An Unusual Way to Generate Conic Sections. Related Euclidean Constructions

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Abstract

It is shown that given two points and a circle, the basic conic sections, except the parabola, can be generated as envelopes using an unusual construction.

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Key words: conic section, envelope, ellipse, hyperbola, circle.

§1. Introduction

Given two points A and B and a circle with center C, conic sections, except the parabola, can be generated as envelopes by the following construction: (Fig.1)

Take a point P on the circle. Draw the lines AP and BP. Find the second intersections Q and S of these lines with the circle.
The family of lines SQ has envelopes which are ellipses, hyperbolas or circles.
I) We shall first present examples for various cases.
II) Next we shall prove that the envelope is a conic section.
III) Finally we shall find the semi-minor and the semi-major axes by two Euclidean constructions.

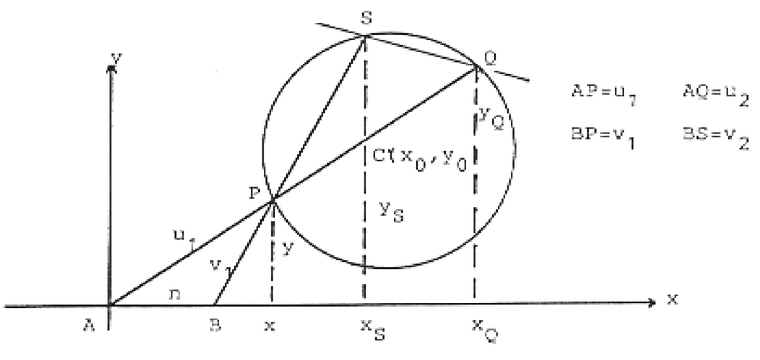

Fig.1

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§2. Examples for Various Cases

a) When the line AB is outside of the circle, the envelope of the lines SQ is an ellipse inside of the circle and does not touch the circle. (Figs.2,3)

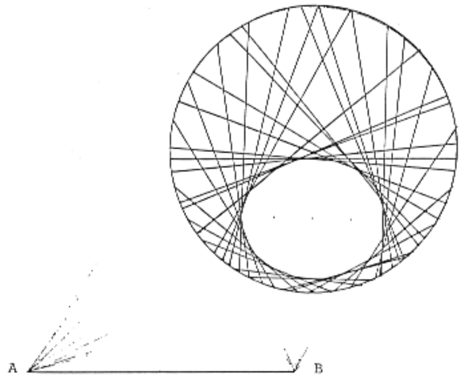

Fig.2

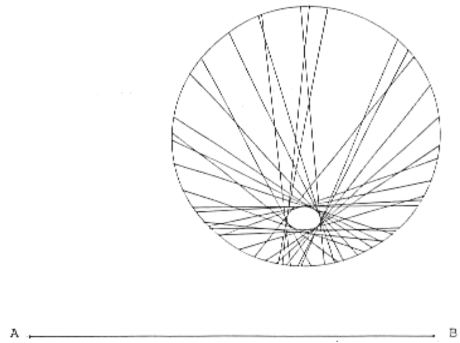

Fig.3

b) When the line AB touches the circle, the envelope ellipse inside of the circle touches the circle at the same point. (Fig.4)

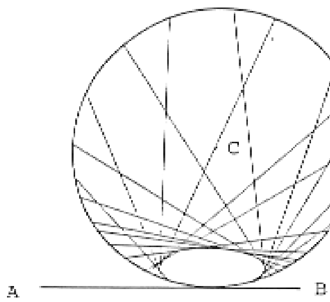

Fig.4

c) When the line AB crosses the circle, with A and B still outside of the circle, the envelope ellipse inside of the circle touches the circle at the points where AB intersects the circle. (Fig.5)

