

SOME PROPERTIES OF THE POINT OPTIMAL INVARIANT TEST FOR THE CONSTANCY OF PARAMETERS

Eiji Kurozumi*

In this paper we consider the time-varying parameter model. Since there is no uniformly most powerful test for the constancy of parameters, the locally best invariant (LBI) test has often been considered in the literature, including the study by Nabeya and Tanaka (1988). We show the existence of the limiting distribution of the point optimal invariant (POI) test statistic when we can derive the limiting distribution of the LBI test statistic. We prove that the limiting characteristic function of the POI test statistic can be expressed using that of the LBI test statistic.

Key words and phrases: Characteristic function, constancy of parameters, locally best invariant test, point optimal invariant test, power envelope.

1. Introduction

In this paper we consider the following time-varying parameter model:

$$(1.1) \quad y_t = x_{1t}\alpha_t + x'_{2t}\beta + \varepsilon_t, \quad \alpha_t = \alpha_{t-1} + u_t,$$

for $t = 1, \dots, T$, where $\{x_{1t}\}$ and $\{x_{2t}\}$ are one and p dimensional nonstochastic sequences, $\{\varepsilon_t\} \sim NID(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon^2 > 0$, $\{u_t\} \sim NID(0, \sigma_u^2)$ with $\sigma_u^2 \geq 0$, $\{\varepsilon_t\}$ and $\{u_t\}$ are independent of each other, and α_0 is an unknown fixed constant. The model (1.1) belongs to the Kalman filter model type and is considered in Harvey (1989), Nyblom (1986), Nyblom and Mäkeläinen (1983), Nabeya and Tanaka (1988), amongst other studies, and it is used to analyze purse snatchings in Harvey (1989) and a macro-econometric model in Cooley (1975).

Stacking each variable from $t = 1$ to T , the model (1.1) can be expressed as

$$(1.2) \quad y = X\gamma + DLu + \varepsilon,$$

where $y = [y_1, \dots, y_T]'$, $X = [x_1, X_2]$ with $x_1 = [x_{11}, \dots, x_{1T}]'$ and $X_2 = [x_{21}, \dots, x_{2T}]'$, $\gamma = [\alpha_0, \beta']'$, $D = \text{diag}\{x_{11}, \dots, x_{1T}\}$, L is a lower triangular matrix with lower elements 1's, $u = [u_1, \dots, u_T]'$ and $\varepsilon = [\varepsilon_1, \dots, \varepsilon_T]'$. If α_0 is known, $\alpha_0 x$ is subtracted from both sides of (1.2) and the model is re-expressed similarly to (1.2).

Our interest lies in whether or not α_t is constant. Since α_t takes a constant value α_0 for all t when $\sigma_u^2 = 0$ and it varies over time when $\sigma_u^2 > 0$, we consider the following testing problem:

$$(1.3) \quad H_0 : \rho^* = 0, \quad \text{vs.} \quad H_1 : \rho^* > 0,$$

where $\rho^* = \sigma_u^2/\sigma_\varepsilon^2$ is a signal-to-noise ratio. For example, when $x_{1t} = 1$ and $x_{2t} = 0$, we may see $x_{1t}\alpha_t = \alpha_t$ as a mean process of y_t . That is, the path of y_t follows $x_{1t}\alpha_t$ on average, but y_t is observed with an error ε_t . Since $x_{1t}\alpha_t = \alpha_0$ under the null hypothesis, we can see that the process y_t has constant mean α_0 and its first difference is zero under the null hypothesis. On the other hand, we have $x_t\alpha_t = \alpha_0 + \sum_{i=1}^t u_i$ under the alternative and then it becomes a random walk process. That is, mean of y_t varies gradually and its first difference is an i.i.d. process with variance σ_u^2 .

