Wave Forces on Offshore Structures: Nonlinear Wave Diffraction by Large Cylinders

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ABSTRACT

A nonlinear theory of wave diffraction is presented and used to evaluate the forces exerted on a cylinder of large diameter. A perturbation technique has been used to solve the problem with the inclusion of second-order terms. Analytical solutions are expressed in the form of an integral, and a numerical technique is applied to solve the resulting integral equation with satisfaction of all necessary hydrodynamic boundary conditions including surface and radiation boundary conditions. Predictions of the present analysis are compared with the experimental data. The comparison shows excellent agreement

1. Introduction

In recent years there has been considerable interest in the study of the hydrodynamic forces that ocean waves may exert on off-shore structures that have the form of circular cylinders, piles, square caissons, and so forth. One motivation for such studies has arisen from the need to build solid off-shore structures in connection with oil and gas recovery and production. Accurate prediction of the wave loadings on the structures is extremely important for design purposes, but the usual theory is deficient because it neglects the effect of nonlinear terms.

The present data is concerned with the forces that water waves may exert on vertical circular cylinders. It is necessary to distinguish between small and large diameter cylinders, the diameter being compared to the characteristic wavelength and amplitude of the wave.

Morison *et al.* (1950) gave an empirical equation for determination of the force on a circular cylinder in terms of an inertia force and a viscous drag force under the assumption that the incident wave field is not significantly affected by the presence of the cylinder. While this assumption may be satisfied approximately in the instance of a cylinder of small diameter, it ceases to be true as the diameter increases in relation to the incident wavelength. The Morison equation is then no longer applicable, and diffraction theory must be used.

In application of diffraction theory the cylinder is assumed to be smooth. Thus the viscous drag forces are negligible and the inertial forces are predominant.

A diffraction theory for calculation of wave loads on a vertical cylinder was formulated by MacCamy and Fuchs (1954). The cylinder was envisaged as extending from a horizontal sea bed to above the water surface. This diffraction theory has been used by a number of investigators, including Mogridge and Jamieson (1976), to predict the wave loads on large submerged cylinders. However, in all instances the analyses have been restricted to the inclusion of linear wave theory only. This is a serious restriction since in practice the character of ocean waves is generally a nonlinear function of the disturbance.

It is to be expected that the correlation between experimental measurements and theoretical predictions will be poor unless the theory includes nonlinear effects. Accordingly a number of investigators have directed their attention toward consideration of a nonlinear theory.

Chakrabarti (1975, 1977, 1978) used Stokes' fifth-order wave theory to derive an expression for the wave forces on a circular cylinder, but the solution failed to satisfy the kinematic free-surface boundary condition in the vicinity of the cylinder. Yamaguchi and Tsuchiya (1974) employed a perturbation theory to derive a velocity potential that included second-order terms. They expressed the wave force in the form of a series, and obtained numerical results that appeared to be in reasonable agreement with experimental results. However, Chakrabarti (1977) concluded that on the basis of his theory and a comparison with experimental results, the solutions of Yamaguchi and Tsuchiya lead to considerable overestimates of the forces involved.

Raman and Venkatanarasaiah (1976) and Raman *et al.* (1975, 1977) also used a perturbation theory to obtain a nonlinear analysis of the propagation of a water wave. They derived an implicit solution by use of generalized Fourier transforms of first- and second-order perturbation equations. The final solutions were obtained for a cylinder of relatively small diameter, for which drag forces are likely to be significant. The solutions did not satisfy the free-surface boundary conditions (Chakrabarti, 1978).

Garrison (1979) considered a second-order theory for the interaction of a regular wave with a fixed object in water of finite depth. The resulting boundary-value problem was then solved numerically.

Isaacson (1977) discussed the physical aspects of the problem and concluded that a nonlinear theory does not exist. However, his assertions appear to be lacking in rigorous mathematical justification. Recently, Hunt and Baddour (1980) have shown that Isaacson's apparent inconsistency follows from a non-analytic property of the solution at the intersection of the cylinder and the free surface, and concluded that this does not invalidate the Stokes' expansion method. Miloh (1980) indicated that Wehausen (1980) has also used a similar argument to refute the Isaacson assertion.

There appears to be a need for a nonlinear theory that satisfies the appropriate boundary conditions and for which numerical results may be obtained and compared with measured values. The present study deals with the nonlinear wave loads on large vertical cylinders that extend from a horizontal sea bed to above the free surface of the water. A perturbation technique is used to solve the nonlinear diffraction problem with inclusion of second-order terms. All solutions are expressed in terms of analytic functions of closed form.

Numerical results based on the predictions of the present analysis are compared with experimental data. It is found that there is excellent agreement between the predicted and measured values.

2. Mathematical formulation

Consider a rigid vertical cylinder of radius a acted on by a regular surface wave of height H that progresses in the direction of the positive x-axis as shown in Fig. 1. In the absence of the wave the water depth is h and in the presence of the wave the free surface height is η above the mean surface level. It is assumed that the fluid is incompressible and that the motion is irrotational. The governing equations of motion are well known (Phillips, 1977; Longuet-Higgins and Cokelet, 1976).

The fluid motion satisfies the equations

$$v_r = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad v_z = \frac{\partial \phi}{\partial z}, \tag{1}$$

where

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2}$$

within the region

$$a \leqslant r < \infty, \quad -h \leqslant z \leqslant \eta, \quad -\pi \leqslant \theta \leqslant \pi.$$

The boundary conditions that correspond to Fig. 1 are as follows: If the equation of the free surface is

$$z = \eta(r, \theta, t),$$

then the dynamic and kinematic boundary conditions are the following:

Dynamic boundary condition

For $z = \eta$ and $r \geqslant a$,

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0, \tag{3}$$

in which g is the gravitational acceleration, and any

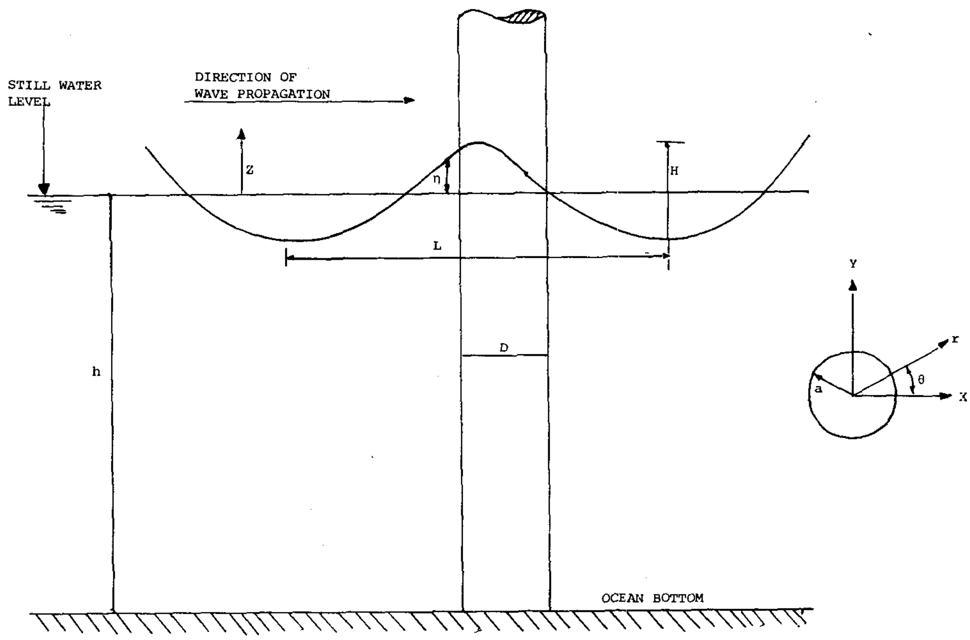

FIG. 1. Definition sketch of wave forces on a vertical circular cylinder.

arbitrary function of time in Eq. (3) is supposed to be included in ϕ.

