Some properties of distinguished vector fields on Riemannian manifolds

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Abstract

The vector fields play an important role in Riemannian (or pseudo-Riemannian) manifolds. In literature it is known the concept of covariant cohomology operator ∇^{α}, where ∇ means the Levi-Civita connection and α a closed Pfaffian. In this note we prove that the properties of torse vector fields [1, 3, 9, 8] and quasi-exterior concurrent vector fields are invariant under the action of ∇^{α}.

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Let $\flat : TM \to T^*M$ be the musical isomorphism defined by the metric tensor g and $\sharp$ the inverse of $\flat$. Following W. Poor [4], let

$$A^q(M, TM) = \Gamma \mathrm{Hom}(\Lambda^q TM, TM)$$

be the set of vector valued q-forms and by

$$d^{\nabla} = A^q(M, TM) \to A^{q+1}(M, TM)$$

the exterior covariant derivative with respect to ∇. Notice that in general $d^{\nabla^2} = d^{\nabla} \circ d^{\nabla} \neq 0$ unlike $d \circ d = 0$.

Next $dp \in A^1(M, TM)$ stands for the soldering form of M (dp is the canonical vector valued 1-form and $d^{\nabla}(dp) = 0$).

The cohomology operator d^{ω} is following [2] defined by

$$d^{\omega} = \alpha + e(\omega),$$

where d is the exterior differentiation, and one has

$$d^{\omega} \circ d^{\omega} = 0.$$

Any form $u \in \Lambda M$ satisfying $d^{\omega}u = 0$ is said to be d^{ω}-closed and ω is called the cohomology form. If u is an exact form, then u is said to be d^{ω}-exact. Among the most remarkable vector fields we point out the following:

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(i) the Killing vector fields X, which satisfy the *Killing equation* ([3, 8])

$$\langle \nabla_u X, \nabla \rangle + \langle u, \nabla_v X \rangle = 0, \qquad u, v \in \mathcal{X}(M).$$

(ii) affine vector fields X:

$$\mathcal{L}_X \nabla = 0,$$

where $\mathcal{L}_X = i \circ d + d \circ i$ is the Lie differentiation.

(iii) torse forming vector fields X ([9]),

$$\nabla X = -sdp + u \otimes X, \qquad s \in C^\infty M; u \in \Lambda^2 M.$$

(iv) quasi-exterior concurrent vector fields X ([6]),

$$\nabla^2 X = v \wedge dp + \varphi \otimes X; \qquad v \in \Lambda^2 M, \varphi \in \Lambda^\alpha M$$

(v) exterior concurrent vector fields ([6, 5]),

$$\nabla^2 X = \mu \wedge dp \qquad \mu \in \Lambda^1 M.$$

Accordingly one has

$$\nabla^2 X = f X^\flat \wedge dp \quad \Rightarrow \quad f = -\frac{1}{n-1} Ric(X),$$

where $n = \dim M$ and $Ric(X)$ denotes the Ricci curvature of M with respect to X.

Let $\mathcal{O} = Span\{e_A \mid A = 1, ..., n\}$ be the local field of adapted vectorial frames over M and let

$$\mathcal{O}^A = Span\{\omega^A\}$$

be the associated coframe. Then the Cartan structure equation written in indexless form are

$$\begin{aligned} a)\ \nabla e &= \theta \otimes e \\ b)\ d\omega &= -\theta \wedge \omega \\ c)\ d\theta &= -\theta \wedge \theta + \Theta. \end{aligned}$$

In the above equation θ (resp. Θ) are the local connection forms in the tangent bundle of M (resp. the curvature 2-forms on M).

If ∇ is the Levi-Civita operator, then by reference to [7] we say that the operator ∇^α defines an autoparallel transformation of ∇. In other words, if X is a vector field, then one has

$$\nabla^\alpha X = \nabla X + \alpha \otimes X,$$

where $\alpha \in \Lambda^1 M$ is a closed associated Pfaffian. One may also say that ∇^α is the operator of covariant cohomology and that α is the parallelism form, and by recurrence one derives

$$\nabla^{q\alpha} X = \nabla(\nabla^{(n-1)\alpha} X) + \alpha \otimes \nabla^{(q-1)\alpha} X.$$

Assume that X is a torse forming, i.e., by reference to (iii) it satisfies

$$\nabla X = sdp + u \otimes X, \qquad \text{with } s \in \Lambda^0 M; u \in \Lambda^1 M.$$

Then applying ∇^α to X, one has

$$\nabla^\alpha X = \nabla X + \alpha \otimes X = sdp + (\alpha + u) \otimes X,$$

and setting $\nabla^x X = \nabla X'$ one gets, after a short calculation,

$$\nabla X' = s' dp + u' \otimes X',$$

which shows that X' is an other torse-forming vector field.

Hence one may say that the property of torse-forming is invariant by the coparellel operator ∇^α.

Consider now the quasi-exterior concurrent vector field, i.e.,

$$\nabla^2 X = v \wedge dp + \varphi \otimes X.$$

Applying the operator $\nabla^{2\alpha}$, from (1.6) one derives

$$\nabla^{2\alpha} X = v \wedge dp + (2\alpha + \varphi) \otimes X$$

and this shows that the property of quasi-exterior concurrence is an invariant of the second operator $\nabla^{2\alpha}$.

Further applying the tensor curvature $\mathcal{R}$ one deduces after some calculations:

$$\mathcal{R}^\alpha (Z, Z') X = f (Z \wedge Z') \otimes X.$$

$Z, Z' \in \mathcal{X}(M)$, $f \in \Lambda^0 M$, $\wedge$, is the skew symmetric operator on vector fields, i.e.,

$$Z \wedge Z' = (Z')^\flat \otimes Z - Z^\flat \otimes Z'.$$

Then we have the following

Theorem. *Let M be a Riemannian manifold, and let ∇^α be the coparellel operator acting on M. Then the property of torse-forming vector field is invariant by ∇^α and the property of quasi-exterior concurrence for a vector field is invariant by the operator $\nabla^{2\alpha}$. Moreover, if R is the tensor curvature on M, then*

$$\mathcal{R}^\alpha (Z, Z') X = f (Z \wedge Z') \otimes X,$$

where $X, Z, Z' \in \mathcal{X}(M)$, $f \in \Lambda^0 M$ and $\wedge$ is the skew symmetric operator on vector fields, i.e.,

$$Z \wedge Z' = (Z')^\flat \otimes Z - Z^\flat \otimes Z'.$$

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