].

For $\alpha^2 < 1$ [dominance of the (*1*, *2*, *3*) RT over the (*4*, *1**, *2*) RT], and for the particular Case, $I = 0$, the dynamics are those of the three-wave problem with *1* as the unstable component and *2* and *3* (or *4*) being stable because the nonlinear coefficient in (2.13a) has a sign opposite to those of the coefficients in (2.13b, c). [This is a general property of the gyroscopic equations (2.13).] For $\alpha^2 > 1$ [dominance of the (*4*, *1**, *2*) RT over the (*1*, *2*, *3*) RT], on the other hand, *1* and *2* are the stable components, and *3* (or *4*) is the unstable one of the reduced three-wave system. Notice that $\alpha^2 < 1$ ($\alpha^2 > 1$) is the condition of instability (stability) of *1*, (2.7a), derived without assuming $I = 0$.

3. Harmonics resonance

An equatorial wave is not an exact nonlinear solution, unlike for the case of the systems of QGF and IGW studied in R81. Thus if the initial fields correspond to the structure of a single mode with parameters (k_1, ω_1, n_1) (namely, $Z_1(0) \neq 0$ and $Z_a(0) = 0$, for $a \neq 1$), then (I-2.25) through (I-2.27) imply that other components with wavenumbers equal to any integer multiplied by k_1 will be excited. For a short enough elapsed time the amplitudes of modes with $k = 0$, $\omega \neq 0$ or $|k| = 2|k_1|$ and n = odd grow like $\sigma|Z_1(0)|^2 t$; those with other values of k and/or parity of n (as well as components with $k = \omega = 0$) grow like higher powers of t and $|Z_1(0)|$. If the amplitude $|Z_1(0)|$ is small, the energy loss will be enhanced if one of the harmonics is in resonance with the initial wave, i.e., if there is a component *2* such that

$$\omega_2 + 2\omega_1 = 0, \quad k_2 + 2k_1 = 0,$$

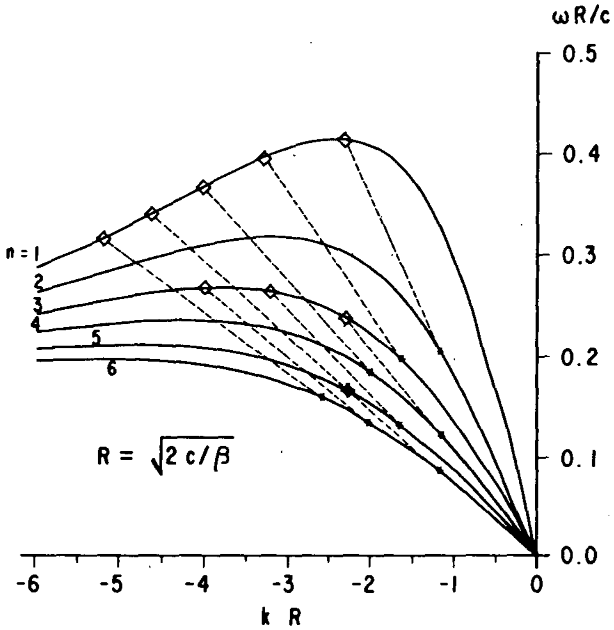

FIG. 1. Examples of Rossby waves (asterisks) that interact resonantly with a harmonic (diamonds) (see Table 1). Notice the possibility of more than two modes coupled this way; e.g., n = 6, 5, and 1 along the line $\omega = -3kc/41$.

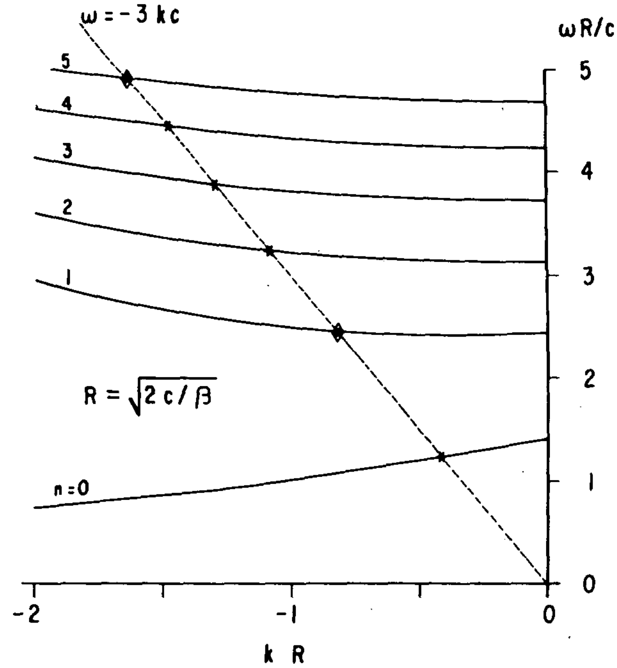

FIG. 2. As in Fig. 1 but for gravity modes. The modes lie on the line $\omega = -3kc$, and are in infinite chains (e.g., n= 0, 1, 5, . . . , or n = 2, 9, 37, · · ·).

and

$$n_2 = \text{odd}. \quad (3.1)$$

This equation is a limiting case of triad of waves with the same speed, whose solution is given by (I-4.7) with $n_a = n_b = n_1$ and $n_c = n_2$, *viz.*,

$$s_2 = s_1 = (2n_2 - 8n_1 - 3)/(3c),$$
$$n_2 < n_1, \quad \text{(Rossby)}, \quad (3.2)$$

or

$$s_2 = s_1 = -1/(3c), \quad n_2 = 4n_1 + 1, \quad \text{(gravity)}. \quad (3.3)$$

The corresponding wavenumber of the fundamental wave is given by

$$k_1^2 = (2\beta/3c)(n_1 - n_2)[1 - (sc)^{-2}]$$
$$\approx (2\beta/3c)(n_1 - n_2), \quad \text{(Rossby)}, \quad (3.4)$$

(the last approximation represents an error smaller than 1.6% in k), or

$$k_1^2 = (\beta/12c)(3n_1 + 1), \quad \text{(gravity)}. \quad (3.5)$$

Several examples of waves that interact resonantly with the second harmonic are shown in Fig. 1 (for Rossby modes) and Fig. 2 (for gravity ones); their frequencies and wavenumbers are marked in the corresponding dispersion curves. Values of k and ω are also presented in Table 1 along with that of the interaction coefficient γ calculated for the ocean model [λ = 1 in (I-2.2)]. For the sake of comparison, the

TABLE 1. Examples of resonance of a wave with a harmonic. The first nine (last six) cases are made up of Rossby (gravity) modes. R is the deformation radius and c is the separation constant; $R^2 = 2c/\beta$. The interaction coefficient γ is evaluated (for Tables 1, 2 and 3) for the ocean model; $\lambda = 1$ in (I-2.34).

n		kR		$\omega R/c$			
1	2	1	2	1	2	sc	γc
2	1	1.173	−2.346	−0.207	0.414	−17/3	0.8744
3	1	1.645	−3.290	−0.197	0.395	−25/3	0.6059
4	3	1.161	−2.322	−0.120	0.240	−29/3	0.6854
4	1	2.008	−4.017	−0.183	0.365	−33/3	0.4482
5	3	1.638	−3.277	−0.133	0.266	−37/3	0.5420
5	1	2.316	−4.631	−0.169	0.339	−41/3	0.3512
6	5	1.158	−2.316	−0.085	0.169	−41/3	0.5889
6	3	2.004	−4.009	−0.134	0.267	−45/3	0.4325
6	1	2.587	−5.174	−0.158	0.317	−49/3	0.2878
0	1	0.408	−0.816	−1.225	2.449	−1/3	0.1656
1	5	0.816	−1.633	−2.449	4.899	−1/3	−0.1079
2	9	1.080	−2.160	−3.240	6.481	−1/3	0.0819
3	13	1.291	−2.582	−3.873	7.746	−1/3	−0.0681
4	17	1.472	−2.944	−4.416	8.832	−1/3	0.0593
5	21	1.633	−3.266	−4.899	9.798	−1/3	−0.0530

value of γc corresponding to any trio of interacting K modes is equal to $3^{1/2}/(2\pi^{1/4}) \approx 0.6505$. (Recall that the importance of nonlinear effects is measured by the value of $|\gamma Z|$.)