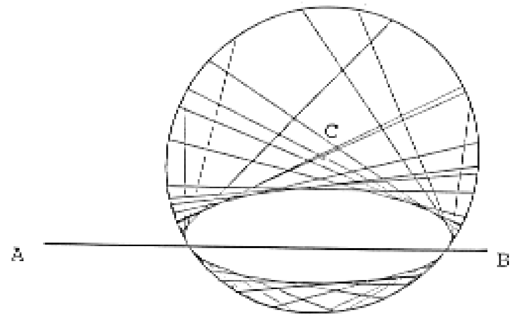

Fig.5

d) When both A and B are inside of the circle, the envelope ellipse inside of the circle touches the circle at two points. (Fig.6)

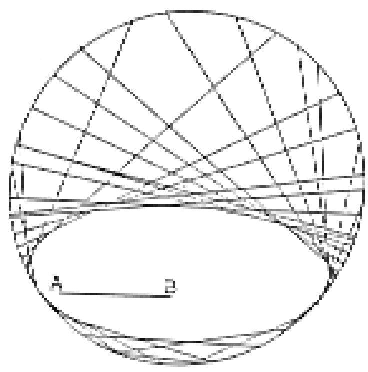

Fig.6

e) When A and B are inside of the circle and are symmetrically located around the center C, the envelope ellipse also is symmetric around AB. (Fig.7)

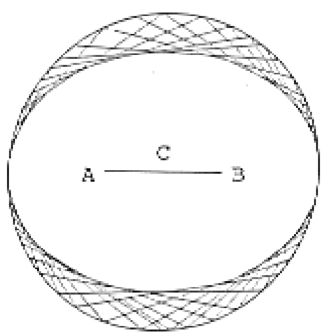

Fig.7

f) When in e) $A \to B$ the ellipse goes over to a circle.

g) When one of A or B is inside of the ellipse and the other outside, the envelope is a hyperbola.

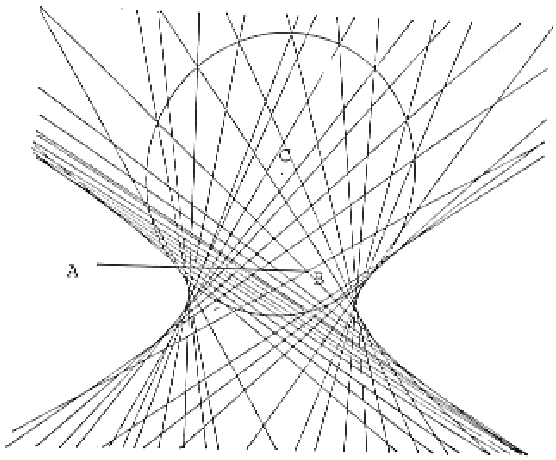

Fig.8

h) When A is outside of the circle, and B is at C, the hyperbola has AC as its axis and is tangent to the circle. (Fig.9)

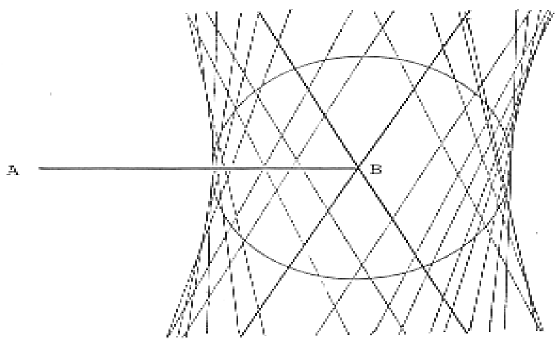

Fig.9

In all cases:
The major axis of the conic section is parallel to AB. The center of the conic section is on the perpendicular line from the center C of the circle to AB.

Proof. The standard process for finding the envelope of a family of lines is to express the lines as a function of a parameter, then to take the derivative with respect to the parameter and eliminate the parameter between the two equations.

In this problem the simplicity of the geometric construction is deceiving. The equation of the line SQ in terms of the parameters we tried, turned out to be very complicated; its derivative even more so. Eliminating the parameter between the two equations is a near impossible task, even for the computer with symbolic operations.