Kinematic boundary condition

For $z = \eta$ and $r \geqslant a$,

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\frac{\partial \eta}{\partial \theta} = \frac{\partial \phi}{\partial z}. \tag{4}$$

Boundary condition on the body surface

For $r = a$ and $-h \leqslant z \leqslant \eta$,

$$\partial\phi/\partial r = 0. \tag{5}$$

Boundary condition on sea bed

For $z = -h$,

$$\partial\phi/\partial z = 0. \tag{6}$$

Radiation boundary condition

$$\lim_{kr\to\infty} (kr)^{1/2}\left(\frac{\partial}{\partial r} \pm ik\right)(\phi - \phi^{(I)}) = 0, \tag{7}$$

where k $(=2\pi/L)$ is the wavenumber of the wave of wavelength L, and the velocity potential ϕ has been expressed in the form $\phi = \phi^{(I)} + \phi^{(R)}$ in which $\phi^{(I)}$ and $\phi^{(R)}$ denote incident and reflected potentials, respectively.

It has been found in an earlier study (Rahman and Moes, 1979) that the radiation condition can be satisfied by the Hankel function of first and second kind provided the negative sign in Eq. (7) is used for the first kind and the positive sign for the second kind.

3. Series solution in terms of parameter ϵ

Let $\phi_1(r, \theta, z, t)$ and $\eta_1(r, \theta, t)$ denote the solution of Eqs. (2)–(7) subject to the neglect of second-order terms such as $(\partial\phi/\partial r)^2$ and $(\partial\phi/\partial r)(\partial\eta/\partial t)$. Then if ϵ denotes a constant, the quantities

$$\epsilon\phi_1(r, \theta, z, t) \quad \text{and} \quad \epsilon\eta_1(r, \theta, t)$$

are solutions of (2)–(7) subject to neglect of terms that involve ϵ^2. Furthermore, it may be seen that the general solution of Eqs. (2)–(7) may be expressed in the form

$$\phi = \sum_{n=1}^{\infty} \epsilon^n \phi_n, \tag{8}$$

$$\eta = \sum_{n=1}^{\infty} \epsilon^n \eta_n, \tag{9}$$

where ϵ is regarded as an additional independent variable. By a nondimensional analysis of the governing equations, it can be easily shown that ϵ is of $O(kH/2)$, where H is the incident wave amplitude. In a similar manner, the incident potential, $\phi^{(I)}$, and the reflected potential, $\phi^{(R)}$, may also be expanded in a formal perturbation series.

For any given value of N the sums of only the first N terms of the series in (8) and (9) may be regarded as the Nth-order approximation to the solutions of Eqs. (2)–(7). The Nth-order approximation is the solution subject to neglect of terms ϵ^m where $m > N$.

Carrying out a Taylor-series expansion of the nonlinear boundary conditions (3) and (4) about $z = 0$, and substitution of the series (8) and (9) into (2)–(7), and comparison of terms that contain the first power of ϵ, leads to the following equations for a first-order solution:

$$\frac{\partial^2\phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi_1}{\partial\theta^2} + \frac{\partial^2\phi_1}{\partial z^2} = 0, \tag{10}$$

$$g\eta_1 + \frac{\partial\phi_1}{\partial t} = 0, \quad \text{for} \quad z = 0, \quad r \geqslant a, \tag{11}$$

$$\frac{\partial\eta_1}{\partial t} - \frac{\partial\phi_1}{\partial z} = 0, \quad \text{for} \quad z = 0, \quad r \geqslant a, \tag{12}$$

$$\frac{\partial\phi_1}{\partial r} = 0, \quad \text{for} \quad r = a, \tag{13}$$

$$\frac{\partial\phi_1}{\partial z} = 0, \quad \text{for} \quad z = -h, \tag{14}$$

$$\lim_{kr\to\infty} (kr)^{1/2}\left[\left(\frac{\partial}{\partial r} \pm ik\right)(\phi_1 - \phi_1^{(I)})\right] = 0, \tag{15}$$

where $\phi_1^{(I)}$ is the first-order term of incident potential, $\phi^{(I)}$.

Likewise, substitution of (8) and (9) into (2)–(7), and comparison of terms that contain the second power of ϵ, leads to the following equations that express the second-order terms ϕ_2 and η_2 as functions of ϕ_1 and η_1:

$$\frac{\partial^2\phi_2}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi_2}{\partial\theta^2} + \frac{\partial^2\phi_2}{\partial z^2} = 0, \tag{16}$$

$$g\eta_2 + \frac{\partial\phi_2}{\partial t} + \eta_1\frac{\partial^2\phi_1}{\partial t\partial z} + \frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi_1}{\partial\theta}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2\right] = 0 \quad \text{for} \quad z = 0, \quad r \geqslant a, \tag{17}$$

$$\frac{\partial\eta_2}{\partial t} + \frac{\partial\phi_1}{\partial r}\frac{\partial\eta_1}{\partial r} + \frac{1}{r^2}\frac{\partial\phi_1}{\partial\theta}\frac{\partial\eta_1}{\partial\theta} - \left(\frac{\partial\phi_2}{\partial z} + \eta_1\frac{\partial^2\phi_1}{\partial z^2}\right) = 0 \quad \text{for} \quad z = 0, \quad r \geqslant a, \tag{18}$$

$$\partial\phi_2/\partial r = 0, \quad \text{for} \quad r = a, \tag{19}$$

$$\partial\phi_2/\partial z = 0, \quad \text{for} \quad z = -h, \tag{20}$$

with the radiation condition

$$\lim_{k_2 r\to\infty} (k_2 r)^{1/2}\left(\frac{\partial}{\partial r} \pm ik_2\right)[\phi_2 - \phi_2^{(I)}] = 0, \tag{21}$$

where k_2 is the wavenumber corresponding to second-order wave theory, and $\phi_2^{(I)}$ is the second-order term of the incident potential $\phi^{(I)}$.

Similar substitution, and comparison of terms in ϵ^n, leads to equations that express ϕ_n and η_n in terms of ϕ_m and η_m where $m < n$.