In the case of Rossby waves, the resonant chains are usually made of two components, although there are solutions with more modes (e.g., $n = 6$, 5, and 1 along $\omega = -3kc/41$). The chains must always have a finite number of components, though, because Rossby modes have bounded $|\omega|$.

Gravity modes, on the other hand, are found in infinite chains along the $\omega = -3kc$ line; zonal and meridional numbers given by $|k/k_1| = 2, 4, 8, \ldots$, and $n = 4n_1 + 1$, $4(4n_1 + 1) + 1$, $4[4(4n_1 + 1) + 1] + 1, \ldots$. In fact, these chains of resonantly coupled modes are more dense: they involve all components with $k = jk_1$ and $n = j^2 n_1 + (j^2 - 1)/3$, where j is any integer different from zero or a multiple of 3, and k_1 and n_1 are the parameters of the fundamental wave, related by (3.5). Even though there are infinitely many harmonics in resonance, these chains are *dynamically* quite different from the system K modes since the value of γ, instead of being a constant, is a function of the n_i, due to the different meridional structures.

[More generally, it follows from the expression for $\omega^2(n, s)$ in Table I-1 that certain harmonics of a fundamental wave (with parameters k', n', and ω') belong to a resonant chain (say, $k_j = jk'$ and $\omega_j = j\omega'$, with j an integer) if $n_j = j^2 n' + (j^2 - 1)(1 + sc)/2$ is also an integer, and (I-2.26) is satisfied for the corresponding triads. This can only happen for certain values of the slowness s of the chain (e.g., sc must be a rational number); the particular cases (3.2) and (3.3) correspond to the condition that the first and second harmonics ($j = 1, 2$) belong to the chain. Resonant chains of R (G) modes have a finite (infinite) number of components, and higher harmonics correspond to modes more (less) trapped to the equator. The components of any coupled triad of high gravity harmonics have also large values of n_i; the results of Section I-2.e may be used to determine whether γ is negligible or not: the slowness s in (3.2) or (3.3) is the solution of $\Delta(2n_i + 1 + sc) = 0$; with (I-2.40) it follows that the stationary-phase latitudes are given by $y_0{}^2 = -sc^2\beta^{-1}$. Consequently, only high harmonics in resonant chains of *westward* propagating G waves ($-1 < sc < 0$) interact, and the interaction region is within one deformation radius from the equator.]

a. Resonance of the first and second harmonics

To illustrate some of the consequences of (3.1), when there are no other states resonantly connected, we consider (I-2.27) truncated to include only components *1* and *2*, i.e., the two-wave problem

$$\left.\begin{aligned}[d_t + i\omega_2]Z_2 &= \tfrac{1}{2}\gamma\omega_2 Z_1^{*2}\\ [d_t + i\omega_1]Z_1 &= \gamma\omega_1 Z_2^* Z_1^*\end{aligned}\right\}. \quad (3.6)$$

The coupling coefficients have been written using (I-3.6). Neglect of the excitation of other components (assuming that they are not initially present) is approximately valid for an elapsed time t such that $\mu^2 t \ll |\omega_1|$ (Bretherton, 1964); μ is the inverse of the nonlinear time scale, defined in (3.8) below. If the resonant pair is resonantly coupled with one or more components, it is shown below that the truncated system (3.6) is then justifiable only for a much shorter time, *viz.*, $|\mu|t \leqslant O(1)$.

A system similar to (3.1)/(3.6) was obtained by McGoldrick (1970) studying the interaction between surface gravity and capillary waves.

The general solution of (3.6) may be found from that of the three-wave problem (e.g., replacing Z_3 by Z_2, and Z_1 and Z_2 by $Z_1 2^{-1/2}$ in the solution of Section 5 of R81; it corresponds to the $m' = 0$ cases of that paper). All possible solutions are classified by a single parameter m that may have any value from 0 to 1. Given any initial condition, m is calculated by $m = \frac{1}{2}[1 + 3^{1/2} \tan(\alpha - \pi/6)]$, where $1 + \cos(3\alpha) = (27/2)[\mathrm{Im}(Z_1^2 Z_2)]^2(|Z_1|^2 + |Z_2|^2)^{-3}$, and $0 \leq \alpha \leq \pi/3$.

The $m = 0$ solution is given by (I-4.5) and (I-4.6) with Z_a and Z_b replaced by $Z_1 2^{-1/2}$ and Z_c by Z_2; it corresponds to the case of constant energies and instantaneous frequencies. Since both fundamental wave and the harmonic suffer the same relative frequency shift, this solution constitutes a first approximation to a nonlinear periodic wave that propagates with speed $(1 \pm \gamma|Z_2|)/s'$ without changing shape. Notice that, unlike the Stokes wave (and due to resonance of the interaction), 1) the energy of the second harmonic is half that of the fundamental component, 2) the change in phase speed $\delta(\omega/k)$ may have either sign, and 3) $\delta(\omega/k)$ is $O(|Z|)$ instead of $O(|Z|^2)$.

The $m = 1$ solution, on the other hand, corresponds to the maximum energy exchange between both components. In particular, this is the solution of (3.6) for the initial condition $Z_2(0) = 0$; *viz.*,

$$\left.\begin{aligned} Z_2 &= Z_1(0)^* \tanh(\mu t) \exp(-i\omega_2 t - i\delta_1) \\ Z_1 &= Z_1(0) \operatorname{sech}(\mu t) \exp(-i\omega_1 t) \end{aligned}\right\}, \tag{3.7}$$

where δ_1 is the phase of $Z_1(0)$, and

$$\mu = \gamma(1, 1, 2^*)\omega_1|Z_1(0)|. \tag{3.8}$$

The fundamental wave looses half of its energy to the resonant harmonic in time $t = 0.88|\mu|^{-1}$ (i.e., $\cosh(\mu t) = 2^{1/2}$); with (3.8) this time is equal to the period of the wave for $|\gamma Z_1(0)| \sim 1/7$. This solution is illustrated in Fig. 3a. Notice that the energy flux is from the component with smaller frequency to that with larger value of $|\omega|$. The opposite is true for the decay of component *2* into *1*, which is also represented by (3.7) but from some initial time t_0 such that $|\mu|t_0 \ll -1$. This decay is not like the case of parametric subharmonic instability of internal gravity waves in which the frequencies are related as in (3.1) but the wavenumbers are very different in magnitude; see R81.

