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ON THE ORBITAL PERIODS FOR A PARTICULAR CASE OF SPHERICAL SYMMETRY

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SUMMARY: A particular case of steady state and spherical symmetry - the so-called logarithmic potential introduced as the first approximation for dark coronae of galaxies - is studied. Both time and angle dependence of the distance to the centre for the orbit of a bound test particle with arbitrary initial conditions are calculated numerically. The main attention is paid to the ratio of the sidereal period to the anomalistic one. It is found that this ratio is only slightly variable for a given mean distance to the centre and to increase with increasing orbital eccentricity. This quantitative result may be explained by the fact that the cumulative mass dependence on the distance corresponding to the logarithmic potential obeys a power law, the case where the ratio of the second derivative of the potential to the square of angular velocity for the same distance is constant. On the other hand, compared to the period of circular motion both periods increase with increasing eccentricity.

Key words. Stellar dynamics

1. INTRODUCTION

The question concerning the integrals of motion in the case of large or statistical stellar systems, like star clusters and galaxies, is among the fundamental ones in stellar dynamics. For such systems, it is usual to assume the steady state. As it is well known, this assumption means that the energy integral should exist and, also, the steady state is expected to be associated with a kind of symmetry. Symmetries allowing some sort of angular-momentum integral are of a special interest (for more details: e.g. Binney and Tremaine 1987, Contopoulos 2002), since for many galaxies and star clusters, as a first approximation, steady state with axial symmetry has been used. Nearly planar orbits, typical for stars of galaxy discs, appear as a point where the approximations of spherical and axial symmetries (both involving the steady state) come close together. Due to low orbital eccentricities such orbits are usually referred to as epicyclic ones. What is especially characteristic for them is that the ratio of the periods, i.e. the frequencies, is given simply through a dimensionless quantity which describes the local properties of the gravitation field (e.g. Ninković 1996). In principle, one has three basic frequencies (periods). One of them - the so-called circular frequency - depends on the assumed potential and it is referred to a certain distance from the centre. The other two - sidereal and epicycle ones - characterize the given orbit, i.e. they are integrals of motion; the epicycle frequency coincides with the anomalistic one, and for this reason it is, perhaps, better to prefer the latter name. The term epicycle is closely connected with the motion along orbits of low eccentricity, therefore it is not suitable to use it when a generalization towards arbitrary eccentricities is intended.

The ratio of the two periods (anomalistic to sidereal) is very important since it determines the character of a planar orbit, more clearly whether an orbit will be periodic or not (Contopoulos 2002). However, this ratio can be studied in terms of the period of circular motion because this period can be related to the sidereal one by means of a transcendental equation (Ninković 1996). Therefore, the present authors want to examine numerically the ratio of the two periods (anomalistic to sidereal), as well as to express both of them in terms of the period of circular motion for the same mean distance (see below), in the case of a given potential without any limitation concerning the orbital eccentricity.

2. FORMALISM

As it is well known, an analytic solution for an orbit in the case of steady state and spherical symmetry can be obtained for two particular cases only: homogeneous sphere and point mass (alternatively referred to as, say, spherical harmonic oscillator and Kepler potential, respectively, Binney and Tremaine 1987 - p. 107). In addition, in these two cases the ratio sidereal-to-anomalistic period is constant - it depends neither on the mean distance nor on the eccentricity - and has an integral value. Due to the latter property, as also well known, for these two potentials the orbits are closed. Of course, this is valid provided that the test particle is bound. Motion of an unbound test particle is not considered in the present paper.