Instead we tried the following approach, which does not involve solving equations, and it worked.
The basic idea is the following:

We write the equation of the envelope ellipse in terms of the three unknown parameters A^2, B^2, and c:

(1) $$\frac{(x'-x_0)^2}{A^2}+\frac{(y'-c)^2}{B^2}=1$$

(We use x' and y', because x and y were used for the coordinates of P.)

This is the equation of an ellipse
a) with its axis parallel to x-axis, and
b) with its center at (x_0, c).
x_0 is the x-coordinate of the center of the circle as well as of the ellipse. The y-coordinate c of the center of the ellipse is unknown.
These two features emerge from geometrical constructions as well as numerical calculations. They will be justified *a posteriori* by the general proof.

We write the line SQ as

(2) $$y' = mx' + b$$

Here m and b are found by first finding the coordinates of S and Q.
For this we use powers of the points A and B with respect to the circle:

$$K_A = u_1u_2 \qquad \frac{K_A}{{u_1}^2}=\frac{u_2}{u_1} \qquad {u_1}^2 = x^2+y^2$$

$$K_B = v_1v_2 \qquad \frac{K_B}{{v_1}^2}=\frac{v_2}{v_1} \qquad {v_1}^2 = (x-n)^2+y^2$$

They give:

$$x_Q = \frac{K_A}{{u_1}^2}x \qquad x_S = \frac{K_B}{{v_1}^2}(x-n)+n$$

$$y_Q = \frac{K_A}{{u_1}^2}y \qquad y_S = \frac{K_B}{{v_1}^2}y$$

Once the coordinates of S and Q are found in terms of x, the equation of SQ is

$$\frac{y'-y_Q}{x'-x_Q}=\frac{y_Q-y_S}{x_Q-x_S}$$

which can be brought into the form:

$$y' = mx' + b$$

where

$$m = \frac{(K_A{v_1}^2 - K_B{u_1}^2)y}{K_A{v_1}^2x - K_B{u_1}^2(x-n) - n{u_1}^2{v_1}^2} b = \frac{K_A}{{u_1}^2}y - \frac{K_A}{{u_1}^2}xm$$

Here

$$K_A = {x_0}^2+{y_0}^2-R^2 \qquad {u_1}^2 = x^2+y^2$$

$$K_B = (x-n)^2+{y_0}^2-R^2 \qquad {v_1}^2=(x-n)^2+y^2$$

$$y = y_0 \pm \sqrt{R^2 - (x - x_0)^2}$$

In brief, both m and b are functions of the single parameter x.

To find the intersections of the line (2) with the ellipse (1) we insert (2) into (1) and find a quadratic equation for x'.

$$A'x'^2 - 2B'x' + C' = 0$$

where

$$A' = B^2 + A^2 m \qquad B' = B^2 x_0 - A^2 m(b - c) \qquad C' = B^2({x_0}^2 - A^2) + A^2(b - c)^2$$

For the line to be tangent to the ellipse the discriminant D should be zero.

$$D = B'^2 - 4A'C' = 0$$

$$A^2 m^2 + B^2 - (b - c)^2 - 2x_0 m(b - c) - m^2 {x_0}^2 = 0 \tag{3}$$

This is the condition for the line to be tangent to the ellipse.

To determine A, B and c we take three lines

$$y' = m_1 x' + b_1$$

$$y' = m_2 x' + b_2$$

$$y' = m_3 x' + b_3$$

Writing the equation (3) for each pair m_i, b_i and subtracting side-by-side we find two equations for A^2 and c.

