The oscillation of linear first order differential systems

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Abstract

Some sufficient conditions are established for the oscillation of first order linear differential systems whose coefficients obey certain conditions. An example is given to illustrate the results.

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§1. Introduction

The purpose of this paper is to establish oscillation criteria for linear first order differential systems with variable coefficients of the form

$$\begin{cases} u'(x) = a(x)u(x) + b(x)v(x) \\ v'(x) = c(x)u(x) + d(x)v(x). \end{cases} \tag{1.1}$$

Throughout this paper we shall focus our attention only to certain restrictions on the coefficients functions of system (1.1) namely, $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are continuously differentiable functions on the interval $[\alpha, \infty)$, where α is a positive real number, such that $b(x) \neq 0$, and

$$a(x) + \frac{b'(x)}{b(x)} + d(x) > 0.$$

Many criteria have been found which involve the behavior of the integral of a combination of the coefficients. This approach has been motivated by some authors, (for example see [1,2,4,7]) and the references contained therein. An example is given in this paper showing the applicability of our theorems.

The following definitions will be used throughout this paper, which are recalled from the earlier literature.

Definitions. a) A solution $(u(x), v(x))$ of the system (1.1) is said to be *nontrivial* if $u(x) \neq 0$, for at least one $x \in [\alpha, \infty)$.

b) A nontrivial solution $(u(x), v(x))$ of system (1.1) is said to be *oscillatory*, if it has arbitrarily large zeros on $[\alpha, \infty)$, otherwise it said to be *non oscillatory*.

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§2. Main Results

We prove the following theorems

Theorem 1 *If* $b(x) \neq 0$, *and* $(a(x) + \frac{b'(x)}{b(x)} + d(x)) > 0$ *on* $[\alpha, \infty)$, *such that*

$$\lim_{x\to\infty}\left[-\frac{1}{4}\int\limits_{\alpha}^{x}\left\{(a(s)-\frac{b'(s)}{b(s)}-d(s))^2+4(a'(s)+b(s)c(s)\right\}ds\right]=\infty \tag{2.2}$$

Then the system (1.1) is oscillatory.

Proof. Suppose that system (1.1) is non oscillatory, then there exists a non trivial solution of (1.1) that has no zero on $[\beta, \infty)$ for some $\beta > \alpha$.

The system (1.1) can be transformed to the second order linear differential equation of the form

$$u''(x) + p(x)u'(x) + q(x)u(x) = 0, \tag{2.3}$$

where

$$p(x) = -(a(x) + \frac{b'(x)}{b(x)} + d(x)), \tag{2.4}$$

and

$$q(x) = \left(\frac{b'(x)a(x)}{b(x)} + d(x)a(x) - a'(x) - b(x)c(x)\right). \tag{2.5}$$

Let $w(x)$ be the function defined by $w(x) = -u^{-1}(x)u'(x)$ for $x \in [\beta, \infty)$,then $w(x)$ is well defined and satisfies the Riccati equation,

$$w'(x) = w^2(x) - p(x)w(x) + q(x).$$

Integrating both sides of this equation from β to x we get

$$\begin{aligned} w(x) &= w(\beta) + \int\limits_{\beta}^{x}\left(w^2(s) - p(s)w(s) + q(s)\right)ds \\ &= w(\beta) + \int\limits_{\beta}^{x}\left((w^2(s) - p(s)w(s) + \frac{1}{4}p^2(s)) - \frac{1}{4}p^2(s) + q(s)\right)ds \\ &= w(\beta) + \int\limits_{\beta}^{x}\left((w(s) - \frac{1}{2}p(s))^2 + \frac{1}{4}(4q(s) - p^2(s))\right)ds. \end{aligned}$$

Now substituting $p(x)$ and $q(x)$ from (4), (5) and simplifying, we get

$$\begin{aligned} w(x) &= w(\beta) + \int\limits_{\beta}^{x}\left[w(s) + \frac{1}{2}\left(a(s) + \frac{b'(s)}{b(s)} + d(s)\right)^2\right]ds \\ &\quad -\frac{1}{4}\int\limits_{\beta}^{x}\left[\left(a(s) - d(s) - \frac{b'(s)}{b(s)}\right)^2 + 4(a'(s) + b(s)c(s))\right]ds. \end{aligned}$$

The hypothesis (2.2) implies that there exists $\gamma > \beta$ such that

$$w(x) \geq \int_{\gamma}^{x} \left[w(s) + \frac{1}{2}\left(a(s) + \frac{b'(s)}{b(s)} + d(s) \right)^2 \right] ds, \quad \text{on } [\gamma, \infty).$$

Define

$$Q(x) = \int_{\gamma}^{x} \left[w(s) + \frac{1}{2}\left(a(s) + \frac{b'(s)}{b(s)} + d(s) \right)^2 \right] ds. \tag{2.6}$$

Then $w(x) \geq Q(x) > 0$, on $[\gamma, \infty)$. Differentiating (2.6) we get

$$Q'(x) = w(x) + \frac{1}{2}\left(a(x) + \frac{b'(x)}{b(x)} + d(x) \right)^2 \geq w^2(x).$$