The motion of a bound test particle in the case under study (steady state + spherical symmetry) in the orbital plane can be characterized by a combination of two independent integrals of motion. For instance, we can use the apocentric and pericentric distances - r_{a} and r_{p}, respectively. Another option is to use the mean distance r_{m} and orbital eccentricity e, which is a dimensionless quantity. These two quantities have already been mentioned, but since they can be defined in various ways, their definitions used herewith will be given.

$$r_{\mathrm{m}} = \frac{r_{\mathrm{a}} + r_{\mathrm{p}}}{2} \quad ,$$

$$e = \frac{r_{\mathrm{a}} - r_{\mathrm{p}}}{r_{\mathrm{a}} + r_{\mathrm{p}}} \quad . \tag{1}$$

The eccentricity, as defined here, is to be understood as a measure of deviation from a circular orbit, not geometrically as in the case of an ellipse since orbits are generally not closed. All such combinations of independent integrals of motion are due to the two fundamental integrals (conservation laws) - energy and angular momentum modulus. The same is true for the two periods - anomalistic P_{a} and sidereal P_{s}. For any particular potential corresponding to the conditions of steady state and spherical symmetry, the period of circular motion depends on the radius only. If the orbital eccentricity is low, the period of circular motion and the anomalistic one are related very simply, (e.g. Ninković 1996)

$$P_{\mathrm{c}} = (3 - \gamma)^{1/2} P_{\mathrm{a}} \quad , \tag{2}$$

where γ is a dimensionless quantity, more precisely the ratio of the second derivative of the potential to the square of the cyclic frequency of circular motion (it corresponds to period P_{c}). In the case where the cumulative mass follows a power law, the ratio γ is constant, i.e. it does not depend on the distance. This means that in such cases for low eccentricities the relation between the two periods is universal, i.e. distance independent. Since the sidereal and circular frequencies (periods) are almost equal to each other for the case of a low orbital eccentricity (Ninković 1996), Eq. (2) yields a kind of lower limit for the ratio of the anomalistic period to the sidereal one for a given potential. Higher orbital eccentricities require this ratio to be treated differently.

In order to shed more light on the ratio of the two periods (sidereal to anomalistic), for the case of higher orbital eccentricities, the present authors carried out some calculations assuming a particular type of stationary and spherically symmetric potential. It is a potential yielding a power law for the cumulative mass, so that the quantity γ, given above, will be constant. This will be the so-called logarithmic potential given by:

$$\Pi(r) = {u_{\mathrm{c}}}^2 (1 + \ln \frac{r_{\mathrm{l}}}{r}) \quad . \tag{3}$$

With this potential, the circular velocity (u_{c}) is constant. For the density it yields a simple power law - it decreases as r^2 being infinite at the centre. Consequently, the cumulative mass is a linear function of r. Therefore, such a mass distribution cannot be valid over an infinite distance from the centre; in other words the limiting radius r_{l} must be finite since otherwise the total mass would be infinite. Therefore, the potential, as seen from Eq. (3), contains two parameters: the circular velocity and the limiting radius. Like the density, it also becomes infinite at the centre. This means that any rectilinear bound orbit (angular momentum equal to zero) should be excluded. Beyond the boundary $r = r_{\mathrm{l}}$ the density is equal to zero, i.e. the potential is that of point mass. Therefore, our calculations have another constraint: a test particle is not only bound, but also its apocentric distance must not exceed the limiting radius - $r_{\mathrm{a}} \leq r_{\mathrm{l}}$.

The mass distribution characterized by potential given by Eq. (3), i.e. density inversely proportional to r^2, is often referred to as isothermal. However, the isothermal solution is not the only one; it is obtained only if the velocity distribution is isotropic. The mass distribution assumed here can be also valid if the velocity distribution is not isotropic (e.g. Antonov et al. 1975).

3. ORBIT CALCULATIONS

The constraints, already mentioned above, are

$$J > 0 \quad , \qquad r_a \le r_l \quad , \tag{4}$$

where J is the specific (per unit mass) angular momentum of the test particle. Since both the specific angular momentum and specific energy E can be related to the mean distance r_m and eccentricity e, a set of orbits with a fixed r_m and various e can be examined. The relations between E and J, r_m and e are (Ninkovich 1986 in Russian)

$$E = \frac{1}{2} f_1(e) {u_c}^2 - {u_c}^2 (1 + \ln \frac{r_l}{r_m}) \quad ,$$

$$J^2 = 2 r_m^2 (1 \pm e)^2 {u_c}^2 [\frac{1}{2} f_1(e) - \ln (1 \pm e)] \quad ,$$

