A Note on Pappus′ Theorem

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Abstract

Let Π be a projective plane coordinatized by a ternary ring $(\mathcal{R}, T)$. Using the notations of [1], three operations $+$, $\cdot$ and $*$ defined by $a + b = T(1, a, b)$, $a \cdot b = T(a, b, 0)$, $a * b = T(a, 1, b), \forall a, b \in R$, where $(x, y) \circ [m, k] \Leftrightarrow T(m, x, y) = k$, $\forall m, x, y, k \in R$. In this paper, we give two configurational characterisations for $(R, *)$ to be Abelian group, using involutory perspectivities.

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§1. Introduction

In 1959, Pickert [3] proved that if the Pappus Theorem holds in the projective plane for two fixed base lines, then the plane is Pappian. In 1966, Buekenhout [4] proved that if Pascal′s Theorem holds for a single oval in a projective plane, then the plane is Pappian. In [1], he exploited the analogy between these two results by using Buekenhout′s methods to prove Pickert′s Theorem. This result was achieved in [1, Chapter III], through in a slightly weaker form than that obtained by Pickert. Basic definition and theorems on projective planes may be found in [2] or [9].

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane and coordinated by a set R $(0, 1 \in R)$ with respect to a coordinatining quadrangles (X, Y, O, E). That is for any two points $A, B \in \mathcal{P}$ the line joining A and B is denoted by $A \vee B$ or AB and for any two lines $u, v \in \mathcal{L}$, the intersection point is denoted by $u \wedge v$ or uv. The points on XY (distinct from Y) are coordinatized as $(m), m \in R$, the points not on XY as $(x, y), x, y \in R$. So we may say $O = (0, 0), E = (1, 1)$ $X = (0)$. Let us say $Y = (\infty)$, $\infty \notin R$. The lines through Y (distinct from XY) are coordinatized as $[a], a \in R$. Let us say $XY = [\infty]$; the lines not passing through Y, as $[m, k], m, k \in R$. Thus $(a, b) \in [a], [m, k] = (m) \vee (0, k)$. Using the notations of [1], a ternary operation T may be defined on the set R as the following [1] :

$$(x, y) \in [m, k] \Leftrightarrow T(m, x, y) = k, \text{ for all } x, y, m, k \in R.$$

The system (R, T) is called a ternary ring on π and π can be coordinatized by this ternary ring. Three different binary operations denoted by $+, ., *$ may be defined on R as follows [9]

$$a + b = T(1, a, b), a.b = T(a, b, 0) \text{ and } a * b = T(a, 1, b) \text{ for all } a, b \in R.$$

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In this paper, we give two configurational characterization for $(\mathcal{R}, *)$ to be Abelian group, using involutory perspectivities.

A (P, l) perspectivity (or central colineation), is a perspectivity with centre P and axis l, that is, a colineation which fixes all the lines through P and all the points on l. A projective plane is a said to be (P, l) transitive if the perspectivities of the plane with centre P and axis l are transitive on the points of one line through P (and hence every line through P), excluding P itself and the points of l.

(**Pappus′ Theorem**). *If u, v and w are distinct lines of a projective plane Π, then the (u, v, w)-Pappus′ Theorem states that if $A, B, C \in u$ and $A', B', C' \in v$ in such a way that*

$$AB' \wedge A'B, AC' \wedge A'C \in w \text{ then } CB' \wedge C'B \in w.$$

Although this statement is not at first sight symmetric in u, v, w, it is easy to see that the (u, v, w)-Pappus′ Theorem implies the (a, b, c)-Pappus′ Theorem, where (a, b, c) is any permuted arrangement of the lines u, v and w.

Definition. Let u and v be any two lines of a projective plane Π, for every point $P(\notin u \vee v)$ of the plane Π, we define the involutory permutation σ_P of the point set $u \vee v$ by

(i) $P,\ X,\ \sigma_P(X)$ colinear, for all $X \in u \vee v$,

(ii) $X \in u \Rightarrow \sigma_P(X) \in v, Y \in v \Rightarrow \sigma_P(Y) \in u.$

(iii) $X \in u \Rightarrow \sigma_P(X) = (P \vee X) \wedge v, Y \in v \Rightarrow \sigma_P(X) = (P \vee Y) \wedge u$

Lemma 1. ([1], Theorem 3.1) *The following statements are equivalent:*

(a) *the $(u, v, w) -$ PappusTheorem;*

(b) $(\sigma_X \sigma_Y \sigma_Z)^2 = 1$ *for any three points $X, Y, Z \in w(\notin u, v)$*

(c) $\sigma_X \sigma_Y \sigma_Z = \sigma_T$ *for some $T \in w$, where $X, Y, Z \in w(\notin u, v)$.*

Lemma 2. ([1], Theorem 3.3) *The $([0, 0], [0], [\infty])$-Pappus Theorem holds in Π if and only if $(R\backslash\{0\}, \cdot)$ is an Abelian group.*

Lemma 3. ([1], Theorem 3.6) *Provided that $a + b = T(a, 1, b)$, for all $a, b \in R$, the $([0], [1], [\infty])$-Pappus Theorem holds in Π if and only if $(R, +)$ is an Abelian group.*

§2. Main Results

Theorem 1. *The $([0], [\infty], [1])$-Pappus Theorem holds in Π if and only if $(R, *)$ is an Abelian group.*

Proof. We denote by σ_X, the permutation of the points of $[0]$ and $[\infty]$ which interchanges pairs collinear with $(1, x)$, where $x \in R$. Assume that the restricted Pappus Theorem holds true.

