Some Asymptotic Results on Extended Sequences Connected with the Regular Continued Fraction

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Abstract

Using a two-dimensional Wirsing method, we give a solution to Gauss' problem related to the extended case of doubly infinite versions of $(s_n)_{n\in\mathrm{N}}$ and $(r_n)_{n\in\mathrm{N}_+}$ connected with the regular continued fraction.

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§1. Introduction

Let Ω denote the collection of irrational numbers in the unit interval $I = [0,1]$. Consider the so-called continued fraction transformation τ of Ω defined as $\tau(\omega) = \omega^{-1} \pmod 1$ = fractionary part of ω^{-1}, $\omega \in \Omega$. Define $\mathbb{N}_+$-valued functions a_n on Ω by $a_{n+1}(\omega) = a_1(\tau^n(\omega))$, $n \in \mathbb{N}_+ = \{1, 2, \ldots\}$, where $a_1(\omega) = [\omega^{-1}]$ = integer part of ω^{-1}, $\omega \in \Omega$. Here τ^n denotes the n-th iterate of τ. For any $n \in \mathbb{N}_+$, writing

$$[x_1] = \frac{1}{x_1}, [x_1, \ldots, x_n] = \frac{1}{(x_1 + [x_2, \ldots, x_n])}, \ n \geq 2,$$

for arbitrary indeterminates x_i, $1 \leq i \leq n$, we have

$$\omega = \lim_{n\to\infty} [a_1(\omega), \ldots, a_n(\omega)], \quad \omega \in \Omega,$$

and this explains the name of τ.

The metric point of view in studying the sequence $(a_n)_{n\in\mathrm{N}_+}$ is to consider that the a_n, $n \in \mathbb{N}_+$ are $\mathbb{N}_+$-valued random variables on $(I, \mathcal{B}_I)$, which are defined almost everywhere with respect to any probability measu μ on $\mathcal{B}_I$ assigning probability 0 to the set $I \setminus \Omega$ of rational numbers in I. (Such a μ is, clearly, Lebesgue measure λ). The probabilistic structure of the sequence $(a_n)_{n\in\mathrm{N}_+}$ under λ is described by the equations

$$\lambda(a_1 = i) = \frac{1}{i(i+1)},$$

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$$\lambda(a_{n+1} = i|a_1, \ldots, a_n) = p_i(s_n), \quad n \in \mathbb{N}_+,$$

where

$$p_i(x) = \frac{x+1}{(x+i)(x+i+1)}, \ i \in \mathbb{N}_+, \ x \in I,$$

and $s_n = [a_n, \ldots, a_1]$. Thus, under λ, the sequence $(a_n)_{n\in\mathbb{N}_+}$ is neither independent nor Markovian. There is a probability measure γ on $\mathcal{B}_I$

$$\gamma(A) = \frac{1}{\log 2}\int_A \frac{dx}{x+1}, \quad A \in \mathcal{B}_I,$$

called Gauss' measure, which makes $(a_n)_{n\in\mathbb{N}_+}$ into a strictly stationary sequence. Moreover, γ is τ-invariant, i.e., $\gamma(\tau^{-1}(A)) = \gamma(A)$ for all $A \in \mathcal{B}_I$. Hence, by its very definition, $(a_n)_{n\in\mathbb{N}_+}$ i s a strictly stationary sequence under γ.

Let us define some related random variables. For any $n \in \mathbb{N}_+$, put $r_n = a_n + [a_{n+1}, a_{n+2}, \ldots]$. Obviously, for any $n \in \mathbb{N}_+$, $r_n = \frac{1}{\tau^{n-1}}$ and $s_n = \frac{1}{a_n + s_{n-1}}$, with τ^0 = identity map and $s_0 = 0$.

Let μ be a probability measure on $\mathcal{B}_I$ absolutely continuous with respect to λ. A great deal of work was done on the asymptotic behaviour of $\mu(s_n \le x, \ r_{n+1}^{-1} \le y)$, $n \to \infty$, $x, y \in I$. For a detailed account we refer the reader to [3]. Taking up a problem raised in [5], in this paper we present a result related to the extended case of doubly infinite versions of $(s_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}_+}$.