$$A^2 p_{21} + 2cq_{21} - r_{21} = 0 \qquad A^2 p_{31} + 2cq_{31} - r_{31} = 0 \tag{4}$$

with

$$p_{ij} = {m_i}^2 - {m_j}^2$$

$$q_{ij} = (b_j + x_0 m_j) - (b_i + x_0 m_i)$$

$$r_{ij} = (b_j + x_0 m_j)^2 - (b_i + x_0 m_i)^2$$

(4) are two linear equations for A^2 and c and are easily solved. Inserting A^2 and c into the equation (3) gives B^2. If the ellipse with A^2, B^2, and c is the envelope, then for any line

$$y' = m_i x' + b_i$$

the equations

$$A^2 p_{21} + cq_{21} + r_{21} = 0$$

$$A^2 p_{31} + cq_{31} + r_{31} = 0$$

$$A^2 p_{ij} + cq_{ij} + r_{ij} = 0$$

should be satisfied. (Here $j = 1$, 2 or 3 or any other index.)
This means that

$$\begin{vmatrix} p_{21} & q_{21} & r_{21} \\ p_{31} & q_{31} & r_{31} \\ p_{ij} & q_{ij} & r_{ij} \end{vmatrix} = 0$$

Of course here we could take for (21) and (31) any other pair.
This determinant does not involve solving equations or eliminating parameters. Its calculation is straightforward and gives identically zero.

We do not repeat the proof for the hyperbola, which only differs in the sign of the equation (1).

§3. Two Euclidean Constructions

From the envelopes resulted two interesting geometrical problems. They serve to box-in the envelope ellipse into a rectangle.

1) Find the semi-minor axis of the ellipse by constructing the top and the bottom horizontal tangents of the ellipse.

Problem: (Figs.10,11)

Given a circle and two points A and B (say outside of the circle), find a point P on the circle such that, when connected to A and B, the second intersections D and E of AP and BP with the circle give a line DE parallel to AB.

Solution:

$$\frac{PA}{PD} = \frac{PB}{PE}$$

is the condition for DE to be parallel to AB.

Thus A and B are on the homothetic of the given circle C_1 with the homothety center at P. Hence C_1 and its homothetic circle are tangent at P.

The problem becomes: Construct a circle which goes through two given points A and B and is tangent to a given circle C_1.

Construction:

The intersection point O of the tangent line at P, with the line AB has the same power wrt both circles. To find O we draw an arbritrary circle C_2 which goes through A and B and intersects C_1 at F and G,

We connect F with G and find O. We draw two tangent lines from O to the circle C_1 and find the points P and P'.

When P (or P') are connected to A and B they intersect the circle C_1 at points D, E (or D', E') such that DE (or $D'E'$) is parallel to AB.

The envelope ellipse is between the lines DE and $D'E'$, touching them.

The distance between DE and $D'E'$ is $2b$, where b is the semi-minor axis of the ellipse.

The center of the ellipse can be found by dropping a perpendicular line from the center of the circle onto the mid-parallel between DE and $D'E'$.

2) Find the semi-major axis of the ellipse by constructing the right and left vertical tangents of the ellipse.

Problem: (Figs.12,13)

Given a circle and two points A and B (say outside of the circle), find a point P on the circle such that, when connected to A and B, the second intersections P_2 and

Q of AP and BP with the given circle give a line P_2Q perpendicular to AB.

Solution:

This problem turned out to be quite challenging. We did not know if a Euclidean construction existed. It required the proof of the orthogonality of the given circle with an auxiliary circle. It also required an inversion and a back-inversion.

Consider the problem solved, with $P_2Q \perp AB$. Take the auxiliary circle which goes through A, B and P. We prove in the appendix that this *circle with center* M_1 *is orthogonal to the given circle with center* M.

The problem reduces to: Given a circle with center M and two points A, B outside of it, find a circle with center M_1, which goes through A and B and is orthogonal to the given circle.

This problem doesn't seem to be any easier than the original one. But then, we remember that inversion inverts circles into circles, and in particular, circles into lines if the inversion center is on the circle. Also, inversion does not change angles between intersecting curves. This turned out to be the key to the solution of the problem.

Take an arbitrary point I on the given circle. Invert the given circle wrt I with an arbitrary power k, obtaining the line ℓ. Invert the auxiliary circle through A, B and P wrt I and with the same power k, obtaining the circle with the center S, which goes through A' and B' (images of A and B). Since the inversion does not change angles between curves, the line ℓ is orthogonal to the circle with center S.