Therefore $Q'(x) \geq Q^2(x)$, and hence

$$1 \leq \frac{Q'(x)}{Q^2(x)};$$

this inequality holds for $x > \gamma$. Integrating both sides of this inequality from γ to x we get

$$\int_{\gamma}^{x} ds < \frac{1}{Q(\gamma)} - \frac{1}{Q(x)}.$$

Therefore, since $Q(x) > 0$, we conclude

$$\lim_{x\to\infty} \int_{\gamma}^{x} ds < \frac{1}{Q(\gamma)}.$$

But this is not true. Hence the system (1.1) is oscillatory and this completes the proof. □

Theorem 2 *If there exists a function $g(x) \in C^1[\alpha, \infty)$, and $g(x) > 0$ such that*

$$\lim_{x\to\infty} \int_{\alpha}^{x} g^{-1}(s) ds = \infty,$$

and

$$\lim_{x\to\infty} [-\tfrac{1}{4}\int_{\alpha}^{x}\{g^{-1}(s)\left[(a(s) + \tfrac{b'(s)}{b(s)} + d(s) - g'(s)\right]^2 + 4g(s)(a'(s) \\ + b(s)c(s) - d(s)a(s) - \tfrac{b'(s)}{b(s)})\}ds + \tfrac{1}{2}g'(x)] = \infty, \tag{2.7}$$

then the system (1.1) is oscillatory.

Proof. Suppose that there exists a nonoscillatory solution of the system (1.1). For $x \geq \alpha$ define

(2.8) $$w(x) = -g(x)u^{-1}(x)u'(x),$$

where $g(x)$ is a non-vanishing function belongs to $C^1[\alpha,\infty)$, and $\frac{1}{g(x)} > 0$ on $[\alpha,\infty)$. Differentiating (2.8) with respect to x and simplifying, $w(x)$ satisfies the Riccati equation

$$w'(x) = \frac{1}{g(x)}\left[w^2(x) + g'(x) - g(x)p(x)w(x)\right] + g(x)q(x).$$

Now defining for $x \in [\alpha,\infty)$ the function

(2.9) $$R(x) = w(x) + \frac{1}{2}g'(x),$$

we have

$$\begin{aligned} w'(x) &= \frac{1}{g(x)}\left[R^2(x) - g(x)p(x)R(x) - \frac{1}{4}(g'(x))^2\right] + \frac{1}{2}g(x)p(x)g'(x) \\ &\quad + g(x)q(x) \\ &= \frac{1}{g(x)}\left[R^2(x) - g(x)p(x)R(x)\right] - \frac{1}{4g(x)}(g'(x))^2 + \frac{1}{2}g(x)p(x)g'(x) \\ &\quad + g(x)q(x) \\ &= \frac{1}{g(x)}\left(R(x) - \frac{1}{2}g(x)p(x)\right)^2 - \frac{1}{4g(x)}\{g^2(x)p^2(x) - 2g(x)p(x)g'(x) \\ &\quad + (g'(x)\}^2 + q(x)g(x). \end{aligned}$$

Now substituting $p(x)$ and $q(x)$ from (2.4), and (2.5), we get

$$\begin{aligned} w'(x) &= \frac{1}{g(x)}\left[R(x) + \frac{g(x)}{2}\left(a(x) + \frac{b'(x)}{b(x)} + d(x)\right)\right]^2 \\ &\quad - \frac{1}{4g(x)}\left\{g(x)(a(x) + \frac{b'(x)}{b(x)} + d(x)) - g'(x)\right\}^2 \\ &\quad + g(x)\left(\frac{b'(x)a(x)}{b(x)} + d(x)a(x) - a'(x) - b(x)c(x)\right). \end{aligned}$$

Integrating both sides of the above equation from α to x we get

$$\begin{aligned} w(x) &= w(\alpha) + \int_\alpha^x g^{-1}(s)\left\{R(s) + \frac{g(s)}{2}\left(a(s) + \frac{b'(s)}{b(s)} + d(s)\right)\right\}^2 ds \\ &\quad - \frac{1}{4}\int_\alpha^x g^{-1}(s)\left\{g(s)\left(a(s) + \frac{b'(s)}{b(s)} + d(s)\right) - g'(s)\right\}^2 ds \\ &\quad + \int_\alpha^x g^{-1}(s)\left\{\frac{b'(s)a(s)}{b(s)} + d(s)a(s) - a'(s) - b(s)c(s)\right\} ds. \end{aligned}$$

Then by (2.9) rewrites

$$\begin{aligned} R(x) &= w(\alpha) + \int_\alpha^x g^{-1}(s)\left\{R(s) + \tfrac{g(s)}{2}\left(a(s) + \tfrac{b'(s)}{b(s)} + d(s)\right)\right\}^2 ds \\ &\quad -\tfrac{1}{4}\int_\alpha^x g^{-1}(s)\left\{g(s)\left(a(s) + \tfrac{b'(s)}{b(s)} + d(s)\right) - g'(s)\right\}^2 ds \\ &\quad +\int_\alpha^x g^{-1}(s)\left\{\tfrac{b'(s)a(s)}{b(s)} + d(s)a(s) - a'(s) - b(s)c(s)\right\} ds + \tfrac{1}{2}g'(x). \end{aligned}$$