Although our interest is the testing problem (1.3), it is well known that there is no uniformly most powerful test for (1.3). Instead, we can construct the most powerful test for a simple fixed alternative in the class of invariant tests, and we call it the point optimal invariant (POI) test. To construct the POI test, we can choose any simple alternative $\rho^* = \rho$ and then there is an infinite number of POI tests, depending on the value of ρ . One possible choice of ρ is to consider the case when $\rho \rightarrow 0$ and this is known as the locally best invariant (LBI) test. The POI and LBI tests for (1.3) were derived in Kariya (1980), King (1980, 1985) and King and Hillier (1985), and the test statistics may be expressed as a weighted sum of independent chi-squared variables.

Given a particular set of x_{1t} and x_{2t} , the limiting distributions of the test statistics were derived in Nyblom and Mäkeläinen (1983) for $x_{1t} = 1$ and $x_{2t} = 0$ and Nyblom (1986) for $x_{1t} = 1$ and $x_{2t} = t$. On the other hand, Nabeya and Tanaka (1988) investigated the LBI test for various specifications of x_{1t} and x_{2t} and found that the limiting characteristic function of the test statistic can be expressed using the Fredholm determinant, while the limiting behavior of the POI test statistic is not necessarily clear for some sets of x_{1t} and x_{2t} considered in Nabeya and Tanaka (1988).

The model (1.1) is sometimes extended to have a serially correlated error and used for testing (trend) stationarity. For example, Kwiatkowski *et al.* (1992) considered the model (1.1) with $x_{1t} = 1$ and $x_{2t} = 0$ or t and investigated the LBI test for the null hypothesis of $\rho = 0$. Their test is seen as the test of (trend) stationarity and applied to macro-economic time series. The model (1.1) is also extended so that x_{1t} is a vector variable and α_t is a multivariate random walk process. Using this extension, Canova and Hansen (1995) considered the LBI test against the seasonal unit roots and investigated industrial production indexes and monthly stock returns. These extensions are interesting and important in a practical analysis, but we consider the stylized model (1.1) to simplify the problem.

The typical method of deriving the limiting distribution may be classified into two approaches. One method is to express the test statistic as a weighted sum of squares of uncorrelated variables and to derive the limiting expression. In this case, the typical weights are eigenvalues of the matrix related to the model as in Nyblom (1986) and Nyblom and Mäkeläinen (1983), and we may get the limiting distributions of both the POI and LBI test statistics. However, in practice, it is difficult to find the explicit expression of a sequence of weights

except for several special cases. As a result, the models for which this approach is applicable are limited. On the other hand, the Fredholm approach used in Nabeya and Tanaka (1988) approximates the test statistic using the kernel function of the integral equation of the second kind. The limiting distribution is expressed as a weighted sum of squares of standard normal variables with weights being eigenvalues of the kernel function, and the characteristic function is expressed as a function of the Fredholm determinant. The advantage of this approach is that we only have to know the Fredholm determinant of the kernel function to derive the limiting characteristic function, and information of the eigenvalues is unnecessary. However, this approach seems applicable only for the LBI test because the POI test statistic cannot be approximated using the kernel function. In this sense, the Fredholm approach is limited for investigation of the LBI test.

In the remaining parts of the paper we investigate the asymptotics of the POI test under the assumption of normality when the limiting distribution of the LBI test exists. We show in Section 2 that the limiting distribution of the POI test statistic exists and the characteristic function can be derived if we have the limiting characteristic function of the LBI test statistic. The characteristic function of the POI test statistic is expressed using that of the LBI test statistic. Then, our result may complement Nabeya and Tanaka (1988), in which only the LBI test is investigated. Some examples are shown in Section 3 by specifying x_{1t} and x_{2t} . Section 4 concludes the paper.