The quantity η_1 may be eliminated from Eqs. (11) and (12) to yield

$$\frac{\partial^2\phi_1}{\partial t^2} + g\frac{\partial\phi_1}{\partial z} = 0, \quad \text{for} \quad z = 0, \quad r \geq a. \tag{22}$$

Similarly η_2 may be eliminated from Eqs. (17) and (18) to give

$$\frac{\partial^2\phi_2}{\partial t^2} + g\frac{\partial\phi_2}{\partial z} = -\eta_1\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_1}{\partial t^2} + g\frac{\partial\phi_1}{\partial z}\right) - \frac{\partial}{\partial t}\left[\left(\frac{\partial\phi_1}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi_1}{\partial\theta}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2\right] \quad \text{for} \quad z = 0, \quad r \geq a. \tag{23}$$

The pressure $P(r, \theta, z, t)$ may be determined from Bernoulli's equation

$$\frac{P}{\rho} + gz + \frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = 0, \tag{24}$$

and substitution for ϕ as a series in powers of ϵ indicates that P may be expressed as

$$P = -\rho g z - \epsilon\rho\frac{\partial\phi_1}{\partial t} - \epsilon^2\rho\left\{\frac{\partial\phi_2}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial r}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi_1}{\partial\theta}\right)^2\right]\right\} + O(\epsilon^3). \tag{25}$$

The total horizontal force is

$$F_x = \int_0^{2\pi}\int_{-h}^{\eta}[P]_{r=a}(-a\cos\theta)dzd\theta, \tag{26}$$

where η is given by the perturbation expansion, Eq. (9).

Substituting Eq. (25) into Eq. (26) and writing the z-integral as the sum of $\int_{-h}^0 + \int_0^\eta$ we obtain

$$F_x = a\rho\int_0^{2\pi}\left\{\int_{-h}^0\left[gz + \epsilon\frac{\partial\phi_1}{\partial t} + \epsilon^2\left\{\frac{\partial\phi_2}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi_1}{\partial z}\right)^2 + \frac{1}{2a^2}\left(\frac{\partial\phi_1}{\partial\theta}\right)^2\right\}\right]_{r=a}dz + \int_0^{\epsilon\eta_1+\epsilon^2\eta_2}\left[gz + \epsilon\frac{\partial\phi_1}{\partial t} + \epsilon^2\left\{\frac{\partial\phi_2}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi_1}{\partial z}\right)^2 + \frac{1}{2a^2}\left(\frac{\partial\phi_1}{\partial\theta}\right)^2\right\}\right]_{r=a}dz\right\}\cos\theta d\theta. \tag{27}$$

It may be noted that condition (13) is used to obtain (27).

It is clear from Eq. (27) that the integral of gz, the hydrostatic term, up to $z = 0$ contains no $\cos\theta$ term and may be ignored. Also, the upper limit of the z-integral of the second-order terms may be taken at $z = 0$ instead of $z = \epsilon\eta_1 + \epsilon^2\eta_2$ which would only introduce higher-order terms ϵ^3, etc. Thus F_x may be written as

$$F_x = \epsilon F_{x1} + \epsilon^2 F_{x2}, \tag{28}$$

where the first-order contribution is

$$\epsilon F_{x1} = a\rho\int_0^{2\pi}\left\{\int_{-h}^0\left(\epsilon\frac{\partial\phi_1}{\partial t}\right)_{r=a}dz\right\}\cos\theta d\theta \tag{29}$$

and the second-order contribution is

$$\epsilon^2 F_{x2} = a\rho\int_0^{2\pi}\left\{\int_0^{\epsilon\eta_1}\left(gz + \epsilon\frac{\partial\phi_1}{\partial t}\right)_{r=a}dz + \epsilon^2\int_{-h}^0\left[\frac{\partial\phi_2}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi_1}{\partial z}\right)^2 + \frac{1}{2a^2}\left(\frac{\partial\phi_1}{\partial\theta}\right)^2\right]_{r=a}dz\right\}\cos\theta d\theta. \tag{30}$$

4. First-order wave theory

The linear diffraction theory potential for an incoming wave train in the presence of a vertical cylinder was given by Havelock (1940) for deep-water waves, and so the solution of Eqs. (10)–(15), applicable for arbitrary depth h, may be expressed in the following complex form (MacCamy and Fuchs, 1954)

$$\phi_1 = \frac{\sigma\cosh k(z+h)}{k^2\sinh kh}e^{-i\sigma t}\sum_{m=0}^{\infty}\delta_m i^m A_m(kr)\cos m\theta, \tag{31}$$

$$\eta_1 = \frac{e^{-i\sigma t}}{k}\sum_{m=0}^{\infty}\delta_m i^{m+1}A_m(kr)\cos m\theta, \tag{32}$$

where

$$A_m(kr) = J_m(kr) - \frac{J'_m(ka)}{H_m^{(1)\prime}(ka)}H_m^{(1)}(kr), \tag{33}$$

$$\delta_m = \begin{cases} 1, & \text{if} \quad m = 0 \\ 2, & \text{if} \quad m \neq 0 \end{cases} \tag{34}$$

and σ is given by the dispersion relation

$$\sigma^2 = gk\tanh kh. \tag{35}$$

Here, $H_m^{(1)}$ is the mth-order Hankel function of the first kind, defined by

$$H_m^{(1)}(kr) = J_m(kr) + iY_m(kr), \tag{36}$$

in which $J_m(kr)$ and $Y_m(kr)$ are Bessel functions of the first and second kind, respectively. The expression $J'_m(ka)$ denotes the derivative of $J_m(ka)$ with respect to ka.

The expression (32) represents the complex form of a plane wave of amplitude k^{-1} travelling in the x direction, and represented by the terms whose radial dependence is described by $J_m(kr)$, together with a reflected component represented by the terms whose radial dependence is described by $H_m^{(1)}(kr)$. The first order term $\epsilon\phi_1$ will include the complex form of an incident plane wave of amplitude ϵ.

5. Second-order theory

With ϕ_1 and η_1 expressed in the forms (31) and (32), Eq. (23) takes the following form for $z = 0$ and $r \geqslant a$:

$$\frac{\partial^2\phi_2}{\partial t^2} + g\frac{\partial\phi_2}{\partial z} = \frac{g\sigma e^{-2i\sigma t}}{2k}\sum_{m=0}^{\infty} B_m(kr)\cos m\theta, \quad (37)$$

where

$$\sum_{m=0}^{\infty} B_m(kr)\cos m\theta = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \delta_m\delta_n i^{m+n+1}A_{mn} \times[\cos(m+n)\theta + \cos(m-n)\theta] + \frac{2\coth kh}{k^2r^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_m\delta_n i^{m+n+1}A_mA_n(mn) \times[\cos(m-n)\theta - \cos(m+n)\theta], \quad (38)$$

in which A_{mn} is defined as

$$A_{mn} = (3\tanh kh - \coth kh)A_mA_n + 2\coth kh A'_mA'_n. \quad (39)$$

In Eq. (39) a prime denotes differentiation with respect to the argument kr.

The form of (37) suggests that the general solution of Eqs. (16), (19), and (20) may be expressed in the form

$$\phi_2 = \frac{\sigma e^{-2i\sigma t}}{2k^2}\sum_{m=0}^{\infty}\left[\int_0^{\infty} D_m(k_2)A_m(k_2r) \times \cosh k_2(z+h)dk_2\right]\cos m\theta. \quad (40)$$

In the above equation k_2 is the wavenumber of a second-order wave. It is to be noted that k_2 does not assume discrete values but is a continuous variable. Thus the function D depends on both the discrete variable m and the continuous wave number k_2 which can assume any real value in the range $(0, \infty)$.