2 Extended Random Variables

Consider the so-called natural extension $\bar{\tau}$ of τ, which is defined as

$$\bar{\tau}(\omega, \theta) = \left(\tau(\omega), \frac{1}{a_1(\omega)+\theta}\right), \ (\omega, \theta) \in \Omega^2$$

(see [4]). This is a one-to-one transformation of Ω^2 with inverse

$$\bar{\tau}^{-1}(\omega, \theta) = \left(\frac{1}{a_1(\theta)+\omega}, \tau(\theta)\right), \ (\omega, \theta) \in \Omega^2.$$

By Theorem 1.3.1 in [3] the transformation $\bar{\tau}$ preserves the extended Gauss measure $\bar{\gamma}$ on $\mathcal{B}_I^2$ defined as

$$\bar{\gamma}(B) = \frac{1}{\log 2}\iint_B \frac{dxdy}{(xy+1)^2}, \quad B \in \mathcal{B}_I^2,$$

that is, $\bar{\gamma}(\bar{\tau}^{-1}(B)) = \bar{\gamma}(B)$ for all $B \in \mathcal{B}_I^2$. We have

$$\bar{\gamma}(A \times I) = \bar{\gamma}(I \times A) = \gamma(A), \quad \forall A \in \mathcal{B}_I.$$

Define random variables $\bar{a}_l$, $l \in \mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, on Ω^2 by $\bar{a}_{l+1}(\omega, \theta) = \bar{a}_1(\bar{\tau}^l(\omega, \theta))$, with $\bar{\tau}^0$ = identity map on Ω^2 and $\bar{a}_1(\omega, \theta) = a_1(\omega)$, $(\omega, \theta) \in \Omega^2$. Clearly, $\bar{a}_n(\omega, \theta) = a_n(\omega)$, $\bar{a}_0(\omega, \theta) = a_1(\theta)$, $\bar{a}_{-n}(\omega, \theta) = a_{n+1}(\theta)$ for all $n \in \mathbb{N}_+$

and $(\omega,\theta) \in \Omega^2$. The doubly infinite sequence $(\bar{a}_l)_{l\in\mathbf{Z}}$ on $(I^2, \mathcal{B}_I^2, \bar{\gamma})$ is a strictly stationary one; and clearly, it is a doubly infinite version of $(a_n)_{n\in\mathbb{N}_+}$ under $\bar{\gamma}$ (see [3], Subsection 1.3.3.).

It has been proved in [1] that for any $x \in I$ and $l \in \mathbf{Z}$ we have

$$\bar{\gamma}([0,x] \times I | \bar{a}_l, \bar{a}_{l-1}, \ldots) = \frac{(a+1)x}{ax+1}, \; \bar{\gamma} - \text{ a.s.,} \tag{2.1}$$

where $a = [\bar{a}_l, \bar{a}_{l-1}, \ldots]$. Hence, for any $i \in \mathbb{N}_+$ and $l \in \mathbf{Z}$,

$$\bar{\gamma}(\bar{a}_{l+1} = i | \bar{a}_l, \bar{a}_{l-1}, \ldots) = p_i(a), \; \bar{\gamma} - \text{ a.s.} \tag{2.2}$$

Define extended associated random variables $\bar{s}_l$ and $\bar{r}_l$ as

$$\bar{s}_l = [\bar{a}_l, \bar{a}_{l-1}, \ldots], \quad \bar{r}_l = \bar{a}_l + [\bar{a}_{l+1}, \bar{a}_{l+2}, \ldots], \; l \in \mathbf{Z}.$$

Clearly $\bar{s}_l = \bar{s}_0 \circ \bar{\tau}^l$ and $\bar{r}_l = \bar{r}_0 \circ \bar{\tau}^l$, $l \in \mathbf{Z}$. Since $\bar{s}_l = \dfrac{1}{(\bar{s}_{l-1} + \bar{a}_l)}$, $l \in \mathbf{Z}$, it follows from the above equations, Theorem 1.3.1 and Corollary 1.3.6 in [3] that $(\bar{s}_l)_{l\in\mathbf{Z}}$ is a strictly stationary Ω-valued Markov process on $(I^2, \mathcal{B}_I^2, \bar{\gamma})$ with the following transition mechanism: from state $\bar{s} \in \Omega$ the possible transitions are to any state $1/(\bar{s}+i)$ with corresponding transition probability $p_i(\bar{s})$, $i \in \mathbb{N}_+$. Clearly, for any $l \in \mathbf{Z}$ we have

$$\bar{\gamma}(\bar{s}_l < x) = \bar{\gamma}(\bar{s}_0 < x) = \bar{\gamma}(I \times [0,x]) = \gamma([0,x]), \; x \in I.$$

Motivated by (2.1), we shall consider the family of probability measures $(\gamma_a)_{a\in I}$ on $\mathcal{B}_I$ defined by their distribution functions

$$\gamma_a([0,x]) = \frac{(a+1)x}{ax+1}, \; x \in I, \; a \in I.$$

In particular, $\gamma_0 = \lambda$. For any $a \in I$ put $s_0^a = a$ and

$$s_n^a = \frac{1}{s_{n-1}^a + a_n}, \quad n \in \mathbb{N}_+.$$

By (2.2), $(s_n^a)_{n\in\mathbb{N}}$, $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$, is an I-valued Markov chain on $(I, \mathcal{B}_I, \gamma_a)$ which starts at $s_0^a = a$ and has the following transition mechanism: from state $s \in I$ the only possible transition are those to states $\dfrac{1}{s+i}$, $i \in \mathbb{N}_+$, the transition probability from s to $\dfrac{1}{s+i}$ being $p_i(s)$, $i \in \mathbb{N}_+$. Clearly, $(s_n^a)_{n\in\mathbb{N}}$ under γ_a is a version of $(\bar{s}_n)_{n\in\mathbb{N}}$ under $\bar{\gamma}(\cdot|\bar{s}_0 = a)$ for any $a \in \Omega$.