Now the problem is: Find a circle which goes through A', B' and is orthogonal to ℓ. Here A', B' and ℓ are all known.

Construction:

We draw the mid-perpendicular to $A'B'$ and find its intersection S with the line ℓ. Next we back-invert the circle S, obtaining the auxiliary circle which goes through A, B and P, thus finding the point P (and P'). Connecting P to A and B we find the second intersections P_2 and Q of AP and BP with the given circle.

$$QP_2 \perp AB$$

The second intersection P' of the auxiliary circle with the given circle gives the left perpendicular tangent $P_2'Q'$ to the ellipse. The envelope ellipse lies between P_2Q and $P_2'Q'$, touching them. The distance between P_2Q and $P_2'Q'$ is $2a$, where a is the semi-major axis of the ellipse.

§4. Appendix

Take two perpendicular line segments AD and DQ (Figs.14,15).
Take B between A and D. Take P_2 between D and Q.
Draw AP_2 and BQ. Call their intersection P.
We prove that the circle which passes through A, B and P is orthogonal to the circle which passes through P_2, Q and P. The angles $\alpha = \gamma + \varepsilon$ at C, and β at A are defined in Fig.15. It follows that $\angle APB = \angle QPP_2 = \gamma$, $\angle CPP_2 = (\beta - \epsilon) + \alpha = (\beta - \alpha + \gamma) + \alpha = \beta + \gamma$ (as external angle of the triangle CAP). Hence $\angle CPQ = \beta = \angle CP_2Q$. Further $\angle AP_2D = 90 - \alpha + \beta$. This makes

$$\angle PP_2C = 180 - \beta - (90 - \alpha + \gamma) = 90 + \alpha - \beta - \gamma,$$

which is equal to half of the central angles at M. Then $\angle M_1MC = 90 + \alpha - \beta - \gamma$.

On the other hand $\angle CAP = \beta - \alpha + \gamma$ is equal to half of the central angle at M_1, $\angle CM_1M = \beta - \alpha + \gamma$. Combining these we find

$$\angle M_1CM = 90^\circ,$$

that is, *the circles with the centers M and M_1 are orthogonal.*

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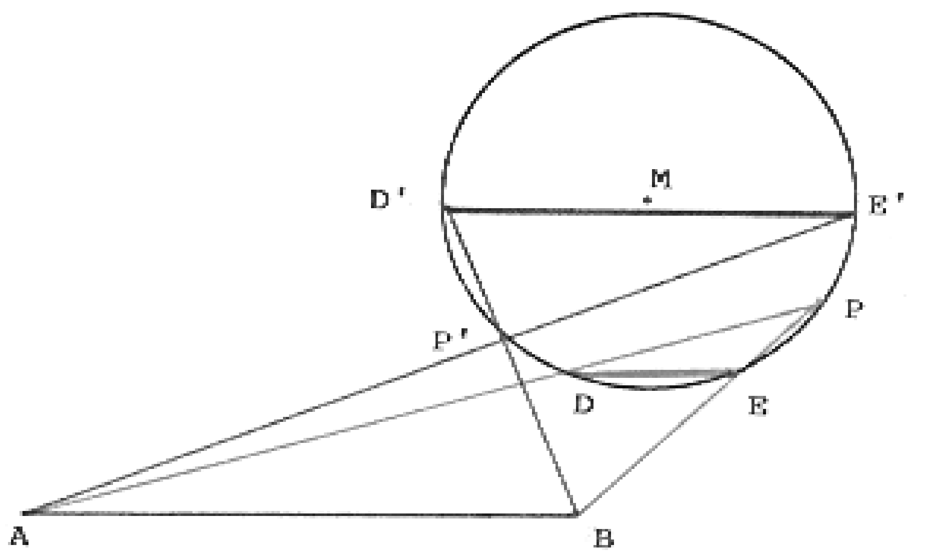