The two-wave system is one extreme case of resonant coupling; the other is given by the Kelvin modes problem (see R82, Section 3): any interacting trio is in resonance and has the same value of γ $[=(2 + \lambda)/(3^{1/2}2\pi^{1/4}c)]$ due to the fact that K components have all identical meridional structure. Therefore, it is interesting to compare the evolution of both systems starting with one single wave at $t = 0$: in (3.6) only the resonant second harmonic is allowed to be excited whereas in the non-dispersive (K) case all harmonics gain energy; see Fig. 3b. For the K modes problem the amplitudes of the fundamental wave and the second harmonic are given by (3.7) with the replacements

$$\left.\begin{aligned} \tanh(\mu t) &\rightarrow [J_2(4\mu t)]/(2\mu t) \\ \operatorname{sech}(\mu t) &\rightarrow [J_1(2\mu t)]/(\mu t) \end{aligned}\right\}, \tag{3.9}$$

and the solution is not valid for $2\mu t > 1$ [R82 Eq. (3.17) for $j = 1, 2$; the nonlinear time scale t^* used in Section 3.3 of that paper is equal to $1/(2\mu)$]. After a time $t = 1/(2\mu)$, the initial wave will have lost 22.5% of its energy and 12.4% will have been gained by the first harmonic in the K modes system, (3.9). For the two-wave problem (3.7), on the other hand, the energy loss is 21.4% and all of it goes to the second harmonic. However, the most important difference between both systems is that in the K modes problem high harmonics gain energy coherently because the value of γ is the same for all triads, building up a front at $\mu t \approx \frac{1}{2}$.

b. Resonance of the first, second, and fourth harmonics

Now we consider the case of a wave *1* that interacts resonantly with the second *2* and fourth *4* harmonics, i.e.,

$$\omega_4 = -2\omega_2 = 4\omega_1, \quad k_4 = -2k_2 = 4k_1,$$
$$n_4, n_2 = \text{odd}. \tag{3.10}$$

One particular example is given by the three R modes $(n_1, n_2, n_4) = (6, 5, 1)$ with $sc = -41/3$. Neglecting (off-resonant) interactions with other components, the evolution of the system is controlled by

$$\left.\begin{aligned} [d_t + i\omega_1]Z_1 &= \gamma\omega_1 Z_2^* Z_1^* \\ [d_t + i\omega_2]Z_2 &= \gamma'\omega_2 Z_4^* Z_2^* + \tfrac{1}{2}\gamma\omega_2 Z_1^{*2} \\ [d_t + i\omega_4]Z_4 &= \tfrac{1}{2}\gamma'\omega_4 Z_2^{*2} \end{aligned}\right\}, \tag{3.11}$$

where γ and γ' are the interaction coefficients of the triads (*1, 1, 2*) and (*2, 2, 4*), respectively (for the R modes just mentioned it is $\gamma c = 0.5889$ and $\gamma' c = 0.3512$; see Table 1).

The system (3.11) does not have the structure of the three-wave problem discussed in R81; it is in fact a limiting case of the five-wave problem, just like system (3.6) is a limiting case of the three-wave problem. The evolution of system (3.11) is constrained by the conservation of total quadratic energy (I-3.13a) and of the value of the Hamiltonian (I-3.14).

The system (3.11) has two equilibrium solutions (constant energies/shifted frequencies) of the form (2.9) corresponding to $\mathrm{Re}(Z_2 Z_1^2) = \mathrm{Re}(Z_4 Z_2^2) = 0$, with

$$\delta = -\gamma' Y_4, \quad 2E_4 = E_2, \quad E_1 = 0, \tag{3.12a}$$

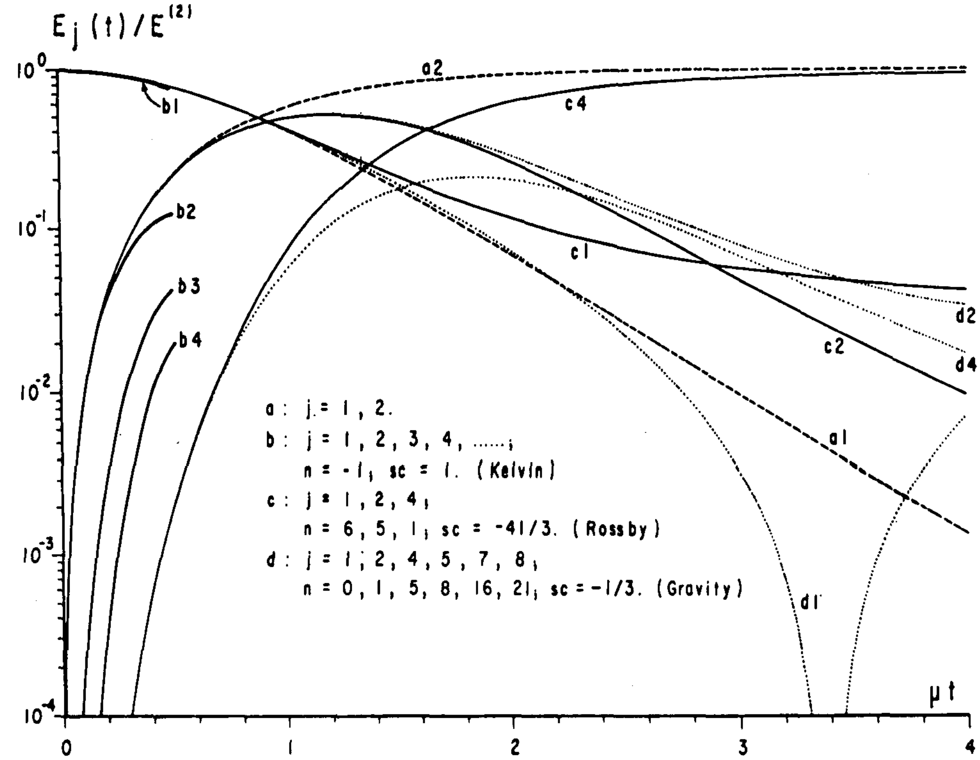

FIG. 3. Evolution of a wave ($j = 1$) that interacts resonantly with harmonics ($j > 1$). The energies of the harmonics with $j < 5$ are shown, relative to the total quadratic energy $E^{(2)}$ (which is a constant of motion). The nonlinear inverse time scale μ is defined in (3.8). (a) Only the fundamental wave and the second harmonic interact resonantly (several examples of pairs are presented in Table 1 and Figs. 1 and 2); the solution is given by (3.7). (b) The Kelvin modes problem (all interacting triads are resonant and with the same interaction coefficient). Neglect of the off-resonant excitation of non-Kelvin modes makes the solution invalid for $|\mu|t > \frac{1}{2}$ (see Ripa, 1982). (c) First, second and fourth harmonics interact resonantly [see equations (3.10) and (3.11)]. (d) A chain of inertia-gravity modes that interact resonantly, truncated down to the first six components; the interaction coefficients are presented in Table 2.

or

$$\delta = -\gamma Y_2, \quad 4\gamma^2 E_4 = \gamma'^2 E_2,$$
$$E_1 = [2 - \gamma'^2/\gamma^2] E_2. \tag{3.12b}$$

The second solution is possible only if $2\gamma^2 \geq \gamma'^2$.