Next, the process $(\bar{r}_l)_{l\in\mathbf{Z}}$ is a strictly stationary Ω'-valued Markov process on $(I^2, \mathcal{B}_I^2, \bar{\gamma})$, where Ω' = the set of irrational numbers in $[1,\infty)$. The transition mechanism of $(\bar{r}_l)_{l\in\mathbf{Z}}$ is as follows: from $\bar{r} \in \Omega'$ the only possible transition is to $\dfrac{1}{\tau(\bar{r}^{-1})}$. Obviously, for any $l \in \mathbf{Z}$,

$$\bar{\gamma}(\bar{r}_l < x) = \bar{\gamma}(\bar{r}_1 < x) = \gamma(r_1 < x), \quad x \in [1,\infty).$$

Finally, the process $(\bar{s}_{l-1}, \bar{r}_l^{-1})_{l\in\mathbf{Z}}$ is a strictly stationary Ω^2-valued Markov process on $(I^2, \mathcal{B}_I^2, \bar{\gamma})$ with the following transition mechanism: from $(\bar{s}, \omega) \in \Omega^2$ the only possible trasition is to

$$\bar{\tau}^{-1}(\bar{s}, \omega) = \left(\frac{1}{\bar{s} + [\omega^{-1}]}, \ \omega^{-1} - [\omega^{-1}] \right).$$

We have

$$\bar{\gamma}(\bar{s}_{l-1} < x, \bar{r}_l^{-1} < y) = \bar{\gamma}(\bar{s}_0 < x, \bar{r}_1^{-1} < y) =$$

$$= \bar{\gamma}([0,y] \times [0,x]) = \frac{1}{\log 2} \int_0^y \int_0^x \frac{dudv}{(uv+1)^2} = \frac{\log(xy+1)}{\log 2}, \ x, y \in I.$$

3 A Two-Dimensional Wirsing Method

In the sequel we shall develop in the two-dimensional case a technique used by Wirsing [6] to give a solution to Gauss' problem.

Let $\bar{\mu}$ be an arbitrary probability measure on $\mathcal{B}_I^2$. For $k, l \in \mathbf{Z}$ put

$$F_{k,l}(x, y) = \bar{\mu}(\bar{s}_k < x, \ \bar{r}_l^{-1} < y).$$

Since

$$(\bar{s}_k < x, \ \bar{r}_l^{-1} < y) = \left(\frac{1}{\bar{a}_k + \bar{s}_{k-1}} < x, \frac{1}{\bar{a}_{l+1} + \bar{r}_{l+1}^{-1}} < y \right)$$

it follows that

$$\begin{aligned} F_{k-1,l+1}(x,y) &= \sum_{i,j\in \mathrm{N}_+} \left(F_{k,l}\left(\frac{1}{i}, \frac{1}{j}\right) - F_{k,l}\left(\frac{1}{x+i}, \frac{1}{j}\right) - \right. \\ &- \left. F_{k,l}\left(\frac{1}{i}, \frac{1}{y+j}\right) + F_{k,l}\left(\frac{1}{x+i}, \frac{1}{y+j}\right) \right). \end{aligned}$$

Assuming that $F_{k,l} \in C^2(I \times I)$ and putting $f_{k,l}(x,y) = \dfrac{\partial^2 F_{k,l}}{\partial x \partial y}(x,y)$, by the above equation we obtain

$$(3.1) \qquad f_{k-1,l+1}(x,y) = \sum_{i,j\in \mathrm{N}_+} \frac{1}{(x+i)^2(y+j)^2} f_{k,l}\left(\frac{1}{x+i}, \frac{1}{y+j}\right),$$

i.e. $f_{k-1,l+1} = T\, f_{k,l}$, where

$$T\, f(x,y) = \sum_{i,j\in \mathrm{N}_+} \frac{1}{(x+i)^2(y+j)^2} f\left(\frac{1}{x+i}, \frac{1}{y+j}\right).$$

Let us consider for $k, l \in \mathbf{Z}$

$$g_{k-1,l+1}(x,y) = (x+1)(y+1) f_{k-1,l+1}(x,y), \quad x, y \in I.$$

Then it follows from (3.1) that

$$g_{k-1,l+1}(x,y) = \sum_{i,j\in \mathrm{N}_+} \frac{(x+1)(y+1)}{(x+i)(x+i+1)(y+j)(y+j+1)} g_{k,l}\left(\frac{1}{x+i}, \frac{1}{y+j}\right),$$

i.e.