Fig.10

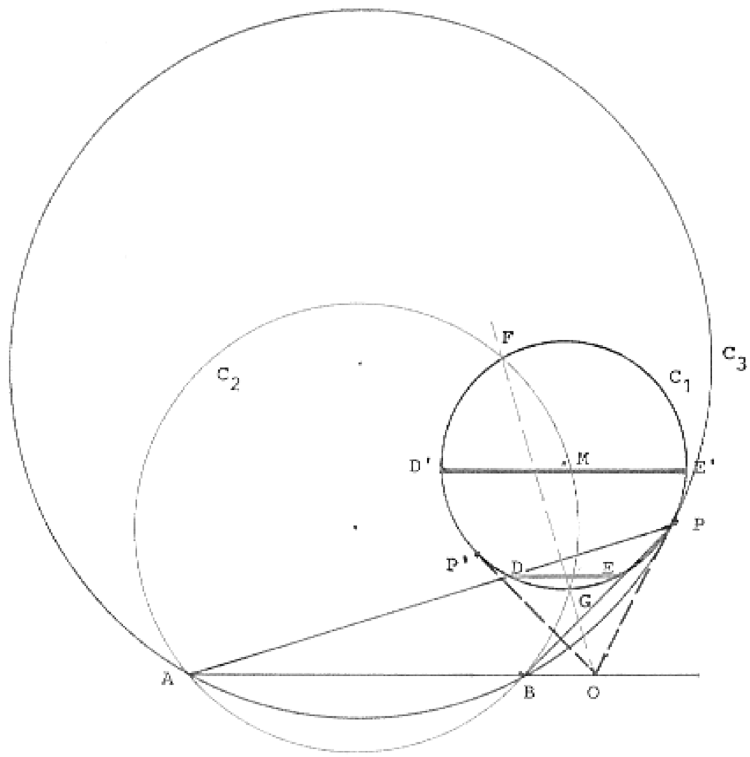

Fig.11

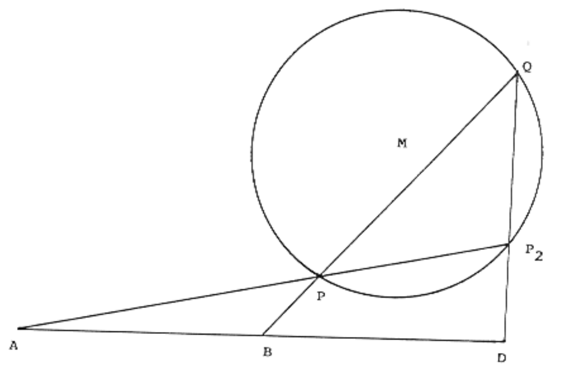

Fig.12

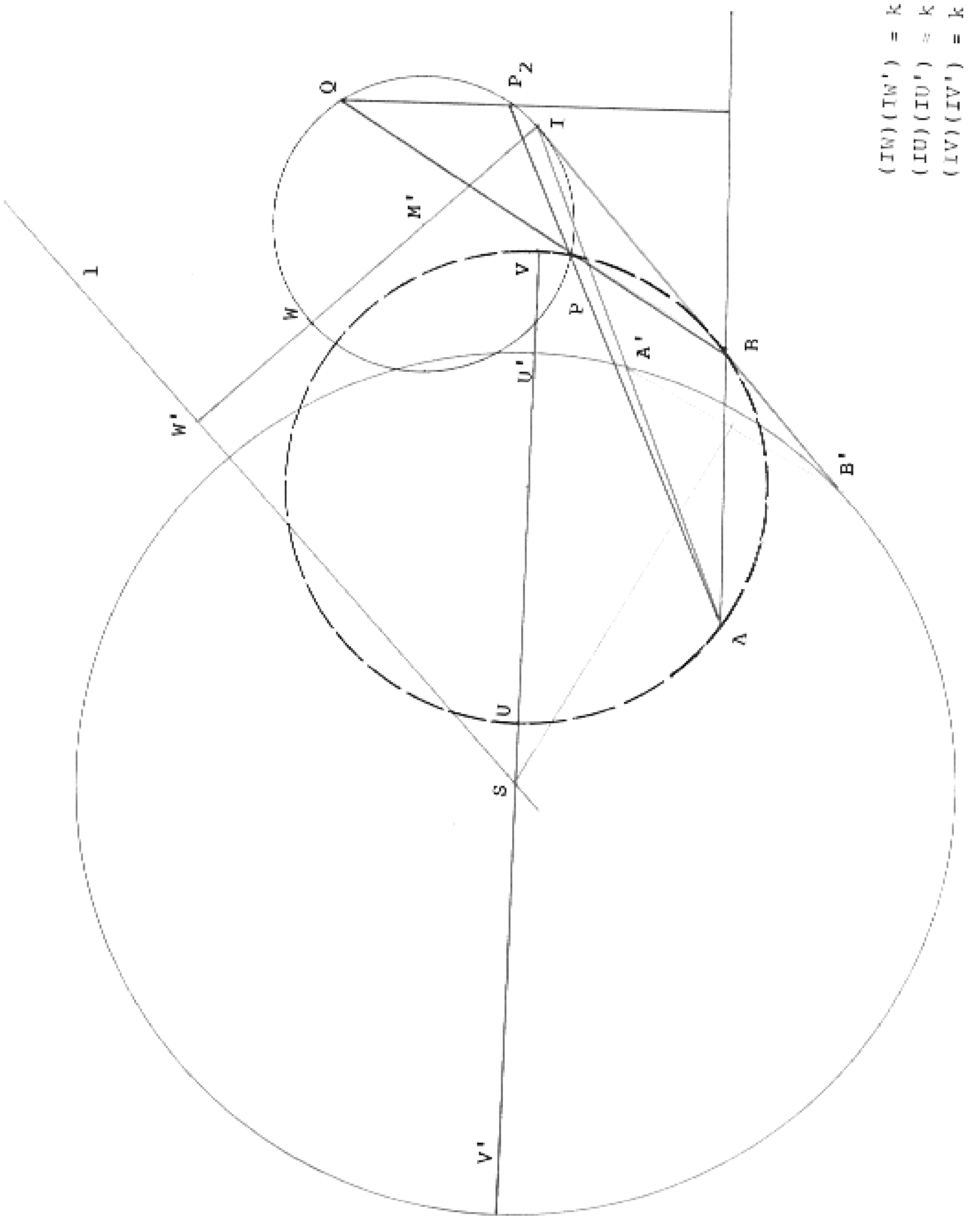

Fig.13