As another extreme example of solution, (3.11) was integrated from an initial condition corresponding to only the first harmonic, *viz.*, $Z_1(0) = E^{1/2}$, $Z_2(0) = Z_4(0) = 0$. It is easy to show that the solution in this case takes the form (I-3.9) with X_a: real, and thus $\mathrm{Im}(Z_2 Z_1^2) = \mathrm{Im}(Z_4 Z_2^2) = 0$.

Evolution of the energies is shown in Fig. 3c. The parameter μ is given by Eq. (3.8); this may be related to the initial velocity field $\mathbf{v}(x, 0)$ (produced by the $n = 6$ Rossby wave) in the form

$$\mu T = 0.6606 \max|s\mathbf{v}|, \tag{3.13}$$

where T and s are the period and slowness of the initial wave. If we make $\gamma' = 0$ in (3.11) we recover the system (3.6); the solution is given by (3.7) [see curve a1 in Fig. 3]: the energy goes rapidly [exponentially, in a time $t = O(\mu^{-1})$] to the second harmonic. With $\gamma' \neq 0$, the energy goes ultimately to the fourth harmonic, but much more slowly [see curve c1 in Fig. 3]; in fact, after a time $t = O(\mu^{-1})$ the system reaches a state of quasi-equilibrium given by

$$\left.\begin{aligned} E_1 &\approx E|\gamma'/\gamma\mu t| = O(t^{-1}) \\ E_2 &\approx E/(2\mu t)^2 = O(t^{-2}) \\ E_4 &\approx E - E_1 = E - O(t^{-1}) \end{aligned}\right\}. \tag{3.14}$$

This type of slow evolution is not possible for the regular three-wave problem (see R81).

c. Resonance of many harmonics

The evolution of the systems (3.6) and (3.11) differ for, say, $\mu t > \frac{1}{2}$ because of the energy gained by the fourth harmonic in the second case. We expect the dynamics of the system of inertia–gravity modes in Fig. 2 to be quite different from these two cases since an infinite number of harmonics are resonantly connected. We also expect the evolution of the G-chain to differ from that of the Kelvin modes problem because the interaction coefficient is not the same for all triads.

A search for equilibrium solutions of the form (3.19) for all components in the chain results in a nonlinear algebraic eigenvalue problem in infinite dimensions (δ is the eigenvalue and Y_a the eigenvector). We suspect that it may have different solutions, as (3.12a) and (3.12b) for the three-harmonics problem (3.11), but we are not aware of a mathematical proof of this statement.

In order to investigate the maximum-energy-exchange solutions, on the other hand, we integrated (I-2.27) truncated to the first six resonant harmonics which have the mixed Rossby–gravity mode as fundamental; the parameters of modes and triads involved in the integration are presented in Table 2. The evolution of the first four energies (starting, as before, with all the energy in the fundamental wave, $n = 0$) is presented in Fig. 3d. The parameter μ is given by Eq. (3.8); this may be related to the initial velocity field $\mathbf{v}(x, 0)$ (produced by the M wave) in the form

$$\mu T = 1.8328 \max|\mathbf{v}|/c, \qquad (3.15)$$

where T is the period of the initial wave. Thus, the ratio of the fundamental wave period to the nonlinear energy transfer time scale (i.e., the time it takes for most of the energy to go to the second harmonic) is approximately equal to the ratio of the maximum particle velocity to the zonal phase speed (for Rossby modes) or to the separation constant c (for inertia-gravity modes).

The curves shown in Fig. 3d are insensitive to truncating the resonant chain to either five or six components, up to $\mu t \approx 3$. The integration was made using a fourth-order Runge-Kutta scheme with $\mu\Delta t = 0.005$. An idea of the errors involved in the numerical integration is illustrated by the change of total quadratic energy which should be constant: its order of magnitude is given by $E^{(2)}(t)/E^{(2)}(0) - 1 \approx 5 \times 10^{-13}\mu t$, at least for $\mu t \leqslant 500$ (100 000 time steps).

Cases a, c, and d in Fig. 3 (with 2, 3, and 6 components respectively), present important differences for $\mu t > 1$. In order to investigate further the dynamical difference between the system of inertia–gravity modes (which has an infinite number of RT, unevenly coupled) and those with two or three harmonics [Eqs. (3.6) or (3.11)]. The former was integrated, starting as in Fig. 3, for different truncations of the resonant chain. Figures 4 and 5 show the evolution, for $\mu t \leqslant 50$, of the envelopes X_a of the G-chain truncated to the first five and six resonant harmonics, respectively, (the corresponding resonant triads are Nos. 1–4, and Nos. 1–6 from Table 2).

With only five harmonics ($M = 5$), Fig. 4, the system rapidly goes into a typical solution of three-wave problem: components ($j = 2$) and ($j = 7$) have no energy, and there is a catalytic energy exchange between the ($j = 1$) and ($j = 5$) harmonics in a time scale determined by the energy of the ($j = 4$) mode which is approximately constant (see R81. It corresponds to the $m' \approx 1$, $m = 1 - m'$ solution). Notice that the extrema of ($j = 1$) correspond to the zeroes of ($j = 5$), and vice versa. (The presence of component *2* is essential to initially transfer energy from *1* to *4*, via the chain generation *1* + *1* → *2*, *2* + *2* → *4*.) Surprisingly, the same asymptotic solution was reached starting from several random initial conditions (constrained to real X_a, with uniform and independent distributions), *i.e.*, this is a stable-limit cycle of the system (Lorenz, 1963).

With one more component ($M = 6$), Fig. 5, the solution is found to be, again quite unexpectedly, aperiodic and unstable.

The trajectories of Figs. 4 and 5 are shown in Fig. 6 in the form of energy partition diagrams for selected triads: a point near, say, the vertex *1* [the side (*4*, *5*)] of the triangle (*1*, *4*, *5*) means that component *1* has much more (less) energy than *4* + *5*. Since total quadratic energy is conserved, a point in $M - 2$ independent triangles, as shown in Fig. 6, gives the instantaneous energy distribution among all components in an M-wave system. That distribution and the signs of the X_a fully determine the state of the system in the cases of Figs. 4 and 5, because the X_a are real.

In a three-wave system, one point in a single triangle describes the energy partition; conservation of pseudomomentum (or action differences) requires the trajectory to be along a straight line. The possible

TABLE 2. Components used in the integrations of Fig. 3d and *ff*. The parameters j and n are the order of harmonic and the meridional quantum number [$k_j^2 = j^2\beta/(12c)$, $\omega_j = -3k_jc$]. The interaction coefficients of each resonant triad involved is presented in the bottom line.

j	n	Triads					
1	0	××		×		×	
2	1	×	××		×		
4	5		×	×			××
5	8			×	×		
7	16				×	×	
8	21					×	×
	γc	0.1656	−0.1079	−0.0894	−0.0614	0.0554	−0.0530