$$g_{k-1,l+1} = U g_{k,l}, \quad k,l \in \mathbf{Z}, \tag{3.2}$$

where

$$Ug(x,y) = \sum_{i,j\in \mathrm{N}_+} \frac{x+1}{(x+i)(x+i+1)} \frac{y+1}{(y+j)(y+j+1)} g\left(\frac{1}{x+i}, \frac{1}{y+j}\right).$$

We assume that the domain of U is $C^2(I\times I)$. We may write the operator U as

$$Ug(x,y) = \sum_{i,j\in \mathrm{N}_+} p_{ij}(x,y) g(u_{ij}(x,y)),$$

where

$$p_{ij}(x,y) = p_i(x)p_j(y)$$

with p_i defined previously, and such that $\sum_{i,j\in \mathrm{N}_+} p_{ij}(x,y) = 1$, and

$$u_{ij}(x,y) = (u_i(x), u_j(y)) = \left(\frac{1}{x+i}, \frac{1}{y+j}\right).$$

Then

$$\begin{aligned} Ug(x,y) &= \sum_{i,j\in \mathrm{N}_+} p_i(x)p_j(y)g(u_i(x),u_j(y)) = \\ &= \sum_{j\in \mathrm{N}_+} p_j(y) \sum_{i\in \mathrm{N}_+} p_i(x) g(u_i(x),u_j(y)) = \sum_{j\in \mathrm{N}_+} p_j(y) U_j g(x,y), \end{aligned}$$

where

$$U_j g(x,y) = \sum_{i\in \mathrm{N}_+} p_i(x) g(u_i(x),u_j(y)) =$$

$$= \sum_{i\in \mathrm{N}_+} \left(\frac{i}{x+i+1} - \frac{i-1}{x+i}\right) g(u_i(x),u_j(y)).$$

We have

$$\frac{\partial Ug}{\partial x}(x,y) = \sum_{j\in \mathrm{N}_+} p_j(y)\cdot \frac{\partial U_j g}{\partial x}(x,y),$$

where

$$\begin{aligned} \frac{\partial U_j g}{\partial x}(x,y) &= -\sum_{i\in \mathrm{N}_+} \left[\left(\frac{i}{(x+i+1)^2} - \frac{i-1}{(x+i)^2}\right) g(u_i(x),u_j(y)) + \right. \\ &+ \left. \frac{x+1}{(x+i)^3(x+i+1)} \cdot \frac{\partial g}{\partial u_i}(u_i(x),u_j(y))\right], \end{aligned}$$

since the series of derivatives is uniformly convergent. Then

$$\frac{\partial^2 Ug}{\partial y \partial x}(x,y) = \sum_{j \in \mathbb{N}_+} \left[p_j'(y) \frac{\partial U_j g}{\partial x}(x,y) + p_j(y) \frac{\partial^2 U_j g}{\partial y \partial x}(x,y) \right].$$

Since

$$p_j'(y) = -\left(\frac{j}{(y+j+1)^2} - \frac{j-1}{(y+j)^2} \right)$$

and

$$\begin{aligned} \frac{\partial^2 U_j g}{\partial y \partial x}(x,y) &= -\sum_{i \in \mathbb{N}_+} \left[\left(\frac{i}{(x+i+1)^2} - \frac{i-1}{(x+i)^2} \right) \frac{\partial g}{\partial u_j}(u_i(x), u_j(y)) + \right. \\ &+ \left. \frac{x+1}{(x+i)^3(x+i+1)} \cdot \frac{\partial^2 g}{\partial u_j \partial u_i}(u_i(x), u_j(y)) \right] \cdot \frac{\partial u_j}{\partial y}(y) \end{aligned}$$

it follows that

$$\frac{\partial^2 Ug}{\partial y \partial x}(x,y) = \sum_{i,j} \frac{i}{(x+i+1)^2} \cdot \frac{j}{(y+j+1)^2} [g(u_i(x), u_j(y)) - g(u_{i+1}(x), u_j(y))] -$$

$$-\sum_{i,j} \frac{i}{(x+i+1)^2} \cdot \frac{j}{(y+j+1)^2} [g(u_i(x), u_{j+1}(y)) - g(u_{i+1}(x), u_{j+1}(y))] -$$

$$-\sum_{i,j} \frac{x+1}{(x+i)(x+i+1)} \cdot \frac{j}{(y+j+1)^2} \left[\frac{\partial g}{\partial x}(u_i(x), u_j(y)) - \frac{\partial g}{\partial x}(u_i(x), u_{j+1}(y)) \right] -$$

$$-\sum_{i,j} \frac{y+1}{(y+j)(y+j+1)} \cdot \frac{i}{(x+i+1)^2} \left[\frac{\partial g}{\partial y}(u_i(x), u_j(y)) - \frac{\partial g}{\partial y}(u_{i+1}(x), u_j(y)) \right] +$$

$$+\sum_{i,j} \frac{x+1}{(x+i)(x+i+1)} \cdot \frac{y+1}{(y+j)(y+j+1)} \cdot \frac{\partial^2 g}{\partial y \partial x}(u_i(x), u_j(y)).$$