2. The existence of the limiting distribution and the characteristic function

Let us consider the model (1.1) and the testing problem (1.3). We can see that (1.3) is invariant under the group of transformations: $y \rightarrow ay + Xb$, $\gamma \rightarrow a\gamma + b$ and $\sigma_\varepsilon^2 \rightarrow a^2\sigma_\varepsilon^2$, where $0 < a < \infty$ and b is a $(p + 1) \times 1$ vector. According to Kariya (1980), King (1980, 1985) and King and Hillier (1985), the POI and LBI test statistics are given by $y'\bar{M}'\Sigma(\rho)^{-1}\bar{M}y/y'My$ and $y'MDLL'DMy/y'My$ respectively, where $\Sigma(\rho) = I_T + \rho DLL'D$ with I_T a $T \times T$ identity matrix, $\bar{M} = I_T - X(X'\Sigma(\rho)^{-1}X)^{-1}X'\Sigma(\rho)^{-1}$ and $M = I_T - X(X'X)^{-1}X'$. We can see that $\sigma_\varepsilon^2\Sigma(\rho)$ is the variance matrix of y . Notice that, to construct the POI test statistic, we have to assume a simple alternative point ρ and then the POI test statistic depends on both ρ and ρ^* , the true signal-to-noise ratio in the process. On the other hand, the LBI test statistic depends only on ρ^* because it is derived when $\rho \rightarrow 0$.

We modify these test statistics in order to consider the asymptotic distributions as follows:

$$\mathcal{P}_T = T \left(1 - \frac{y'\bar{M}'\Sigma(\rho)^{-1}\bar{M}y}{y'My} \right) = \frac{y'(M - \bar{M}'\Sigma(\rho)^{-1}\bar{M})y}{y'My/T}$$

and

$$\mathcal{L}_T = \frac{y'MDLL'DMy/s(T)}{y'My/T},$$

where $s(T)$ is some scaling factor so that the numerator of \mathcal{L}_T can converge in distribution. Note that the numerator of \mathcal{L}_T can be expressed as

$$\sum_{j=1}^T \left(\sum_{t=j}^T x_t \tilde{y}_t \right)^2,$$

where \tilde{y}_t are regression residuals of y_t on x_t . In a typical case the summation over j is approximated by integration and the invariance principle holds for the summation in parentheses. Then, intuitively, $S(T)$ is T times the square of the convergence order of the summation in parentheses. For example, when $x_{1t} = 1$ and $x_{2t} = 0$, we have $x_t = 1$ and then $\sum_{t=j}^T \tilde{y}_t$ is of order $T^{1/2}$, so that $s(T) = T \times (T^{1/2})^2 = T^2$. Note that $s(T)$ corresponds to $c(T)T$ in Nabeya and Tanaka (1988).

To investigate power properties, it is often assumed that H_1 consists of a sequence of local alternatives: $\rho = c^2/s(T)$, where $c > 0$ is a constant. In this case, the true value of ρ in the process is $\rho^* = c^{*2}/s(T)$. Consequently, the POI test statistic depends on c as well as c^* . To show explicitly the dependence of the statistics on local parameters c and c^* , we denote \mathcal{P}_T and \mathcal{L}_T by $\mathcal{P}_T(c, c^*)$ and $\mathcal{L}_T(c^*)$.

As explained in the previous section, the asymptotic properties of \mathcal{L}_T have been investigated in the literature for various sets of x_{1t} and x_{2t} and then we assume such information in this paper.

ASSUMPTION 1. The limiting distribution of \mathcal{L}_T exists under the null hypothesis and the characteristic function is given by $\phi_L(\theta)$.

For example, \mathcal{L}_T converges in distribution for $x_{1t} = 1$ and $x_{2t} = 0$ with $s(T) = T^2$ and $\phi_L(\theta)$ is given by $(\cos \sqrt{2i\theta})^{-1/2}$ when α_0 is known, while it becomes $(\sin \sqrt{2i\theta}/\sqrt{2i\theta})^{-1/2}$ for unknown α_0 . See Nabeya and Tanaka (1988) for the expression of the characteristic function. On the other hand, Assumption 1 excludes explanatory variables that decay slowly to zero. For example, when $x_{1t} = 1/T$ and $x_{2t} = 0$, the LBI test statistic converges in distribution with $S(T) = 1$, in which case we cannot consider the local alternative because $S(T)$ does not diverge. Moreover, when $x_{1t} = 1/T^2$, we can show that \mathcal{L}_T converges to zero in probability under the null hypothesis. In general, we do not consider variables whose order is less than T^{-1} .