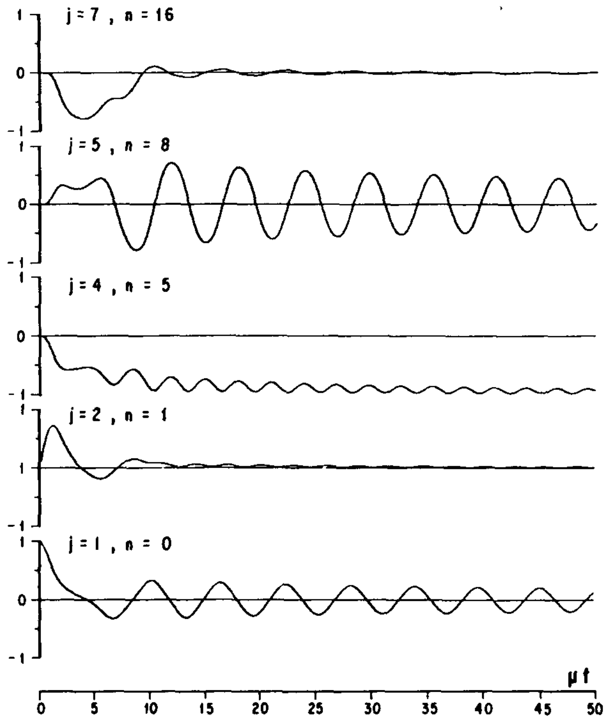

FIG. 4. Evolution of the envelopes of the resonant harmonics which have the mixed Rossby–gravity wave as fundamental (see Table 2). The ordinates are $X_a(t)/[E^{(2)}]^{1/2}$. The system is truncated to five components; the solution is regular and stable.

lines are parallel, the longest emerging from the vertex corresponding to the unstable component (maximum frequency) and the shorter are infinitesimals at the other two vertices (see Fig. 6 in R81). The trajectory at the (*1*, *4*, *5*) triangle in Fig. 6a takes precisely this form, after a short transient, as pointed out in the analysis of Fig. 4 (however, one can appreciate a slow drift towards the vertex *4*). On the (*2*, *5*, *7*) triangle, the system is most often in the vertex *5*, and there is a sudden excursion to the side (*2*, *7*) each time the amplitude of component *5* goes through a zero.

The orbit of the $M = 6$ case, Fig. 6b, on the other hand, looks quite ergodic. The system passes through all available points in the triangles. [This was firmly demonstrated integrating the system from several random initial conditions and for an elapsed time much larger than that of Fig. 6b: the triangles are effectively (and always) filled up by the orbit drawn on paper.] Notice that there is not a minimum of energy, or any other critical parameter, for the onset of chaos in this case: a change of the total energy only modifies the nonlinear time scale through the definition (3.8) (see Section I-3.b).

Similar chaotic behavior has been found, after the pioneering work of Lorenz (1963), in several models with few components (e.g., Lorenz, 1980; Pedlosky and Frenzen, 1980; Pedlosky, 1981; Gent and McWilliams, 1982; in geophysical fluid dynamics problems). However, unlike for the $M = 6$ case, the systems studied by these authors are dissipative, forced and (in the cases of Lorenz, 1963; Pedlosky and Frenzen, 1980; Pedlosky, 1981) linearly unstable. The chaos in these systems is related to the presence of strange attractors, which need dissipation for their existence (for a review article, see Ott, 1981). On the other hand, the solution with $M = 6$ is an example of chaotic evolution in a conservative, unforced, few-components system, and, furthermore, with the phases locked due to the resonance. We believe that this is a case of conservative chaos, first discovered by Hénon and Heiles (1963) in the study of the rotation of a star in an axial symmetric galaxy. (For a review article see Tabor, 1981). Meiss (1981) also found chaotic behavior in a resonant five-wave system when all triads (not just those where the test-wave is present) are allowed to interact; see Section I-3.b.

In order to further stress the dramatic difference between the $M = 5$ and $M = 6$ cases, variance spectra of the X_a are presented in Fig. 7. The spectra were calculated from records of length $\mu T = 163.68$ in order to increase the frequency resolution and, starting from a random initial condition to minimize the effect of transients. Groups of four consecutive bands were averaged in the $M = 6$ case because otherwise

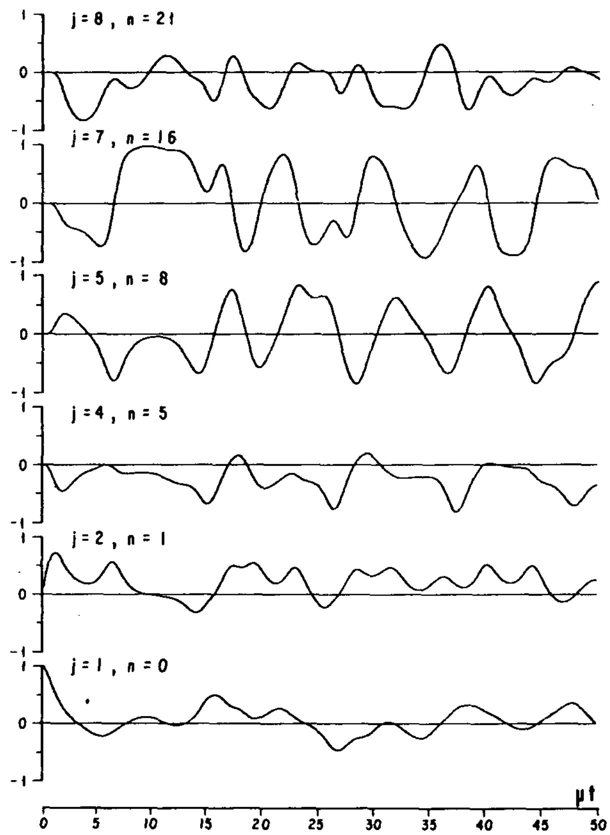

FIG. 5. As in Fig. 4 but with the system truncated to six components; the solution is irregular, aperiodic, and unstable.