$$f_1(e) = \frac{(1+e)^2 \ln (1+e) - (1-e)^2 \ln (1-e)}{2e} \quad .$$

Using these relations we can study the ratio of the two periods for a given r_m and various eccentricities, varying the mean distance r_m afterwards. We solve numerically the differential equations of motion (Lagrange equation and the angular momentum one)

$$\ddot{r} - \frac{J^2}{r^3} = \frac{d\Pi}{dr} \quad ,$$

$$\dot{\psi} = \frac{J}{r^2} \quad ,$$

where Eq. (3) is applied for the potential; ψ is the position angle in the orbital plane. By solving the first equation one obtains the dependence of the distance on time, and combining it subsequently with the second equation one obtains the dependence $r(\psi)$, i.e. the orbit. This dependence does not mean that for every ψ one has only one value for r since, generally, the orbit is not closed. In our calculations the initial conditions always correspond to the apocentric position which means that the initial value of r is equal to $r_m(1+e)$ (formula (1)). Then the radial velocity component $\dot{r}$, as well known, is equal to zero, whereas the transverse one (v_t), is easily obtained as J/r. In our calculations we use r_l and u_c as the units for the distance and velocity, respectively. Consequently, the unit of specific energy will be u_c^2, i.e. $r_l u_c$ in the case of specific angular momentum. For each pair r_m, e we evaluate the angle difference $\Delta\psi$. This is the difference between the position angle of the pericentre and that of the next apocentre. It is constant and it corresponds to half the anomalistic period. Having in mind that half the sidereal period corresponds to the angle difference exactly equal to π, the ratio $\pi/\Delta\psi$ will be equal to the ratio of the two periods looked for in the present paper. Finally, the period of the circular motion corresponding to the same distance r_m is easily obtained as $2\pi r_m/u_c$. As mentioned previously, in the limiting case of low eccentricities the sidereal period is approximately equal to that of the circular motion, whereas the anomalistic one is related to the period of circular motion by means of Eq. (2); it is to be added that for the case of potential (3) the ratio γ is constant, since the cumulative mass follows a power law,the exponent equals to unity exactly. Thus, in the particular case studied here, for low eccentricities the ratio of the period of circular motion, i.e. sidereal, to the anomalistic one is equal to $2^{1/2}$, an irrational number. Therefore, for low eccentricities the angle difference $\Delta\psi$ is about $\pi/\sqrt{2}$.

4. RESULTS AND DISCUSSION

Our main results are given in the Table 1. As it is clearly seen, the increase in the orbital eccentricity brings about a slight decrease in the angle difference $\Delta\psi$, which means that for higher eccentricities the ratio sidereal-to-anomalistic periods is only slightly different from $\sqrt{2}$, the value characteristic for low eccentricities. In addition, the variation of the mean distance has no influence here.

Table 1. Eccentricity (e), semi-anomalistic period ($\frac{1}{2}P_a$) and angle difference ($\Delta\psi$) for a given values of the mean distance (r_m)

		$r_m = 0.3$			
e	0.10	0.30	0.50	0.70	0.90
$\frac{1}{2}P_a$	0.67	0.68	0.69	-	0.73
$\Delta\psi$	2.23	2.26	2.26	-	2.17
		$r_m = 0.5$			
e	0.10	0.30	0.50	0.70	0.90
$\frac{1}{2}P_a$	1.12	1.12	1.14	1.17	1.22
$\Delta\psi$	2.24	2.21	2.20	2.18	2.34
		$r_m = 0.7$			
e	0.10	0.30	0.50	0.70	0.90
$\frac{1}{2}P_a$	1.56	1,57	-	-	-
$\Delta\psi$	2.23	2.21	-	-	-

It should be pointed out that, in a realistic case of spherical symmetry, this angle difference is limited to values within $[\frac{\pi}{2}, \pi]$ (Contopoulos 2002, p. 379). It may be added here that the lower limit corresponds to the homogeneous sphere and the upper one to the point mass. This can be easily seen from Eq. (2), though it holds for low eccentricities, because in the case of these two models the ratio γ is an integer, to be equal to -1 and 2, respectively. This means that for any realistic case of spherical symmetry, γ varies as a monotonous function of r between the limits of -1 and +2 from the centre towards the periphery. Only if the cumulative mass follows a power law, γ is constant. In the case of a cuspy mass distribution (e.g. Dehnen 1993, Tremaine et al. 1994) the central value of γ is greater than -1 depending on the density slope near the centre of the system. Also, if a black hole dominates the potential in the central part (e.g. Tremaine et al. 1994), the central value of γ is 2, but this quantity in such a case is no longer a monotonous function of the distance to the centre r.