We can write $\frac{\partial^2 Ug}{\partial y \partial x}(x,y)$ as

$$\begin{aligned} &\frac{\partial^2 Ug}{\partial y \partial x}(x,y) = \\ &= \sum_{i,j} ij u_{i+1}'(x) u_{j+1}'(y) \int_{u_{i+1}(x)}^{u_i(x)} \left[\frac{\partial g}{\partial u}(u, u_j(y)) - \frac{\partial g}{\partial u}(u, u_{j+1}(y)) \right] du + \\ &+ \sum_{i,j} i p_j(y) u_{i+1}'(x) \int_{u_{i+1}(x)}^{u_i(x)} \frac{\partial}{\partial u} \left(\frac{\partial g}{\partial y}(u, u_j(y)) \right) du + \\ &+ \sum_{i,j} j p_i(x) u_{j+1}'(y) \int_{u_{j+1}(y)}^{u_j(y)} \frac{\partial}{\partial v} \left(\frac{\partial g}{\partial x}(u_i(x), v) \right) dv + \\ &+ \sum_{i,j} p_i(x) p_j(y) \frac{\partial^2 g}{\partial x \partial y}(u_i(x), u_j(y)). \end{aligned}$$

Since $g \in C^2(I \times I)$ we have

$$g(u, u_j(y)) - g(u, u_{j+1}(y)) = \frac{\partial g}{\partial v}(u, v)\Big|_{(u, c_j)} (u_j(y) - u_{j+1}(y))$$

where $c_j \in (u_{j+1}(y), u_j(y))$. Thus, we can write

$$\frac{\partial^2 U g}{\partial x \partial y}(x, y) = V \frac{\partial^2 g}{\partial x \partial y}(x, y), \quad g \in C^2(I \times I), \tag{3.3}$$

where $V : C^0(I \times I) \to C^0(I \times I)$ is defined as

$$\begin{aligned} Vg(x,y) &= \sum_{i,j} i j u'_{i+1}(x) u'_{j+1}(y) \int_{u_{i+1}(x)}^{u_i(x)} g(x,y)|_{y=c_j} (u_j(y) - u_{j+1}(y)) dx + \\ &+ \sum_{i,j} i p_j(y) u'_{i+1}(x) \int_{u_{i+1}(x)}^{u_i(x)} g(x, u_j(y)) dx + \\ &+ \sum_{i,j} j p_i(x) u'_{j+1}(y) \int_{u_{j+1}(y)}^{u_j(y)} g(u_i(x), y) dy + \\ &+ \sum_{i,j} p_i(x) p_j(y) g(u_i(x), u_j(y)). \end{aligned}$$

Clearly,

$$\frac{\partial^2 U^n g}{\partial x \partial y}(x, y) = V^n \frac{\partial^2 g}{\partial x \partial y}(x, y), \tag{3.4}$$

for all $n \in \mathbb{N}_+$ and $g \in C^2(I \times I)$.

We are going to show that V^n takes certain functions into functions with very small values when $n \in \mathbb{N}_+$ is large.

Proposition 3.1. *There are positive constants* $v > 0.02036$ *and* $w < 0.17319$, *and a real-valued function* $\psi \in C^0(I \times I)$ *such that*

$$v\psi \leq V\psi \leq w\psi.$$

Proof. Let us write

$$Ug(x, y) = \sum_{i,j \in \mathbb{N}_+} p_i(x) p_j(y) g(u_i(x), u_j(y)) =$$

$$= \sum_{\substack{i \in \mathbb{N}_+ \\ j = i+k, k \in \mathbb{N}}} p_i(x) p_{i+k}(y) g(u_i(x), u_{i+k}(y)) = \sum_{k \in \mathbb{N}} U_k g(x, y),$$

where

$$U_k g(x, y) = \sum_{i \in \mathbb{N}_+} p_i(x) p_{i+k}(y) g(u_i(x), u_{i+k}(y)).$$

Let $h_k : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$, $k \in \mathbb{N}$, be a $C^2(\mathbf{R}_+ \times \mathbf{R}_+)$-function such that

$$\lim_{\substack{x\to\infty\\ y\to\infty}} \frac{h_k(x,y)}{(x+1)(y+1)} = 0.$$

We look for a function $g : (0,1] \times (0,1] \to \mathbf{R}$ such that $U_k g = h_k$, assuming that the equation

$$U_k g(x,y) = h_k(x,y) \tag{3.5}$$

holds for $x, y \in \mathbf{R}_+$. Then (3.5) yields

$$\frac{h_k(x,y)}{(x+1)(y+1)} - \frac{h_k(x+1,y+1)}{(x+2)(y+2)} =$$

$$= \frac{1}{(x+1)(x+2)(y+k+1)(y+k+2)} g\left(\frac{1}{x+1}, \frac{1}{y+k+1}\right)$$