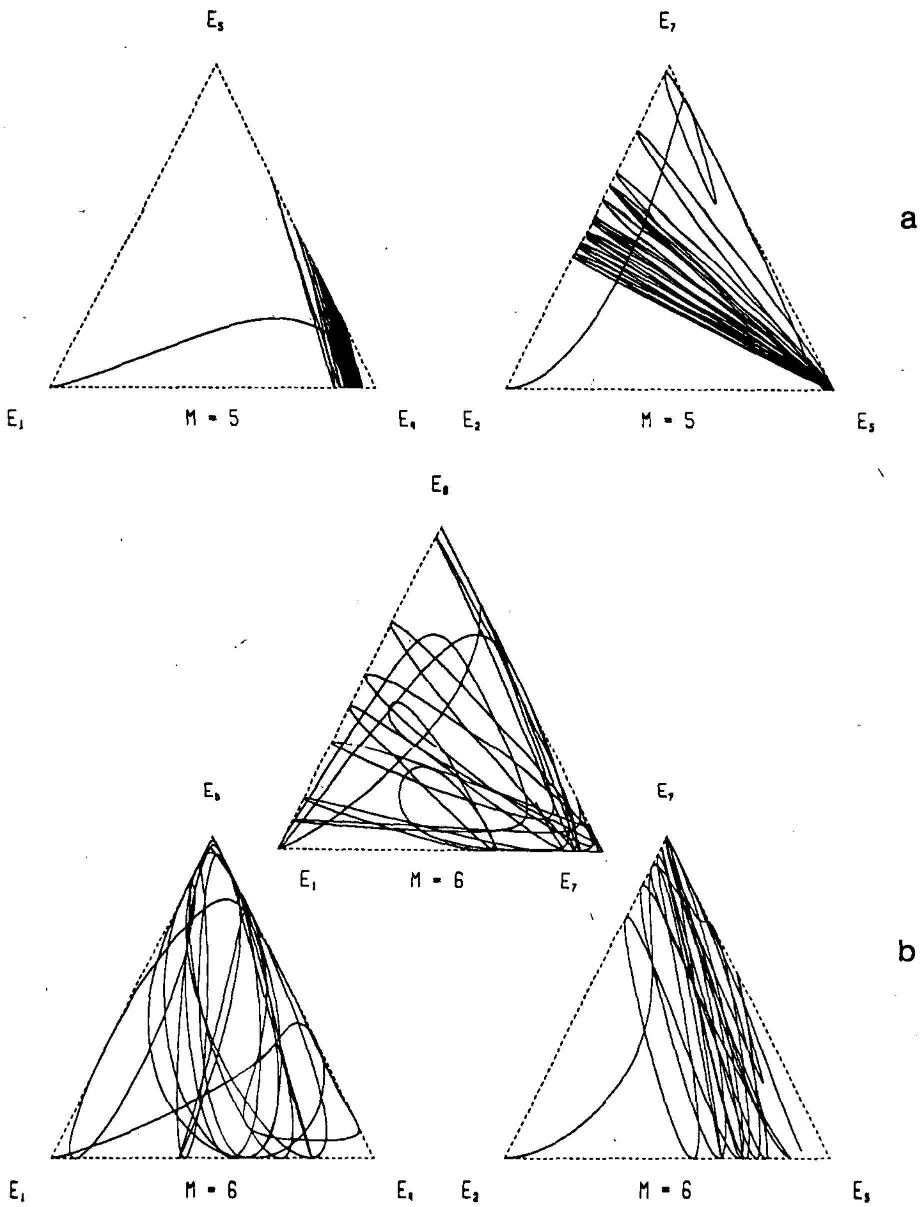

FIG. 6. Trajectories of the systems in (a) Fig. 4 and (b) Fig. 5 in a energy partition diagram for selected triads.

the spectra are too noisy; no band averaging was done for $M = 5$.

The most outstanding feature in the spectra of the $M = 5$ case, Fig. 7a, are the peaks at the frequency $f_1 = 0.183\mu$ cycles, for components *1* and *5*, which correspond to the catalytic energy exchange between these two components, evident in Fig. 4. This nonlinear frequency is indistinguishable from the three-wave prediction, $[\gamma(1, 4, 5)^2 5\omega_1{}^2\langle E_4\rangle]^{1/2} = 0.1838\mu$ cycles. The amplitude of component *4* is modulated at twice this frequency. One can also perceive a peak for *2* at $2f_1$ produced by the generation $1 + 1 \rightarrow 2$, a peak at $3f_1$ for *1* and *5* produced by the interaction of *1* with *2* and *4*, respectively, and a peak at f_1 for *7* produced by the interaction of *2* and *5*.

No significant peaks were ever observed in the spectra of the $M = 6$ case, Fig. 7b.

It may be argued that $\mu t = 50$ is too large an elapsed time for boundary effects to be neglected in the case of the ocean. Moreover, we expect the solution to be quantitatively quite different if more resonant harmonics are included in the integration. Nevertheless, these calculations are done in the spirit of arbitrarily isolating one part of a complex problem, as stated in the Introduction of Part I. The unexpected results of the $M = 5$ and 6 cases justifies this philosophy,

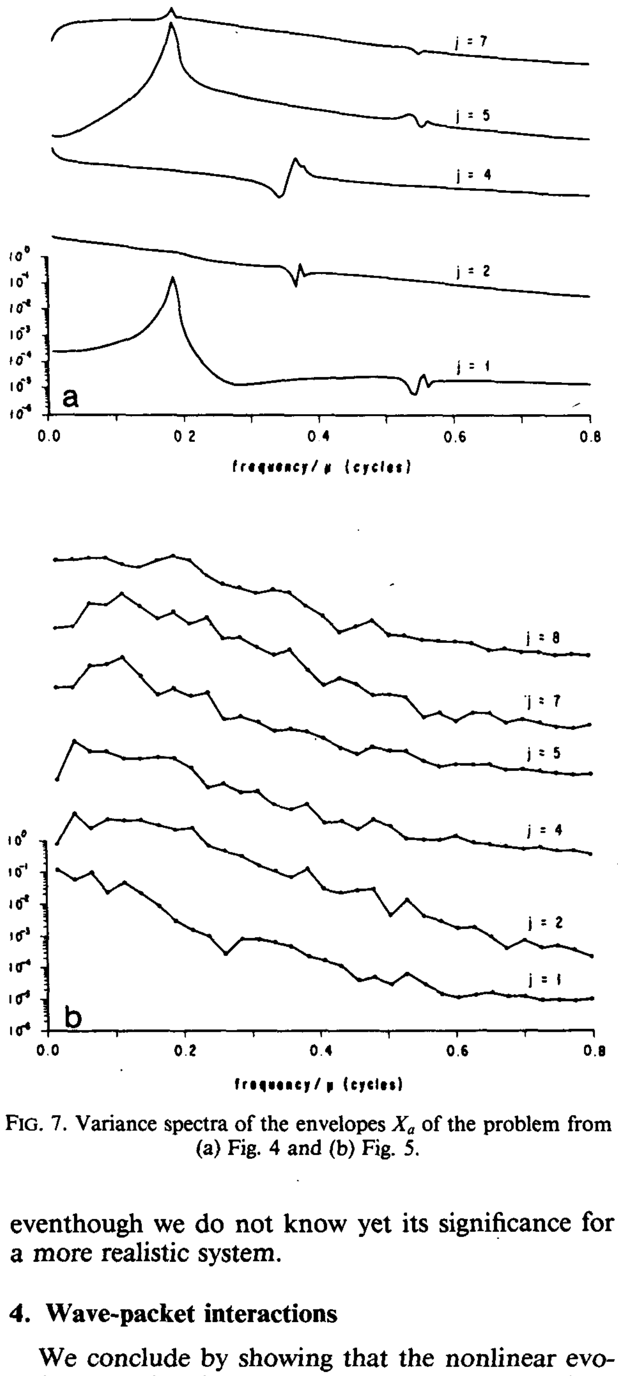

FIG. 7. Variance spectra of the envelopes X_a of the problem from (a) Fig. 4 and (b) Fig. 5.

eventhough we do not know yet its significance for a more realistic system.

4. Wave-packet interactions

We conclude by showing that the nonlinear evolution equation for the wave-packets can be derived directly from that of the Z_a, (I-2.27). Let us define the amplitude of a wave-packet by

$$A_{a_0}(x, t) = i(2\pi)^{-1}(\beta c^{-1})^{1/2} \times \int dk_a Z_a(t) \exp(ik_a x), \quad (4.1)$$

where $a = (a_0, k_a)$ and the factor i is introduced for convenience. This is not the most general, or even useful, definition (e.g., suitable filters may be included in the integrand): it is used here just to illustrate the method; a more detailed description will be presented elsewhere.