The hypothesis (2.7) implies there exists $\beta > \alpha$ such that

$$R(x) > \int_\beta^x g^{-1}(s)\left\{R(s) + \frac{g(s)}{2}\left(a(s) + \frac{b'(s)}{b(s)} + d(s)\right)\right\}^2 ds$$

holds for $x > \beta$. Define a function $Q(x)$ for $x > \beta$ by

$$Q(x) = \int_\beta^x g^{-1}(s)\left\{R(s) + \frac{g(s)}{2}\left(a(s) + \frac{b'(s)}{b(s)} + d(s)\right)\right\}^2 ds, \tag{2.10}$$

then for $x > \beta$ we have $R(x) > Q(x) > 0$. Differentiating (2.10) with respect to x, we have

$$Q'(x) = g^{-1}(x)\left\{R(x) + \frac{g(x)}{2}\left(a(x) + \frac{b'(x)}{b(x)} + d(x)\right)\right\}^2$$

$$\begin{aligned} Q'(x) &\geq g^{-1}(x)\left\{Q(x) + \frac{g(x)}{2}\left(a(x) + \frac{b'(x)}{b(x)} + d(x)\right)\right\}^2 \\ &\geq g^{-1}(x)Q^2(x). \end{aligned}$$

Therefore

$$g^{-1}(x) \leq \frac{Q'(x)}{Q^2(x)}.$$

Integrating both sides of this inequality with respect to x (with x replaced by s) form x to β we get

$$\int_\beta^x g^{-1}(s)ds \leq \frac{1}{Q(\beta)} - \frac{1}{Q(x)}.$$

Since $Q(x) > 0$, we infer

$$\int_\beta^x g^{-1}(s)ds \leq \frac{1}{Q(\beta)},$$

which contradicts the hypothesis of the theorem. Hence the system (1.1) is oscillatory.
□

§3. Example

In the following we present an illustrative example showing the applicability of both theorems. Consider the system of first order differential equations

$$\begin{aligned} u'(x) &= v(x) \\ v'(x) &= -\left(1+\frac{2}{x^2}\right)u(x)+\frac{2}{x}v(x) \end{aligned}$$

For this system $a(x)=0$, $b(x)=1$, $c(x)=-\left(1+\frac{2}{x^2}\right)$, and $d(x)=\frac{2}{x}$. To show the applicability of Theorem 1, it is clear that all its hypotheses are satisfied, as follows

$$b(x)=1\neq 0, \text{ and } a(x)+\tfrac{b'(x)}{b(x)}+d(x)=\tfrac{2}{x}>0,$$

on $[\alpha,\infty)$, $\alpha>0$ and

$$\begin{aligned} &\lim_{x\to\infty}\left[-\frac{1}{4}\int_{\alpha}^{x}\left\{\left(a(s)-\frac{b'(s)}{b(s)}-d(s)\right)^2+4(a'(s)+b(s)c(s))\right\}ds\right] \\ = &\lim_{x\to\infty}\left[-\frac{1}{4}\int_{\alpha}^{x}\left\{-\frac{4}{s^2}-4\left(1+\frac{2}{s^2}\right)\right\}ds\right]=\lim_{x\to\infty}\int_{\alpha}^{x}\left(1+\frac{3}{s^2}\right)ds=\infty \end{aligned}$$

Therefore Theorem 1 implies that system (1.1) is oscillatory. This fact is directly verified by noting that the solution of the system is given by

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}=\begin{pmatrix} k_1x\cos x+k_2x\sin x \\ k_1(\cos x-x\sin x)+k_2(\sin x+x\cos x) \end{pmatrix}.$$

To show the applicability of Theorem 2, choose $g(x)=x$. It is clear that the hypothesis (2.7) is satisfied as follows

$$\begin{aligned} &\lim_{x\to\infty}\left\{-\frac{1}{4}\int_{\alpha}^{x}\left[g^{-1}(s)\left(a(s)+\frac{b'(s)}{b(s)}+d(s)-g'(s)\right)^2+4g(s)(a'(s)\right.\right. \\ &\qquad\left.\left.+b(s)c(s)-d(s)a(s)-\frac{b'(s)}{b(s)}\right)\right]ds+\frac{1}{2}g'(x)\right\} \\ = &\lim_{x\to\infty}\left\{-\frac{1}{4}\int_{\alpha}^{x}\left[s^{-1}\left(\frac{2}{s}-1\right)^2-4s\left(1+\frac{1}{s}\right)\right]ds+\frac{1}{2}\right\} \\ = &\lim_{x\to\infty}\left[\int_{\alpha}^{x}\left(-\frac{1}{s^3}+\frac{1}{s^2}-\frac{1}{4s}+s+2\right)ds+\frac{1}{2}\right]=\infty. \end{aligned}$$

Hence Theorem 2 is applicable.

Moreover, $g(x)$ can be chosen to be any polynomial with positive coefficient of leading term and it is greater than zero for $x>\alpha$.

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