As proved in Lemma 3 of Nabeya (1989), we can show that $y'My/T$ converges to σ_ε^2 in probability under a sequence of local alternatives. Therefore we can replace $y'My/T$ of \mathcal{P}_T and \mathcal{L}_T by σ_ε^2 as far as the limiting distribution is concerned. We denote \mathcal{P}_T and \mathcal{L}_T as $\tilde{\mathcal{P}}_T$ and $\tilde{\mathcal{L}}_T$ when $y'My/T$ is replaced by its probability limit, σ_ε^2 . In addition, as proved in the appendix, both test statistics can be expressed as a weighted sum of squared independent variables as follows:

$$(2.1) \quad \tilde{\mathcal{P}}_T(c, c^*) = \sum_{j=1}^{T-p-1} \left(1 - \frac{1}{1 + c^2 \tilde{\lambda}_j} \right) \left\{ \tilde{\varepsilon}_j + (c^{*2} \tilde{\lambda}_j)^{1/2} \tilde{u}_j \right\}^2,$$

$$(2.2) \quad \tilde{\mathcal{L}}_T(c^*) = \sum_{j=1}^{T-p-1} \tilde{\lambda}_j \left\{ \tilde{\varepsilon}_j + (c^* \tilde{\lambda}_j)^{1/2} \tilde{u}_j \right\}^2,$$

where $\tilde{\lambda}_j = \lambda_j/s(T)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{T-p-1}$ being non-zero eigenvalues of $MDLL'DM$, $\{\tilde{\varepsilon}_j\}$ and $\{\tilde{u}_j\}$ are independent standard normal variables and they are independent each other.

To derive the limiting characteristic functions of (2.1) and (2.2), we first investigate the LBI test under the null hypothesis. Since $\{\tilde{\varepsilon}_j\} \sim NID(0, 1)$, the characteristic function of $\mathcal{L}_T(0)$ is given by

$$\phi_{L,T}(\theta) = \prod_{j=1}^{T-p-1} \left(1 - 2i\theta \tilde{\lambda}_j\right)^{-1/2},$$

which converges to $\phi_L(\theta)$ by Assumption 1.

Next, since $\{\tilde{\varepsilon}_t\} \sim NID(0, 1)$, $\{\tilde{u}_j\} \sim NID(0, 1)$ and they are independent each other, the characteristic function of $\mathcal{P}_T(c, c^*)$ is given by

$$\begin{aligned} \phi_{P,T}(\theta) &= \prod_{j=1}^{T-p-1} \left\{ 1 - 2i\theta \left(1 + c^* \tilde{\lambda}_j\right) \left(1 - \frac{1}{1 + c^2 \tilde{\lambda}_j}\right) \right\}^{-1/2} \\ &= \frac{\prod_{j=1}^{T-p-1} \left\{ 1 - (a_1 + a_2) \tilde{\lambda}_j \right\}^{-1/2} \prod_{j=1}^{T-p-1} \left\{ 1 - (a_1 - a_2) \tilde{\lambda}_j \right\}^{-1/2}}{\prod_{j=1}^{T-p-1} \left\{ 1 - (-c^2) \tilde{\lambda}_j \right\}^{-1/2}} \\ &= \frac{\phi_{L,T}((a_1 + a_2)/(2i)) \phi_{L,T}((a_1 - a_2)/(2i))}{\phi_{L,T}(-c^2/(2i))}, \end{aligned}$$

where $a_1 = c^2(2i\theta - 1)/2$ and $a_2 = (c/2)\{(2i\theta - 1)^2 c^2 + 8ic^* \theta\}^{1/2}$. The expansion of the second equality is similar to Nabeya (1989) and Tanaka (1996). Since $\phi_{L,T}(\theta)$ converges to $\phi_L(\theta)$, we obtain

$$\phi_{P,T}(\theta) \rightarrow \frac{\phi_L((a_1 + a_2)/(2i)) \phi_L((a_1 - a_2)/(2i))}{\phi_L(-c^2/(2i))}.$$

Similarly, we get the limiting characteristic function of \mathcal{L}_T as follows.