for $x, y \in \mathbf{R}_+$. Since $\frac{1}{x+1} = u \in (0,1]$ and $\frac{1}{y+1} = v \in (0,1]$ it follows that

$$g\left(u, \frac{v}{1+kv}\right) = \frac{1+kv}{v}\left(\frac{1+kv}{v}+1\right)\cdot$$

$$\cdot\left[h_k\left(\frac{1}{u}-1, \frac{1}{v}-1\right)\left(\frac{1}{u}+1\right)v - h_k\left(\frac{1}{u}, \frac{1}{v}\right)\frac{1}{u}\left(1-\frac{1}{1+v}\right)\right].$$

Hence

$$g(u,v) = \left(\frac{1}{v}+1\right)\left[h_k\left(\frac{1}{u}-1, \frac{1-kv}{v}-1\right)\left(\frac{1}{u}+1\right)\frac{1}{1-kv} - \right.$$

$$\left. - h_k\left(\frac{1}{u}, \frac{1-kv}{v}\right)\frac{1}{u}\frac{1}{1-(k-1)v}\right]$$

for $u, v \in (0,1]$ and we indeed have $U_k g = h_k$ since

$$U_k g(x,y) = (x+1)(y+1)\sum_{i\in\mathbb{N}_+}\left[\frac{h_k(x+i-1,y+i-1)}{(x+i)(y+i)} - \frac{h_k(x+i,y+i)}{(x+i+1)(y+i+1)}\right] =$$

$$= (x+1)(y+1)\left[\frac{h_k(x,y)}{(x+1)(y+1)} - \lim_{i\to\infty}\frac{h_k(x+1,y+1)}{(x+i+1)(y+i+1)}\right] = h_k(x,y),$$

for $x, y \in \mathbf{R}_+$.

In particular, for any fixed $a \in I$ we consider the function $h_k^a : \mathbf{R}_+ \times \mathbf{R}_+ \to \to \mathbf{R}$ defined as

$$h_k^a(x,y) = \frac{y+1}{x+a+1}\left(\frac{1}{y+k+2} - \frac{1}{y+k+3}\right).$$

By the above, the function $g^a : (0,1] \times (0,1] \to \mathbf{R}$ defined as

$$g^a(x,y) = \frac{y+1}{2y+1}\left[\frac{x+1}{ax+1}\cdot\frac{1}{y+1} - \frac{1}{(a+1)x+1}\cdot\frac{1}{3y+1}\right]$$

satisfies

$$Ug^a(x,y) = \sum_{k\in\mathrm{N}} U_k g^a(x,y) = \sum_{k\in\mathrm{N}} h_k^a(x,y) = h^a(x,y),\ x,y \in I,$$

where

$$h^a(x,y) = \frac{y+1}{(x+a+1)(y+2)},\ x,y \in I.$$

We come to V via (3.3). Setting

$$\varphi^a(x,y) = \frac{\partial^2 g^a}{\partial x \partial y}(x,y) =$$

$$= -\frac{1}{(2y+1)^2}\left[\frac{1-a}{(ax+1)^2}\frac{1}{y+1} + \frac{a+1}{((a+1)x+1)^2}\frac{1}{3y+1}\right] -$$

$$-\frac{y+1}{2y+1}\left[\frac{1-a}{(ax+1)^2}\frac{1}{(y+1)^2} + \frac{a+1}{((a+1)x+1)^2}\frac{3}{(3y+1)^2}\right]$$

we have

$$V\varphi^a(x,y) = \frac{\partial^2(Ug^a)}{\partial x\partial y}(x,y) = -\frac{1}{(x+a+1)^2(y+2)^2},\ x,y \in I.$$

Let us choose a by asking that

$$\frac{\varphi^a}{V\varphi^a}(0,1) = \frac{\varphi^a}{V\varphi^a}(1,1).$$

This amounts to

$$16(a-1)(a^2+a-1)(a^2+3a+3) = 11a(a+1)^3(a+2)$$

or

$$5a^5 - 7a^4 - 83a^3 - 157a^2 - 70a + 48 = 0,$$

which yields as unique acceptable solution

$$a = 0.352972898\ldots.$$

For this value of a, the function $\frac{\varphi^a}{V\varphi^a}$ attains its maximum equal to

$$M(a) = 8(a+1)^2(a+3) = 49.10189173\ldots$$

at $(x,y) = (0,0)$, and has a minimum equal to

$$m(a) = 5.774226917\ldots$$

at $(x,y) = (0,1)$ and at $(x,y) = (1,1)$.