Let $X = (1, x), Y = (1, y), Z = (1, z)$ and $I = (1, 0), x, y, z \in R\backslash\{0\}$. Therefore

$$\begin{aligned}\sigma_X\sigma_I\sigma_Y((0)) &= \sigma_X\sigma_I(((1, y) \vee (0)) \wedge [0]) = \sigma_X\sigma_I((0, y)) \\ &= \sigma_X(((0, y) \vee (1, 0)) \wedge [\infty]) = \sigma_X((y)) \\ &= ((y) \vee (1, x)) \wedge [0] = (0, y * x).\end{aligned}$$

Similarly, $\sigma_Y\sigma_I\sigma_X((0)) = (0, x * y)$. By Lemma 1, $x * y = y * x\ \forall x, y \in R$. Thus $(R, *)$ is commutative. Again,

$$\sigma_Z\sigma_I\sigma_Y\sigma_I\sigma_X((0)) = \sigma_Z\sigma_I(0, x * y) = \sigma_Z((x * y)) = (0, (x * y) * z)$$

and

$$\sigma_Y\sigma_I\sigma_Z\sigma_I\sigma_X((0)) = \sigma_Y\sigma_I(0, x * z) = \sigma_Y((x * z)) = (0, (x * z) * y).$$

By Lemma 1, $\sigma_Z\sigma_I\sigma_Y\sigma_I\sigma_X = \sigma_Y\sigma_I\sigma_Z\sigma_I\sigma_X$ and therefore $(x * y) * z = (x * z) * y$. But by the commutativity just proved, this becomes $(x * y) * z = x * (y * z)$, namely the associative law. So $(R, *)$ is an Abelian group.

Conversely, assume that $(R, *)$ is an Abelian group. Therefore, $\forall p \in R$,

$$\begin{aligned}\sigma_X\sigma_Y\sigma_Z((p)) &= \sigma_X\sigma_Y((p) \vee (1, z) \wedge [0]) = \sigma_X\sigma_Y((0, p * z)) \\ &= \sigma_X((0, p * z) \vee (1, y) \wedge [\infty]) = \sigma_X(((p * z) * y^{-1}) \\ &= (((p * z) * y^{-1} \vee (1, x)) \wedge [0]) = (0, (p * z) * y^{-1} * x)\end{aligned}$$

and hence

$$\begin{aligned}(\sigma_X\sigma_Y\sigma_Z)^2((p)) &= \sigma_X\sigma_Y\sigma_Z((0, (p * z) * y^{-1} * x) \\ &= \sigma_X\sigma_Y((0, (p * z) * y^{-1} * x) \vee (1, z) \wedge [\infty]) \\ &= \sigma_X\sigma_Y((z^{-1} * (p * z) * y^{-1} * x)) \\ &= \sigma_X\sigma_Y((y^{-1} * p) * x) \\ &= \sigma_X(((y^{-1} * p) * x \vee (1, y)) \wedge [0]) \\ &= ((0, p * x) \vee (1, x)) \wedge [\infty] = (p).\end{aligned}$$

Thus $(\sigma_X\sigma_Y\sigma_Z)^2 = 1$ and the restricted Pappus Theorem holds in Π by Lemma 1.□

Theorem 2. *Provided that* $a * b = T(1, a, b)$, $\forall a, b \in R$, *the* $((0), (1), (\infty))$ *dual Pappus′ Theorem holds in* π *iff* $(R, *)$ *is an Abelian group.*

Proof. We denote by σ_X, the permutation of the lines through (0) or (1) which interchanges pairs meeting on $[x]$, $x \in R\backslash\{0\}$, let σ_0 be the corresponding permutation for $[0]$.

Assuming the restricted dual Pappus′ Theorem, we have:

$$\sigma_Y\sigma_0\sigma_X([0, 0]) = \sigma_Y\sigma_0([1, x]) = \sigma_Y([0, x]) = [1, y * x].$$

In this last line we have to appeal to the special assumption $T(1, a, b) = a * b$.

Similarly,

$$\sigma_X\sigma_0\sigma_Y([0,0]) = [1, x*y].$$

By the dual of Lemma 1, $x*y = y*x$, and

$$\begin{aligned}\sigma_Z\sigma_0\sigma_Y\sigma_0\sigma_X([0,0]) &= [1, z*(y*x)]\\ \sigma_Y\sigma_0\sigma_Z\sigma_0\sigma_X([0,0]) &= [1, y*(z*x)]\end{aligned}$$

By the dual of Lemma 1, we have $z*(y*x) = y*(z*x)$. But by the commutativity just proved, this becomes $(y*x)*z = y*(x*z)$, namely the associativity. Hence $(R,*)$ is an Abelian group.

Conversely, assume that $(R,*)$ is an Abelian group. Therefore, $\forall k \in R$,

$$\begin{aligned}(\sigma_Z\sigma_Y\sigma_X)^2([0,k]) &= \sigma_Z\sigma_Y\sigma_X\sigma_Z\sigma_Y([1, x*k])\\ &= \sigma_Z\sigma_Y\sigma_X\sigma_Z[0, y^{-1}*x*k]\\ &= \sigma_Z\sigma_Y\sigma_X([1, z*y^{-1}*x*k])\\ &= \sigma_Z\sigma_Y([0, x^{-1}*z*y^{-1}*x*k])\\ &= \sigma_Z([1, y*z*y^{-1}*k]) = [0,k].\end{aligned}$$

So, we have $(\sigma_Z\sigma_Y\sigma_X)^2 = 1$, and by the dual of Lemma 1, the $((0),(1),(\infty))$ dual Pappus$'$ Theorem holds in π. It is easy to see that the $((0),(1),(\infty))$ dual Pappus Theorem implies the $((0),(\infty),(1))$ dual Pappus Theorem since $((0),(\infty),(1))$ is any permuted arrangement of the points $(0),(1)$ and (∞). □

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