Substitution of (40) into (37) leads to the following relation between D_m and B_m:

$$\frac{1}{k}\int_0^{\infty}[k_2\sinh k_2h - 4k\tanh kh\cosh k_2h] \times A_m(k_2r)D_m(k_2)dk_2 = B_m(kr), \quad (41)$$

where $B_m(r)$ may be obtained by equating similar terms of the Fourier series in (38).

The expression for the second-order surface-elevation term η_2 may be found by writing (17) in the form

$$\eta_2 = -\frac{1}{g}\left[\frac{\partial\phi_2}{\partial t} + \eta_1\frac{\partial^2\phi_1}{\partial t\partial z} + \frac{1}{2}\left\{\left(\frac{\partial\phi_1}{\partial r}\right)^2 + \left(\frac{\partial\phi_1}{r\partial\theta}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2\right\}\right] \quad (42)$$

for $r \geqslant a$, and substituting the previously determined expressions for η_1, ϕ_1, and ϕ_2 when $z = 0$.

In order to compute the total horizontal force on the cylinder as given by (28), with use of (29) and (30), it is necessary to solve (41) for the case of $m = 1$.

A tedious, but straightforward, algebraic calculation yields

$$B_1(r) = 8\sum_{m=0}^{\infty}(-1)^{m+1} \times\left[A_{m,m+1} + \frac{2\coth kh}{k^2r^2}m(m+1)A_mA_{m+1}\right]. \quad (43)$$

It is to be noted that $B_1(r)$ is a complex quantity.

Since Eqs. (10)–(15) are linear in ϕ_1 they are also satisfied by $\frac{1}{2}(\phi_1 + \phi_1^*)$ where ϕ_1^* is the complex conjugate of ϕ_1. Then $\frac{1}{2}(\phi_1 + \phi_1^*)$ is real and represents a physical solution of Eqs. (10)–(15). Unfortunately, since Eq. (23) is nonlinear in ϕ_1, the expression $\frac{1}{2}(\phi_2 + \phi_2^*)$, with ϕ_2 according to (40), does not satisfy Eq. (23) with ϕ_1 replaced by $\frac{1}{2}(\phi_1 + \phi_1^*)$. However, the procedure of the previous section may be modified as described in the following section in order to obtain a real solution for ϕ_2 that corresponds to the first order solution $\frac{1}{2}(\phi_1 + \phi_1^*)$. The following section is thus concerned with finding a real solution to the physical problem.

6. Physical solution of second-order theory

After determining a complex solution ϕ_1 in the form (31), a real solution $\frac{1}{2}(\phi_1 + \phi_1^*)$ may be expressed in the form

$$\phi_1 = \frac{\sigma\cosh k(z+h)}{2k^2\sinh kh} \times\sum_{m=-\infty}^{\infty}\delta_m i^m e^{-i\sigma_m t}A_m(kr)\cos m\theta, \quad (44)$$

where δ_m is defined by (34), and

$$\sigma_m = \begin{cases}\sigma, & \text{if } m \geqslant 0_+ \\ -\sigma, & \text{if } m \leqslant 0_-.\end{cases} \quad (45)$$

The function $A_m(kr)$ is defined by (33) for positive values of m, and by the relation

$$A_{-m}(kr) = A_m^*$$

for negative values of m where A_m^* is the complex conjugate of A_m. The summation in (44) includes both $m = 0_+$ and $m = 0_-$ in order to include terms $e^{-i\sigma t}A_0(kr)$ and $e^{i\sigma t}A_0^*(kr)$. The corresponding real solution for η_1 is given by

$$\eta_1 = \frac{1}{2k}\sum_{m=-\infty}^{\infty}\delta_m s_m i^{m+1}e^{-i\sigma_m t}A_m(kr)\cos m\theta, \quad (46)$$

where

$$s_m = \begin{cases} 1, & \text{if } m \geqslant 0_+ \\ -1, & \text{if } m \leqslant 0_-. \end{cases} \quad (47)$$

Eq. (23) then assumes the following form which is analogous to (37):

$$\frac{\partial^2\phi_2}{\partial t^2} + g\frac{\partial\phi_2}{\partial z} = \frac{g\sigma}{4k}\{e^{-2i\sigma t}\sum_{m=0}^{\infty}B_m(kr)\cos m\theta + e^{2i\sigma t}\sum_{m=0}^{\infty}B_m^*(kr)\cos m\theta\}, \quad (48)$$

where

$$e^{-2i\sigma t}\sum_{m=0}^{\infty}B_m(kr)\cos m\theta + e^{2i\sigma t}\sum_{m=0}^{\infty}B_m^*(kr)\cos m\theta$$
$$= \frac{1}{4}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta_m\delta_n i^{m+n+1}A_{mn}e^{-(\sigma_m+\sigma_n)t}[\cos(m+n)\theta + \cos(m-n)\theta] + \frac{1}{4}\frac{\coth kh}{(kr)^2}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\delta_m\delta_n i^{m+n+1}$$
$$\times\left(\frac{\sigma_m+\sigma_n}{\sigma}\right)mne^{-i(\sigma_m+\sigma_n)t}A_mA_n \times[\cos(m-n)\theta - \cos(m+n)\theta], \quad (49)$$

and in which A_{mn} denotes

$$A_{mn} = \left[s_m(\tanh kh - \coth kh) + \left(\frac{\sigma_m+\sigma_n}{\sigma}\right)\tanh kh\right] \times A_mA_n + \left(\frac{\sigma_m+\sigma_n}{\sigma}\right)\coth kh A'_mA'_n. \quad (50)$$

It may be noted that for $m, n \geqslant 0$ the expression (50) is identical to (39) and so the definition of A_{mn} is consistent with the previous definition. It may also be noted that

$$A_{-m,-n} = A_{m,n}^*. \quad (51)$$

It may be verified that the double summations in (49) contain terms that are independent of time t and hence correspond to standing waves. However, it may be verified that these terms add to zero and hence no standing waves appear.

The right-hand side of (48) is real. A solution for ϕ_2, that also satisfies the required boundary conditions, may be written in the form

$$\phi_2 = \frac{\sigma e^{-2i\sigma t}}{4k^2}\sum_{m=0}^{\infty}\left[\int_0^{\infty}D_m(k_2)A_m(k_2r)\times\cosh k_2(z+h)dk_2\right]\cos m\theta + \frac{\sigma e^{+2i\sigma t}}{4k^2}\times\sum_{m=0}^{\infty}\left[\int_0^{\infty}D_m^*(k_2)A_m^*(k_2r)\times\cosh k_2(z+h)dk_2\right]\cos m\theta. \quad (52)$$

The above equation is analogous to (40), and the function $D_m(k_2)$ is the solution of the Eq. (41) with $B_m(kr)$ defined by (49) which is also analogous to (38).