Since $\varphi^a(x,y) < 0$ and $V\varphi^a(x,y) < 0$ for all $x,y \in I$, it follows that for $\psi = -\varphi^a$ with $a = 0.352972898\ldots$ we have

$$\frac{\psi}{M(a)} \le V\psi \le \frac{\psi}{m(a)},$$

that is,

$$v\psi \leq V\psi \leq w\psi,$$

where

$$v = \frac{1}{M(a)} > 0.02036, \; w = \frac{1}{m(a)} < 0.17319.$$

□

Corollary 3.2. *Let* $g \in C^2(I \times I)$ *such that* $\dfrac{\partial^2 g}{\partial x \partial y} > 0$. *Put*

$$\alpha = \min_{(x,y)\in I\times I} \frac{\psi(x,y)}{\dfrac{\partial^2 g}{\partial x \partial y}(x,y)} \quad \textit{and} \quad \beta = \max_{(x,y)\in I\times I} \frac{\psi(x,y)}{\dfrac{\partial^2 g}{\partial x \partial y}(x,y)}.$$

Then

$$\frac{\alpha}{\beta} v^n \frac{\partial^2 g}{\partial x \partial y} \leq V^n \frac{\partial^2 g}{\partial x \partial y} \leq \frac{\beta}{\alpha} w^n \frac{\partial^2 g}{\partial x \partial y}. \tag{3.6}$$

Proof. Since V is a positive operator (that is, takes non-negative functions into non-negative functions) we have

$$v^n \psi \leq V^n \psi \leq w^n \psi, \quad n \in \mathbb{N}_+.$$

Noting that

$$\alpha \frac{\partial^2 g}{\partial x \partial y} \leq \psi \leq \beta \frac{\partial^2 g}{\partial x \partial y},$$

we then can write

$$\frac{\alpha}{\beta} v^n \frac{\partial^2 g}{\partial x \partial y} \leq \frac{1}{\beta} v^n \psi \leq \frac{1}{\beta} V^n \psi \leq V^n \frac{\partial^2 g}{\partial x \partial y} \leq \frac{1}{\alpha} V^n \psi \leq$$

$$\leq \frac{1}{\alpha} w^n \psi \leq \frac{\beta}{\alpha} w^n \frac{\partial^2 g}{\partial x \partial y}, \quad n \in \mathbb{N}_+ ,$$

which shows that (3.6) holds.

□

4 A Solution to Gauss' Problem

Let $\bar{\mu}$ be a probability measure on $\mathcal{B}_I^2$, absolutely continuous with respect to $\bar{\lambda} =$ Lebesgue measure on $\mathcal{B}_I^2$. Now, using the notation in Section 3, let us fixed $k, l \in \mathbf{Z}$ such that

$$g_{k,l}(x,y) = (x+1)(y+1) \frac{\partial^2 F_{k,l}}{\partial x \partial y}(x,y)$$

satisfies $\dfrac{\partial^2 g_{k,l}}{\partial x \partial y}(x,y) > 0$, $x, y \in I$.

It follows from (3.2) that

$$g_{k-n,l+n} = U^n g_{k,l}, \quad k,l \in \mathbf{Z}, \; n \in \mathbb{N}.$$

Then

$$F_{k-n,l+n}(x,y)=\bar{\mu}(\bar{s}_{k-n}<x,\bar{r}_{l+n}^{-1}<y)=\int_0^x\int_0^y\frac{U^n g_{k,l}}{(u+1)(v+1)}dudv, \tag{4.1}$$

with $k,l\in\mathbf{Z}$, $n\in\mathbb{N}$ and $x,y\in I$, where $\dfrac{\partial^2 F_{k,l}}{\partial x\partial y}=\dfrac{d\bar{\mu}}{d\bar{\lambda}}$.

Putting $G(x)=\gamma([0,x])$, we obtain the following solution to Gauss problem.

Theorem 4.1. *Let* $g_{k,l}\in C^2(I\times I)$, $k,l\in\mathbf{Z}$, *such that* $\dfrac{\partial^2 g_{k,l}}{\partial x\partial y}>0$. *For any* $n\in\mathbb{N}_+$ *and* $x,y\in I$ *we have*

$$\frac{(\log 2)^4\alpha \displaystyle\min_{(x,y)\in I\times I}\frac{\partial^2 g_{k,l}}{\partial x\partial y}(x,y)}{4\beta}v^n G(x)G(y)(G(x)G(y)-1)\leq$$
$$\leq\left|\bar{\mu}(\bar{s}_{k-n}<x,\bar{r}_{l+n}^{-1}<y)-G(x)G(y)\right|\leq$$
$$\leq\frac{(\log 2)^4\beta \displaystyle\max_{(x,y)\in I\times I}\frac{\partial^2 g_{k,l}}{\partial x\partial y}(x,y)}{\alpha}w^n G(x)G(y)(G(x)G(y)-1),$$

where α,β,v *and* w *are defined in Proposition 3.1 and Corollary 3.2.*

In particular, for any $n\in\mathbb{N}_+$ *and* $x,y\in I$ *we have*

$$0.000118302v^nG(x)G(y)(G(x)G(y)-1)\leq$$
$$\leq\left|\bar{\mu}(\bar{s}_{k-n}<x,\bar{r}_{l+n}^{-1}<y)-G(x)G(y)\right|\leq$$
$$\leq 0.142502939w^nG(x)G(y)(G(x)G(y)-1).$$