We now: 1) Expand the linear and nonlinear coefficients in (I-2.27) at some central wavenumbers from a discrete set (k_{a_0}), in the form

$$\omega_a = \sum_j \omega_{a,j}(k_a - k_{a_0})^j, \quad (4.2a)$$

$$\sigma_a^{bc} = \sum_{ij} \sigma_{a,ij}^{bc}(k_b - k_{b_0})^i(k_c - k_{c_0})^j. \quad (4.2b)$$

Here σ_a^{bc} need not be expanded in k_a, because $k_a = -k_b - k_c$, in virtue of (I-2.25). 2) Multiply (I-2.27) by $\exp(ik_a x)$ $[=\exp(-ik_b x - ik_c x)]$ and integrate in dk_a. 3) Use (4.1) and (4.2), obtaining

$$i\partial_t A_{a_0} = \sum_j \omega_{a,j}A_{a,j} + \tfrac{1}{2} \sum_{bc} \sum_j \sigma_{a,ij}^{bc} A_{b,j}^* A_{c,j}^*, \quad (4.3)$$

where

$$A_{a,j} = (-i\partial_x - k_{a_0})^j A_{a_0}. \quad (4.4)$$

In principle, eq. (4.3) is equivalent to (I-2.27) and thus an exact representation of the original model equations (I-2.1), but the latter involve only a few dynamical fields and are of first order in ∂_x. Clearly, (4.3) represents no improvement over (I-2.1). It was obtained with the purpose of constructing approximations, by a suitable truncation of 1) the set of modes (a_0, k_{a_0}), 2) the expansion (4.2a) of the linear frequencies ω_a, and 3) the expansion (4.2b) of the nonlinear coupling coefficients σ_a^{bc}. The dynamical fields are obtained by expanding the coefficients of the linear eigenfunctions in powers of $(k_a - k_{a_0})$ and replacing (4.1) in (I-2.14). This expansion is also truncated at an order consistent with the approximation done to (4.3).

The advantage of obtaining the approximate equation in this way instead of the customary perturbative multiple time-scale expansion is that the coefficients are but derivatives of known functions; it is easier to calculate them and, more importantly, to improve the first approximation by including extra terms.

One particularly simple example is that of the self-interaction of the Kelvin mode, because the expansion (4.2) is in this case exact at the first order, *viz.*, $\omega_a = k_a c$ and $\sigma_a^{bc} = -(2 + \lambda)(3^{1/2}2\pi^{1/4})^{-1}(k_b + k_c)$. Moreover, $A(x, t)$ is real in this case, and thus (4.3) takes the form of the one-dimensional advection equation $\partial_t A + c\partial_x A + (2 + \lambda)(3^{1/2}2\pi^{1/4})^{-1}A\partial_x A = 0$. This way of obtaining the equation, used in R82, should be contrasted with the strained coordinates perturbation method of Boyd (1980a).

A couple of other examples of use of this method are included next, with the purpose of clarifying it.

a. Self-interaction of long Rossby modes

We consider the self-interaction of a long-Rossby wave, i.e., the central component has parameters k_0

$= \omega_0 = 0$ or $s_0 c = -2n - 1$; see Table I-2 (we drop the subscript a for simplicity, since we are dealing with a single mode). Iterating twice the equation $sc = -2n - 1 - \omega^2(1 - s^2c^2)/\beta c$ (from Table I-1), with $\omega = k/s$ starting at $k = 0$, we get

$$\left.\begin{aligned} \omega &= -\frac{(1-\alpha k^2)kc}{(2n+1)} + O(k^5) \\ \alpha &= \frac{\beta^{-1}c4n(n+1)}{(2n+1)^3} \end{aligned}\right\}. \quad (4.5)$$

The detuning $\omega_a + \omega_b + \omega_c$, of any interacting triad, $k_a + k_b + k_c = 0$, of Rossby waves with the same value of n is then $O(\alpha k_i^3)$. Therefore, using the expression (I-2.33) for the evaluation of the coupling coefficients we get

$$\begin{aligned} \sigma_a^{bc} &= \gamma\omega_a + O(k_i^3) \\ &= -\gamma c(2n+1)^{-1}(k_b + k_c) + O(k_i^3), \quad (4.6) \end{aligned}$$

where γ, which is evaluated at $k_a = k_b = k_c = 0$, vanishes unless n ($=n_a = n_b = n_c$) is odd in virtue of Eq. (I-2.26). The amplitude $A(x, t)$ is again real, to lowest order in k. Therefore, using (4.5) and (4.6) in (4.3), including only Rossby modes of the same value of n, we get

$$\partial_t A - c(2n+1)^{-1}(1 + \alpha\partial_{xx} + \gamma A)\partial_x A = 0, \quad (4.7)$$

which is the Korteweg–deVries equation (see Miles, 1981). For a review of the solutions of this equation, see Miura (1976). Notice that for $n = -1$ it is $\alpha = 0$, and we recover the nonlinear advection equation which controls the self-interaction of Kelvin modes; whereas for n = even, it is $\gamma = 0$ because (I-2.26) is violated. The self-interaction of even-n, long-R modes is governed by the modified Korteweg–deVries equation, with cubic nonlinearity (Boyd, 1980b).

Clarke (1971) and Boyd (1980b) derived this equation in a β-plane model like the one in this paper, for a midlatitude channel and for an unbounded equatorial ocean respectively. These authors make the usual multiple time-scale perturbation expansion with the original model equations [formally, A is $O(\delta^2)$ and is a function of $\delta(x - Ct)$ and $\delta^2 t$, with $\delta \ll 1$]. The point here is that it is easier to derive (4.7) from the evolution equations in phase-space, (I-2.27). As a result, the nonlinear coefficient of the Korteweg–deVries equation is but γ. From the expression (I-2.34) and for the ocean model ($\lambda = 1$) we calculate $(n, \gamma c)$ = (−1, 0.6504937), (1, 1.3327326), (3, 0.8253934), (5, 0.6677558), (7, 0.5807777), (9, 0.5231703), (11, 0.4812273), (13, 0.4488387), (15, 0.4231557).

b. Long–short wave resonance

As a second example, we consider the case of a long-Rossby mode, with label c, interacting with a packet of shorter Rossby or inertia–gravity waves, with label a_0. The resonance condition (I-4.2a) indicates that the group velocity of the packet matches the phase-speed of the long wave, *viz.*,

$$C_{a_0} = -\frac{c}{(2n_c + 1)}, \quad (4.8)$$

where n_c must be odd as required by (I-2.26) and positive (i.e., the long mode cannot be a K, $n_c = -1$, because $C_a < c$ for $n_a \neq -1$). The solution of (4.8) is reduced to finding a root of a third degree polynomial since

$$\left.\begin{aligned} C = d\omega/dk &= \frac{c[1 + sc(4n + 2 + sc)]}{[4n + 2 + sc(3 - s^2c^2)]} \\ C' = d^2\omega/dk^2 &= \frac{\omega[c^2(1 + 2sC) - 3C^2]}{(\omega^2 + \tfrac{1}{2}\beta sc^2)} \end{aligned}\right\}. \quad (4.9)$$

If c is an R mode, it must be $n_a < n_c$, because $C_a > -c/(2n_a + 1)$ for $k_a \neq 0$. There are solutions for any n_a, though, when c is a G; asymptotically it is $s_a c \to -(2n_c + 1)^{-1}$ as $n_a \to \infty$. For large values of n_a and n_c, the interaction occurs mainly at the turning latitude of the long wave, if $n_c < n_a$, or where the local meridional wavelength of the long mode is half that of the short wave, for $n_c > n_a$; cf., Eq. (I-2.43a) with $a \leftrightarrow c$, or (I-2.43b), respectively. Several examples of long–short wave resonance are presented in Table 3.