7. Resultant horizontal forces on cylinder

For a diffracted wave whose first-order potential is of the form (44), the hydrodynamic pressure evaluated at the cylinder $r = a$ depends on ϕ_1, ϕ_2, etc. The first-order, or linear wave theory, yields the following expression for the horizontal force on the cylinder after integrating according to (29):

$$F_{x1} = \frac{4\rho g}{k^3}\frac{\tanh kh}{|H_1^{(1)\prime}(ka)|}\cos(\sigma t - \alpha_1), \quad (53)$$

where

$$\alpha_1 = \tan^{-1}\left[\frac{J'_1(ka)}{Y'_1(ka)}\right]. \quad (54)$$

The expression (53) has been obtained by many researchers, including MacCamy and Fuchs (1954), Lighthill (1979), and Rahman (1981). Lighthill (1979) obtained this result for deep-water waves only by use of the linear diffraction theory of Havelock (1940).

To evaluate the second-order contribution to the total force the calculation is as summarized below.

Substitution of ϕ_1 and η_1 from (44) and (46), and carrying out the z-integrations, the coefficient of $\cos\theta$ in (30), apart from the $\partial\phi_2/\partial t$ term, may be expressed in the form

$$\frac{g\epsilon^2}{16k^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\delta_m\delta_nR_mR_n\Bigg[\Bigg\{\left(3 - \frac{2kh}{\sinh 2kh}\right)\times\cos[-2\sigma t + \alpha_m + \alpha_n + \tfrac{1}{2}\pi(m+n)] - \left(1 + \frac{2kh}{\sinh 2kh}\right)\cos(\alpha_m - \alpha_n + \tfrac{1}{2}\pi(m-n))\Bigg\}$$
$$\times\{\cos(m+n)\theta + \cos(m-n)\theta\} - \frac{mn}{a^2k^2}\times\left(1 + \frac{2kh}{\sinh 2kh}\right)\{\cos[-2\sigma t + \alpha_m + \alpha_n + \tfrac{1}{2}\pi(m+n)] + \cos[\alpha_m - \alpha_n + \tfrac{1}{2}\pi(m-n)]\}\times\{\cos(m+n)\theta - \cos(m-n)\theta\}\Bigg], \quad (55)$$

where the Wronskian property of Bessel functions gives

$$A_m(ka) = \frac{2i}{(\pi ka)H_m^{(1)\prime}(ka)} = R_m e^{i\alpha_m}, \quad (56)$$

in which

$$R_m = \left(\frac{2}{\pi ka}\right)[J'_m(ka)^2 + Y'_m(ka)^2]^{-1/2}, \quad (57)$$

$$\alpha_m = \tan^{-1}\left[\frac{J'_m(ka)}{Y'_m(ka)}\right]. \quad (58)$$

In view of the subsequent θ-integration described in (30), we require only the coefficient of $\cos\theta$ in the double summation in (55), and this yields to the following:

$$-\frac{2g}{k^2}\left(\frac{\epsilon}{\pi ka}\right)^2 \sum_{l=0}^{\infty}\left\{\left(1-\frac{l(l+1)}{a^2k^2}\right)\left(1+\frac{2kh}{\sinh 2kh}\right)E_l + (-1)^l\left[\left(3-\frac{2kh}{\sinh 2kh}\right)+\frac{l(l+1)}{a^2k^2}\left(1+\frac{2kh}{\sinh 2kh}\right)\right] \times [C_l\cos 2\sigma t - S_l \sin 2\sigma t]\right\}, \quad (59)$$

where

$$E_l = \frac{J'_lY'_{l+1} - J'_{l+1}Y'_l}{(J'^2_l + Y'^2_l)(J'^2_{l+1} + Y'^2_{l+1})}, \quad (60)$$

$$C_l = \frac{Y'_lJ'_{l+1} + Y'_{l+1}J'_l}{(J'^2_l + Y'^2_l)(J'^2_{l+1} + Y'^2_{l+1})}, \quad (61)$$

$$S_l = \frac{Y'_lY'_{l+1} - J'_lJ'_{l+1}}{(J'^2_l + Y'^2_l)(J'^2_{l+1} + Y'^2_{l+1})}, \quad (62)$$

the Bessel function arguments being ka. The part of $\partial\phi_2/\partial t$ proportional to $\cos\theta$, given by (52), contributes to the z integral in (30) the term

$$\frac{g \tanh kh}{k\pi a} e^{-2i\sigma t}\int_0^\infty \frac{D_1(k_2)\sinh k_2h}{k_2^2H_1^{(1)\prime}(k_2a)}dk_2 + \text{c.c.}, \quad (63)$$

where $D_1(k_2)$ is connected with $B_1(ka)$ by the relation (41). Combining (59) and (63), integrating with respect to θ, the result may be expressed as the sum of steady and oscillatory parts:

$$F_{x2} = F_{x2}^{SS} + F_{x2}^{OS}, \quad (64)$$

in which

$$F_{x2}^{SS} = -\frac{2a\rho g}{\pi k^2}\left(\frac{1}{ka}\right)^2 \times \sum_{l=0}^{\infty}\left\{\left[1-\frac{l(l+1)}{a^2k^2}\right]\left(1+\frac{2kh}{\sinh 2kh}\right)E_l\right\} \quad (65)$$

$$F_{x2}^{OS} = \left\{\frac{\rho g \tanh kh}{k} e^{-2i\sigma t}\int_{k_2=0}^\infty \frac{D_1(k_2)\sinh k_2h}{k_2^2H_1^{(1)\prime}(k_2a)}dk_2 + \text{c.c.}\right\} - \frac{2a\rho g}{\pi k^2}\left(\frac{1}{ka}\right)^2\sum_{l=0}^{\infty}(-1)^l\left\{\left(3-\frac{2kh}{\sinh 2kh}\right) + \left(1+\frac{2kh}{\sinh 2kh}\right)\frac{l(l+1)}{a^2k^2}\right\} \times (C_l\cos 2\sigma t - S_l\sin 2\sigma t). \quad (66)$$

The expressions for F_{x2}^{SS} and F_{x2}^{OS} correspond to Eqs. (61), (62) and (63), as obtained by Hunt and Williams (1982) for the shallow water case.

8. Evaluation of $D_1(k_2)$

The complex form of $D_1(k_2)$ must be evaluated from the integral Eq. (41) corresponding to $m = 1$ and in conjunction with (43). As pointed out by Hunt and Baddour (1980), and Hunt and Williams (1982), it is necessary to employ a modification of Weber's integral theorem. Titchmarsh (1924), and subsequently Watson (1952), have discussed the modification of Weber's integral theorem, and reference may be made to their work. Using the A_m function as defined in Eq. (33), the integral theorem states that an arbitrary function $g(r)$ may be represented in the form of Hankel's repeated integral

$$g(r) = \int_0^\infty A_\nu(k_2r)k_2dk_2\int_a^\infty A_\nu(k_2R)Rg(R)dR, \quad (67)$$

provided that $\sqrt{r}g(r)$ is integrable over the range (a, ∞). Fortunately, the $g(r)$ function in the present instance is Hankel's function and has asymptotic form $O(r^{-1/2})$, which confirms that $r^{1/2}g(r)$ is absolutely convergent in the range (a, ∞). Application of this result to Eq. (41) yields

$$B_1(r) = \int_0^\infty A_1(k_2r)k_2dk_2\int_a^\infty A_1(k_2R)RB_1(R)dR. \quad (68)$$