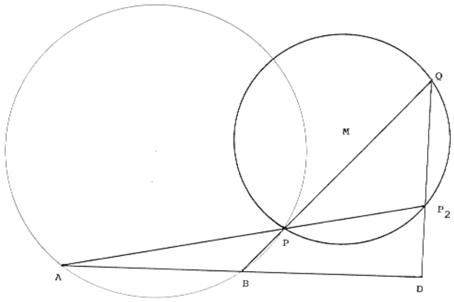

Fig.14

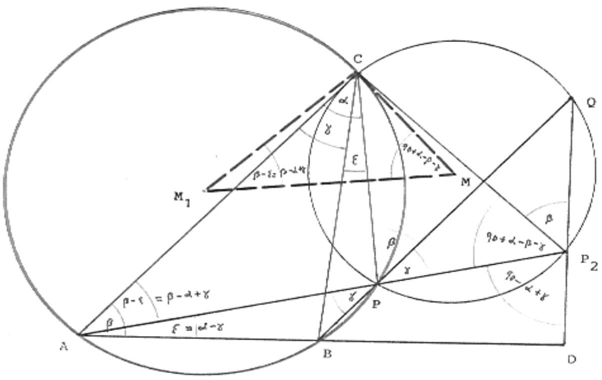

Fig.15

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