Proof. For all $n\in\mathbb{N}$, $x,y\in I$ and $k,l\in\mathbf{Z}$ fixed, set

$$d_n(G(x)G(y))=\bar{\mu}(\bar{s}_{k-n}<x,\bar{r}_{l+n}^{-1}<y)-G(x)G(y).$$

Then by (4.1) we have

$$d_n(G(x)G(y))=\int_0^x\int_0^y\frac{U^n g_{k,l}(u,v)}{(u+1)(v+1)}dudv-G(x)G(y).$$

Differentiating with respect to x and y yields

$$\frac{d_n''(G(x)G(y))G(x)G(y)+d_n'(G(x)G(y))}{(\log 2)^2}=U^n g_{k,l}-\frac{1}{(\log 2)^2}.$$

Differentiating again with respect to x and y we get

$$\frac{d_n^{iv}(G(x)G(y))G^2(x)G^2(y)+4d_n'''(G(x)G(y))G(x)G(y)+2d_n''(G(x)G(y))}{(\log 2)^4(x+1)(y+1)}= \tag{4.2}$$
$$=\frac{\partial^2 U^n g_{k,l}}{\partial x\partial y}(x,y)$$

$n \in \mathbb{N}$, $x, y \in I$. Since $d_n(0) = d_n(1) = 0$, it follows from a well-known interpolation formula that

$$d_n(u) = \frac{u(u-1)}{2} d_n''(\theta), \ n \in \mathbb{N}, \ u \in I,$$

for a suitable $\theta = \theta(n, u) \in I$. Therefore, from (3.3), (4.2) becomes

$$d_n''(G(x), G(y)) = \frac{(\log 2)^4}{2}(x+1)(y+1)V^n \frac{\partial^2 g_{k,l}}{\partial x \partial y}(x, y),$$

which implies

$$d_n(G(x)G(y)) =$$
$$\frac{(\log 2)^4}{2}(\theta'+1)(\theta''+1)V^n \frac{\partial^2 g_{k,l}}{\partial x \partial y}(\theta', \theta'')G(x)G(y)(G(x)G(y)-1)$$

for any $n \in \mathbb{N}$ and $x, y \in I$, and for suitable $\theta' = \theta'(n, x) \in I$ and $\theta'' = \theta''(n, y) \in I$. The result stated follows now from Corollary 3.2.

In the special case $\bar{\mu} = \bar{\lambda}$ we have $g_{k,l}(x, y) = (x+1)(y+1)$, $x, y \in I$. Then, with $a = 0.352972898\ldots$ we have

$$\alpha = \min_{(x,y)\times I \times I} \frac{\psi(x,y)}{\frac{\partial^2 g_{k,l}}{\partial x \partial y}(x,y)} = -\varphi^a(1,1) = 0.386433943\ldots,$$

$$\beta = \max_{(x,y)\times I \times I} \frac{\psi(x,y)}{\frac{\partial^2 g_{k,l}}{\partial x \partial y}(x,y)} = -\varphi^a(0,0) = 6.705945796\ldots,$$

so that

$$\frac{(\log 2)^4 \alpha}{4\beta} = 0.000118302\ldots, \quad \frac{(\log 2)^4 \beta}{\alpha} = 0.142502939\ldots.$$

The proof is complete.

□

5 Concluding Remarks

Theorem 4.1 points out to a kind of asymptotic stochastic independence of $\bar{s}_{k-n}$ and $\bar{r}_{l+n}^{-1}$ under $\bar{\mu}$.

It is interesting to compare this result with Corollary 3 in [2] which shows that for any probability measure μ on $\mathcal{B}_I$, absolutely continuous with respect to λ

$$\mu(s_n \leq x, \tau^0 \leq) \to G(x)\mu([0, y]) = \frac{\log(x+1)}{\log 2}\mu([0, y]),$$

as $n \to \infty$. Moreover, Corollary 2′ in [2] shows that

$$\gamma_a(s_n \leq x, \tau^0 \leq y) \to G(x)\gamma_a([0, y]) = \frac{\log(x+1)}{\log 2}\gamma_a([0, y]),$$

as $n \to \infty$. Again, we have a kind of asymptotic stochastic independence of s_n^a and τ^0 under γ_a for any given $a \in I$, while the following relation in [1]

$$\gamma_a(s_n^a \le x, \tau^n \le y) \to \bar{\gamma}([0,x] \times [0,y]) = \frac{\log(xy+1)}{\log 2},$$

as $n \to \infty$, does not allow for such an interpretation.

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