Using the expression for $B_1(r)$ from (43), and after necessary integration, one obtains

$$D_1(k_2) = \frac{kk_2\int_a^\infty A_1(k_2r)rB_1(r)dr}{k_2\sinh k_2h - 4k\tanh kh\cosh k_2h}. \quad (69)$$

The expression (69) may also be written as

$$\frac{D_1(k_2)\sinh k_2h}{k_2^2H_1^{(1)\prime}(k_2a)} = \frac{k\int_a^\infty A_1(k_2r)rB_1(r)dr}{k_2\left[k_2 - \dfrac{4k\tanh kh}{\tanh k_2h}\right]H_1^{(1)\prime}(k_2a)}, \quad (70)$$

which may be rewritten in nondimensional form as

$$G(k_2a) = \frac{D_1(k_2a)\,\sinh k_2h}{(k_2a)^2 H_1^{(1)\prime}(k_2a)}$$

$$= \frac{\displaystyle\int_{ka}^{\infty} A_1(k_2r)B_1(kr)(kr)d(kr)}{(k_2a)\left[(k_2a) - \dfrac{4ka\,\tanh kh}{\tanh k_2h}\right]H_1^{(1)\prime}(k_2a)}. \quad (71)$$

The function $G(k_2a)$ is a regular function near, and at, $k_2 = 0$. However, it is singular when k_2 is the root of the equation

$$k_2 \tanh k_2h = 4k \tanh kh. \quad (72)$$

Griffith (1956) has demonstrated the singular behaviour of certain Hankel transforms by asymptotic analysis of their integrands, and similar consideration of (71) shows that the integrand in $\int_0^\infty G(k_2)dk_2$ is singular at $k_2 = 4k$ for a particular deep water wave, and at $k_2 = 2k$ for a particular shallow water wave. The solution k_2 of Eq. (72) lies between $2k$ and $4k$ and so may be regarded as corresponding to an ocean of intermediate depth.

The nondimensional forms of the first- and second-order components of the total horizontal forces may now be written as

$$\frac{F_{x1}}{\rho g D^3} = \left[\frac{\tanh kh}{2(ka)^3}\right]\left[\frac{\cos(\sigma t - \alpha_1)}{|H_1^{(1)\prime}(ka)|}\right], \quad (73)$$

$$\frac{F_{x2}}{\rho g D^3} = \left\{\frac{\tanh kh e^{-2i\sigma t}}{8(ka)}\int_0^\infty G(k_2)dk_2 + \text{c.c.}\right\}$$

$$- \frac{1}{4\pi(ka)^4}\sum_{l=0}^{\infty}(-1)^l\left\{\left(3 - \frac{2kh}{\sinh 2kh}\right) + \frac{l(l+1)}{a^2k^2}\right.$$

$$\left.\times\left(1 + \frac{2kh}{\sinh 2kh}\right)\right\}\{C_l \cos 2\sigma t - S_l \sin 2\sigma t\}$$

$$- \frac{1}{4\pi(ka)^4}\sum_{l=0}^{\infty}\left\{\left(1 - \frac{l(l+1)}{a^2k^2}\right)\left(1 + \frac{2kh}{\sinh 2kh}\right)E_l\right\}. \quad (74)$$

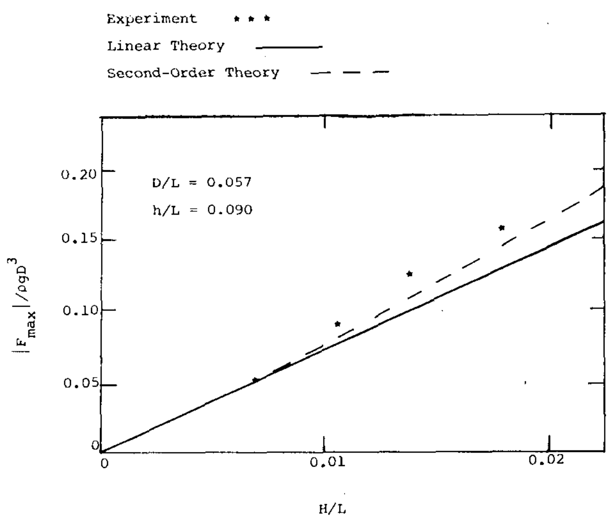

FIG. 2. Comparison of linear and second-order wave forces with experimental data of Mogridge and Jamieson (1976).

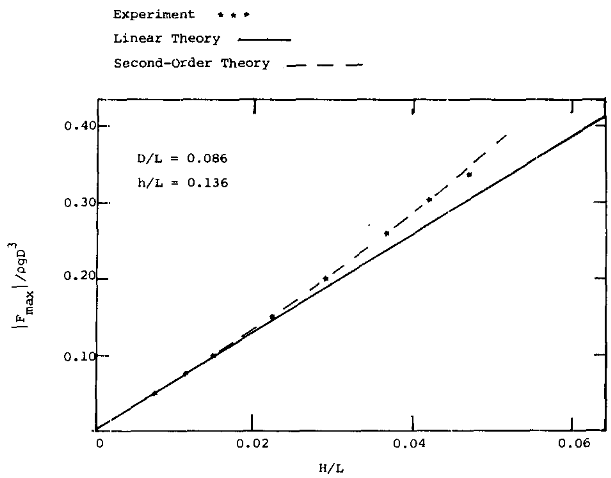

FIG. 3. Comparison of linear and second-order wave forces with experimental data of Mogridge and Jamieson (1976).

9. Discussion of results

A second-order diffraction theory has been developed and tested by comparison with the results of experimental data. Closed form integral solutions satisfying all necessary hydrodynamic boundary conditions were obtained by using the perturbation method. The maximum horizontal force was expressed as the average of the absolute values of the maximum positive force and the maximum negative force.

The total horizontal force in nondimensional form may be expressed as

$$F = F_1 + F_2, \quad (75)$$

where

$$F_1 = C_M \frac{(\pi/8)(H/L)}{(D/L)} \tanh kh \cos(\sigma t - \alpha_1), \quad (76)$$

$$F_2 = \left\{\frac{(\pi/8)(H/L)^2}{(D/L)} \tanh kh e^{-2i\sigma t}\int G(k_2)dk_2 + \text{c.c.}\right\}$$

$$- \frac{(H/L)^2}{4(D/L)(ka)^3}\sum_{l=0}^{\infty}(-1)^l\left\{\left(3 - \frac{2kh}{\sinh 2kh}\right)\right.$$

$$\left. + \frac{l(l+1)}{a^2k^2}\left(1 + \frac{2kh}{\sinh 2kh}\right)\right\}$$

$$\times\{C_l \cos 2\sigma t - S_l \sin 2\sigma t\} - \frac{(H/L)^2}{4(D/L)(ka)^3}$$