Now it is more clear why the interval of the period ratio in the present case is rather limited, compared to a potential in which γ is not constant. In this latter case, the ratio of the periods covers practically the entire interval (1, 2).

The anomalistic period for the same mean distance, as compared to the period of circular motion for this distance, shows an increasing trend; its ratio to the period of circular motion amounts to about $(\sqrt{2})^{-1}$ at low eccentricities (Eq. (2)) and attains a value of about 1.29^{-1} at very high eccentricities (about 0.9). With regard to the above comment concerning the ratio of the anomalistic period to the sidereal one for the same mean distance and eccentricity, this means that the sidereal period also increases with increasing orbital eccentricity: being exactly equal to the period of circular motion at the zero eccentricity its ratio to the period of circular motion attains about 1.1 at high eccentricities. This behaviour does not depend on the mean distance.

The potential used here (formula (3)) has been considered most frequently in connection with the galactic subsystems constituted of dark matter, referred to usually as dark halos or (dark) coronae. In the paper the subject of which was the distribution in the phase space within such a system provided that it is self-consistent, Antonov et al. (1975) also assumed the mass distribution characterized by potential (3). This is the most simple version of a mass distribution law describing the coronae of galaxies since a dark corona has been assumed most frequently to be spherically symmetric (e.g. Samurović et al. 1999).

5. CONCLUSIONS

The period ratio for a special case of stationary and spherically symmetric potential (the so-called logarithmic potential) was examined. This potential was characterized by the constant value of the ratio of second potential derivative to square of angular velocity of circular motion and also by the constant circular velocity. The latter property leads to a linear dependence of a period on the radius for circular motion. The anomalistic period for a given mean distance increases with eccentricity, but its behaviour is independent of the mean distance. On the other hand, the sidereal period being practically equal to that of circular motion for low eccentricities shows an increasing trend with eccentricity so that its ratio to the corresponding anomalistic period slightly increases, i.e. it remains close to $\sqrt{2}$, the value found theoretically for low eccentricities. This result may be ascribed to the power law valid for the cumulative mass, i.e. to constant quantity γ (Eq. (2)). The behaviour of the period ratio also does not show any dependence on the mean distance of the orbit. Therefore, in the case of the logarithmic potential it is almost impossible to find an orbit for which the ratio sidereal-to-anomalistic period is a numerically stable rational number. Clearly, the case of a rational ratio of the periods is of a special interest because then the fifth independent integral of motion is isolating (e.g. Contopoulos 2002).

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О ОРБИТАЛНИМ ПЕРИОДИМА ЗА КОНКРЕТАН СЛУЧАЈ СФЕРНЕ СИМЕТРИЈЕ

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Оригинални научни рад

Проучава се конкретан случај стационарног стања и сферне симетрије - тзв. логаритамски потенцијал који је као прва апроксимација нарочито применљив на тамне короне галаксија. Израчунавана је нумерички за орбиту пробне материјалне тачке и зависност растојања од времена и његова зависност од положајног угла за произвољне почетне услове. Главна пажња је посвећена односу сидеричког и аномалистичког периода. Нађено је да се овај однос незнатно мења за дату средњу вредност растојања до центра система, да би потом растао са растућом ексцентричношћу. Овај квантитативни резултат може да се објасни чињеницом да за логаритамски потенцијал одговарајућа кумулативна маса расте по степеном закону, а то је случај када је однос другог извода потенцијала и квадрата угаоне брзине за исто растојање константан. С друге стране, у поређењу са периодом кружног кретања, оба периода расту са растућом ексцентричношћу.