The expansion (4.2a) takes the form

$$\omega_a \approx \omega_{a_0} + C_{a_0}(k_a - k_{a_0}) + \tfrac{1}{2}C'_{a_0}(k_a - k_{a_0})^2, \quad (4.10a)$$

$$\omega_c \approx C_{a_0}k_c, \quad (4.10b)$$

where C_{a_0} and C'_{a_0} are evaluated at $k_a = k_{a_0}$. The triads that build up the nonlinear term have wavenumbers of the form $(k_a, k_b, k_c) = (k_{a_0} + \delta_a, -k_{a_0} + \delta_b, -\delta_b - \delta_c)$. The detuning is then $\tfrac{1}{2}C'_{a_0}(\delta_a^2 - \delta_b^2)$, and therefore the dominant term for the coupling coefficients is given by the first term in (I-2.33), namely

$$\begin{aligned} \sigma_a^{bc} &\approx \gamma\omega_{a_0}, \\ \sigma_c^{ab} &\approx -\gamma C_{a_0}(k_a + k_b), \quad (4.11) \end{aligned}$$

where γ [$=\gamma(a_0, a_0^*, c)$] is evaluated at $k_a = k_{a_0}$, $k_b = -k_{a_0}$ and $k_c = 0$. Using (4.10) and (4.11) in (4.3) we then get

$$\left.\begin{aligned} &[\partial_t + C_{a_0}\partial_x - \tfrac{1}{2}iC'_{a_0}\partial_{xx} - i\gamma\omega_{a_0}A_c]A_a = 0 \\ &[\partial_t + C_{a_0}\partial_x]A_c = \gamma C_{a_0}\partial_x(|A_a|^2) \end{aligned}\right\}. \quad (4.12)$$

Once again, there are no *ad hoc* nonlinear coefficients to evaluate; rather, these are given by $\gamma\omega_{a_0}$ and γC_{a_0}. Boyd (1983) has obtained (4.12) making a multiple time-scale perturbation expansion with the original model equations.

Ma and Redekopp (1979) have found many inter-

TABLE 3. Examples of a long, almost non-dispersive, Roosby mode that interacts with a shorter inertia-gravity ($-1 < sc$) or Rossby ($sc < -3$) wave. The phase-speed of the long wave, $-c/(2n + 1)$, matches the group velocity C of the short one; $C' = dC/dk$.

Long	Short					
n	n	sc	$\omega R/c$	kR	C'/Rc	γc
1	1	−0.4641	2.5425	−1.1800	0.4136	0.1281
1	2	−0.4114	3.3237	−1.3675	0.2941	0.1063
1	3	−0.3892	3.9474	−1.5363	0.2406	0.0988
1	4	−0.3768	4.4834	−1.6895	0.2086	0.0924
1	5	−0.3690	4.9611	−1.8305	0.1868	0.0864
1	∞	−0.3333				
3	2	−5.5662	0.1943	−1.0817	−0.0889	0.6265
3	1	−3.8392	0.3495	−1.3419	−0.1914	−0.7688
3	1	−0.3039	2.4374	−0.7407	0.4435	0.0626
3	2	−0.2385	3.1776	−0.7579	0.3245	0.0045
3	3	−0.2111	3.7698	−0.7958	0.2690	0.0140
3	4	−0.1959	4.2791	−0.8385	0.2350	0.0282
3	5	−0.1863	4.7334	−0.8818	0.2113	0.0360
3	∞	−0.1429				
5	4	−9.6073	0.1153	−1.1081	−0.0327	0.5565
5	3	−8.0703	0.1827	−1.4744	−0.0555	−0.0556
5	2	−6.3293	0.2609	−1.6513	−0.0886	−0.9585
5	1	−4.2601	0.3834	−1.6331	−0.1631	0.7294
5	1	−0.2572	2.4237	−0.6233	0.4410	0.0153
5	2	−0.1894	3.1590	−0.5983	0.3258	0.0557
5	3	−0.1611	3.7473	−0.6036	0.2712	0.0043
5	4	−0.1455	4.2535	−0.6187	0.2374	−0.0052
5	5	−0.1355	4.7048	−0.6376	0.2138	0.0045
5	∞	−0.0909				

esting solutions of (4.12); side-band instability, phase-jump and envelope solitons, envelope-hole solitary waves, interacting solitons, and bound-states. It is easy to generalize (4.12) including more than one short wave that interacts with the same long one (see Table III); Ma (1981) has found several solutions for the case with two short waves.

5. Summary

There are isolated cases of a double resonant triad, namely, a pair of resonant triads with two components in common making a total of four independent waves. A few solutions for the evolution of such a system are calculated neglecting (off-resonant) interactions with other modes. The main results are:

1) Two of the four waves belong to only one triad; their stability is that dictated by the three-wave problem criteria (i.e., the component with intermediate slowness is unstable and the other two are stable; see Ripa, 1981). One wave is one of the stable components of each triad (if decoupled from the other) and it is also stable when both triads are taking together. Finally, the remaining wave is the unstable component of one triad and one of the stable components of the other; its stability depends on which triad is stronger.

2) There are equilibrium solutions, i.e., with constant energies and constant shifted frequencies. For the particular one with equal relative shifts, (2.9), there is no equi-partition of energy (as in the three-wave problem) and the shift is larger than the one obtained when only one triad is considered, (2.11).

3) The solutions with unshifted frequencies (and, presumably, maximum energy exchange) are completely integrable. In particular, a subclass of these problems can be reduced to the maximum-energy-exchange solution for the three-wave problem, see Eq. (2.13).

There are equatorial waves that are resonantly coupled with several harmonics; indeed, with an infinite number of them for inertia–gravity waves. This is not like the Kelvin mode problem (Ripa, 1982), though, because the different triads interact with uneven strength. The difference in the evolution of the system in the cases of a wave interacting with (a) only the second harmonic, (c) the second and fourth harmonics, (d) many harmonics, and (b) the Kelvin modes case (all interacting triads resonate and with the same strength) may be appreciated in Fig. 3. An important qualitative difference is found in the behavior of the chain of resonant harmonics, truncated to either the first five or six components. With five modes (Fig. 4) the energy rapidly goes into three of them, the evolution is regular, quasi-periodic and stable. On the other hand, with six modes (Fig. 5) the evolution is irregular, aperiodic and unstable. This is quite an unexpected result, considering that the system is unforced, energy-conserving and with the phases locked due to the resonance.

The interaction coefficient γ is the only nonlinear parameter in the equations of long-short wave resonance and Korteweg–deVries. These equations govern the interaction of a packet of Rossby and inertia–gravity waves with a long Rossby mode of the same group velocity and the self-interaction of long Rossby waves, respectively. The derivation of these equations from the evolution equations in phase-space is straightforward, i.e., there is no need of the usual (and cumbersome) perturbation expansion in physical space.

Acknowledgments. I am grateful to S. Ramos for drafting the figures. Part of this work was done while working at the Pacific Marine Environmental Laboratory, in Seattle. This is a contribution to the Equatorial Pacific Ocean Climatic Studies (EPOCS).

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