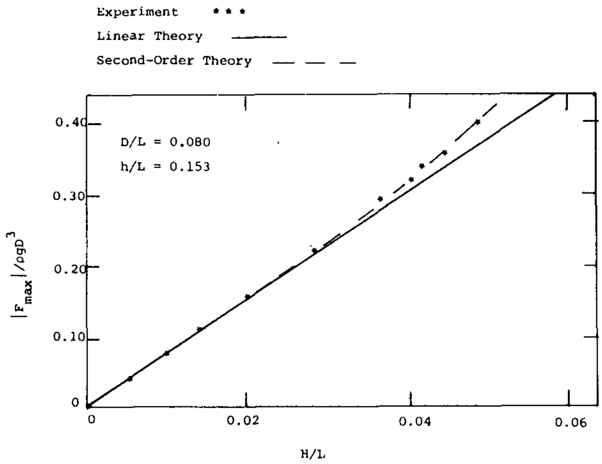

FIG. 4. Comparison of linear and second-order wave forces with experimental data of Mogridge and Jamieson (1976).

$$\times \sum_{l=0}^{\infty} \left\{\left[1 - \frac{l(l+1)}{a^2k^2}\right]\left(1 + \frac{2kh}{\sinh 2kh}\right)E_l\right\}, \quad (77)$$

in which D is the diameter of the cylinder, H is the wave amplitude and C_M is defined to be the coefficient of added mass due to linear theory and is equal to

$$C_M = 4/\{\pi(ka)^2|H_1^{(1)\prime}(ka)|\}. \quad (78)$$

By dimensional analysis it has been observed that wave forces on structures depend on three dimensionless parameters H/L, D/L, and h/L. The importance of these parameters is well recognized by ocean engineers concerned with the design of structures. As correctly enumerated by Hunt and Williams (1982), many experimental studies of wave forces have been published with such a variation of parameters that a concise experimental verification is not possible. In the present paper, however, the predictions of the present analysis are compared with some experimental data collected by the Division of Mechanical Engineering of the National Research Council of Canada. Figs. 2–4 show a comparison of the predicted results with the experimental data. It may be noted that the second-order theory leads to results in excellent agreement with measured values. A remark may be made concerning the substantial discrepancy between the results presented in Fig. 3 of this paper and the Fig. 3 of Rahman and Chehil (1982). This discrepancy may be attributed to two different theoretical developments. It is worth mentioning here that the present theory gives better correlation with the measured values. Another comparison is made in Fig. 5 with the experi-

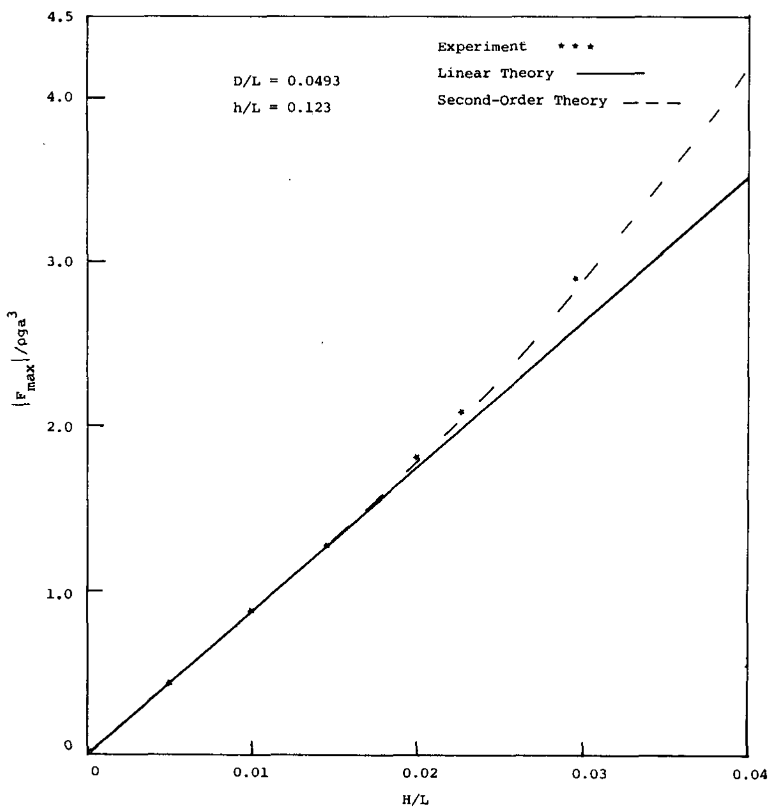

FIG. 5. Comparison of linear and second-order wave forces with experimental data of Raman and Venkatanarasaiah (1976).

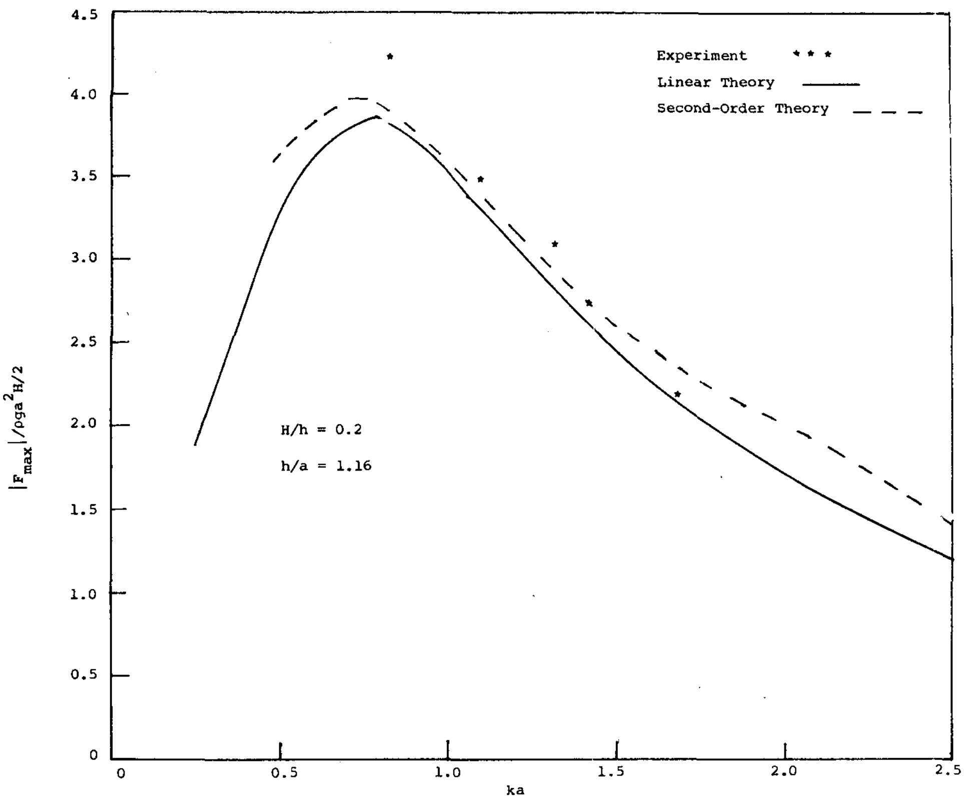

FIG. 6. Comparison of linear and second-order wave forces with experimental data of Chakrabarti (1975).

mental data of Raman and Venkatanarasaiah (1976). The present second-order theory appears to compare well with these experimental data. In Fig. 6, both the first-order and second-order solutions are compared with force measurements of Chakrabarti (1975), which are generally seen to be closer to the second-order predictions.

A further comment regarding the integrand $G(k_2)$ may be made. It is clear that for a cylindrical structure the wave number k_2 of the second-order wave theory must not coincide with the root of $k_2 \tanh k_2 h = 4k \times \tanh kh$ unless the corresponding integral in (71) vanishes. Otherwise the structure is, in effect, being driven at a condition of resonance. This type of resonance behavior is predicted by the second-order wave solution but not by linear wave theory. However, in real situations involving oceanic waves, the resonance phenomenon is observed frequently. Thus the linear theory must be extended by incorporation of second-order theory in order to predict correctly the forces on off-shore structures. The present diffracted theory predicts that the cylinder will be driven to a condition of resonance when $k_2 = 2k$ in the instance of shallow-water waves and when $k_2 = 4k$ in the instance of deep-water waves. For these instances the present mathematical model must be modified. A partial answer to the first case (shallow water) was reported by Rahman (1981).

Considerable difficulties have been encountered in evaluating the integrals in Eq. (74). The numerical procedure is summarized below. The r-integration in the G function is evaluated by a Simpson's integration scheme from the surface of the cylinder to a certain distance λ_1, where λ_1 is large but finite. Because of oscillatory behaviour of the r-integrand, contribution beyond this limiting distance is very small and may be neglected. In the second integral with respect to k_2, the integration is performed in two parts: first the integration is carried out from $k_2 = 0$ to a small neighborhood of the left hand side of the singular point of the G function and then the integration is continued from a small neighborhood of the right hand side of the singular point to a certain distance λ_2, which depends on k_2; the G function is found to be a highly oscillatory function of distance. However, Lighthill (1979) has indicated that beyond this limiting distance the contribution of this infinite integral is insignificant. In the present study, we have verified that this is also true.

The solution method for the integral equation was programmed in Fortran V for the CDC Cyber 170 digital computer at the Technical University of Nova

Scotia. The commercially available IMSL routine for computation of Bessel functions was used to obtain the forces on the structure. A Simpson's integration scheme has been formulated and used to integrate the complex integrands with a very fine grid size.

10. Conclusions

Second-order solutions of nonlinear wave diffraction in the presence of a vertical cylinder in water of finite depth have been derived and compared with available experimental data. Maximum horizontal forces on the cylinder have been predicted by the theory, and the predictions have been compared with the experimental data to show excellent agreement. A close examination of the mathematical solutions predicts that the structure will be driven to resonance for $k_2 = 4k$ in the case of deep water waves and $k_2 = 2k$ in the case of shallow water waves.

Note added in revision. Since submission of this work two papers have been published (Hunt and Baddour (1981), and Hunt and Williams (1982)) with almost identical methods except that we have followed the complex variable analysis where the analysis of the other papers has been performed in terms of real variables. It is to be noted that both these analyses led to the same result which confirms the correctness of the second-order contribution presented in this paper.

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REFERENCES

Chakrabarti, S. K., 1975: Second order wave force on large vertical cylinder. *Proc. ASCE,* **101WW3,** 311–317.

——, 1977: Discussion on forces due to nonlinear waves on vertical cylinders. *J. Waterway, Port, Coastal Ocean Div., ASCE,* **103,** 404–406.

——, 1978: Comments on "Second order wave effects on large diameter vertical cylinder." *J. Ship Res.,* **22,** No. 4, 266–268.

Garrison, C. J., 1979: The consistent second-order theory of wave/structure interaction. Tech. Rep., Naval Postgraduate School, 39 pp.

Griffith, J. L., 1956: On Weber transforms. *J. Proc. Roy. Soc., London,* **89,** 232–248.

Havelock, T. H., 1940: The pressure of water waves on a fixed obstacle. *Proc. Roy. Soc. London,* **A175,** 409–421.

Hunt, J. N., and R. E. Baddour, 1980: Nonlinear standing waves bounded by cylinders. *Quart. J. Mech. Appl. Math.,* **33,** Part 3, 357–371.

——, and ——, 1981: The diffraction of nonlinear progressive waves by a vertical cylinder. *Quart. J. Mech. Appl. Math.,* **34,** Part 1, 69–87.

——, and A. N. Williams, 1982: Nonlinear diffraction of Stokes' water waves by a circular cylinder for arbitrary uniform depth. *J. Mec. Theor. Appl.,* **1,** 429–449.

Isaacson, M. de St. Q., 1977: Nonlinear wave forces on large offshore structures. *J. Waterway, Port, Coastal Ocean Div., ASCE,* **103WW1,** 166–170.

Lighthill, M. J., 1979: Waves and hydrodynamic loading. *Second Int. Conf. on Behaviour of Off-Shore Structures,* Imperial College, London, 1–40.

Longuet-Higgins, M. S., and E. D. Cokelet, 1976: The deformation of steep surface waves on water. I. A numerical method of computation. *Proc. Roy. Soc. London,* **A350,** 1–26.

MacCamy, R. C., and R. A. Fuchs, 1954: Wave forces on piles: A diffraction theory. U.S. Beach Erosion Board, Tech. Memo. No. 69, 17 pp.

Miloh, Touvia, 1980: Irregularities in solutions on nonlinear wave diffraction problem by vertical cylinder. *J. Waterway, Port, Coastal Ocean Div., Proc. ASCE,* **106WW2,** 279–284.

Mogridge, G. R., and W. W. Jamieson, 1976: Wave loads on large circular cylinders: A design method. National Research Council of Canada, Division of Mechanical Engineering, Rep. MH-111, 34 pp.

Morison, J. R., M. P. O'Brien, J. W. Johnson and S. A. Schaaf, 1950: The force exerted by surface waves on piles. *J. Petroleum Tech.,* **189,** 149–154.

Phillips, O. M., 1977: *The Dynamics of the Upper Ocean.* Cambridge University Press, London, 33–36.

Rahman, M., 1981: Nonlinear wave loads on large circular cylinders: A perturbation technique. *Advances in Water Resources,* Vol. 4, Academic Press, 9–19.

——, and J. Moes, 1979: Mathematical modelling of harbour oscillation. National Research Council Canada, DME Hydraulics Laboratory, Rep. No. HY-203, Ottawa, 73 pp.

——, and D. S. Chehil, 1982: Second-order wave force on a large cylinder. *Acta Mech.,* **44,** 127–136.

Raman, H., and P. Venkatanarasaiah, 1976: Forces due to nonlinear waves on vertical cylinders. *Proc. ASCE,* **102WW3,** 301–316.

——, G. V. Prabhakara Rao and P. Venkatanarasaiah, 1975: Diffraction of nonlinear surface waves by circular cylinder. *Acta Mech.,* **23,** 145–158.

——, N. Jothishankar and P. Venkatanarasaiah, 1977: Nonlinear waves interaction with vertical cylinder of large diameter. *J. Ship Res.,* **21,** No. 2, 120–124.

Titchmarsh, E. C., 1924: Weber's integral theorem. *Proc. London Math. Soc.,* **22,** 15–28.

Watson, G. N., 1952: *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, Chap. 14, 451–476.

Wehausen, J. V., 1980: Perturbation methods in diffraction. *Proc. ASCE,* **106WW2,** 290–291.

Yamaguchi, M., and K. Tsuchiya, 1974: Nonlinear effect of waves on wave pressure and wave force on a large circular pile" (in Japanese). *Proc. Civil Eng. Soc. Japan,* No. 229, 41–53.