$$\begin{aligned} \phi_{L,T}(\theta) &= \prod_{j=1}^{T-p-1} \left\{ 1 - 2i\theta \tilde{\lambda}_j \left(1 + c^* \tilde{\lambda}_j\right) \right\}^{-1/2} \\ &= \prod_{j=1}^{T-p-1} \left\{ 1 - (b_1 + b_2) \tilde{\lambda}_j \right\}^{-1/2} \prod_{j=1}^{T-p-1} \left\{ 1 - (b_1 - b_2) \tilde{\lambda}_j \right\}^{-1/2} \\ &= \phi_{L,T}((b_1 + b_2)/(2i)) \phi_{L,T}((b_1 - b_2)/(2i)) \\ &\rightarrow \phi_L((b_1 + b_2)/(2i)) \phi_L((b_1 - b_2)/(2i)), \end{aligned}$$

which is the same expression as in Nabeya (1989) and Tanaka (1996), where $b_1 = i\theta$ and $b_2 = \sqrt{2ic^*2\theta - \theta^2}$.

Weak convergences of \mathcal{P}_T and \mathcal{L}_T are ensured by the continuity theorem.

PROPOSITION 1. *Under Assumption 1, \mathcal{P}_T and \mathcal{L}_T converge in distribution under a sequence of local alternatives, and the limiting characteristic functions are given by*

$$\frac{\phi_L((a_1 + a_2)/(2i))\phi_L((a_1 - a_2)/(2i))}{\phi_L(-c^2/(2i))}$$

and

$$\phi_L((b_1 + b_2)/(2i))\phi_L((b_1 - b_2)/(2i)),$$

respectively.

According to Proposition 1, if we find the limiting characteristic function of the LBI test statistic under the null hypothesis, we can also obtain those of the POI and LBI test statistics under the local alternative. The characteristic functions obtained in Proposition 1 are useful for various purposes such as calculations of percentage points and power, and derivation of the power envelope.

3. Examples

In this section we compare the POI and LBI tests with the power envelope for several models. The following cases are examined here:

(A) $x_{1t} = t^m$, $m = 1, 2$ and $x_{2t} = 0$;

(B) $x_{1t} = 1$ for all t and $x_{2t} = t^m$, $m = 1, 2$.

For case (A) we choose $s(T) = T^4$ and T^6 for $m = 1$ and 2 , while for case (B) we choose $s(T) = T^2$ for $m = 1$ and 2 . Nabeya and Tanaka (1988, Theorems 3 and 6) derived the characteristic functions of these cases, which are given by

$$\phi_L(\theta) = \begin{cases} \Gamma\left(\frac{4m+3}{2(m+1)}\right) J_{1-1/\{2(m+1)\}}\left(\frac{\sqrt{2i\theta}}{m+1}\right) / \left(\frac{\sqrt{2i\theta}}{2(m+1)}\right)^{1-1/\{2(m+1)\}}, & \text{case (A), } m = 1, 2, \\ \frac{12}{(2i\theta)^2} \left(2 - \sqrt{2i\theta} \sin \sqrt{2i\theta} - 2 \cos \sqrt{2i\theta}\right), & \text{case (B), } m = 1, \\ \frac{45}{(2i\theta)^3} \left(\sqrt{2i\theta} \left(1 - \frac{2i\theta}{3}\right) \sin \sqrt{2i\theta} - 2i\theta \cos \sqrt{2i\theta}\right), & \text{case (B), } m = 2, \end{cases}$$

where $\Gamma(z)$ is the gamma function and $J_\nu(z)$ is the Bessel function of the first kind. In Figures 1(a)–(d) we draw the power functions of the POI and LBI tests as a function of c^* . Each figure also has the power envelope that attains the maximum power at each point of c^* . These powers are calculated by numerical

integration using Lévy’s inversion formula,

$$(3.1) \quad 1 - \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{1 - e^{-i\theta x^*}}{i\theta} \phi(c, c^*; \theta) \right] d\theta,$$

where $\phi(c, c^*; \theta) = \phi_P(c, c^*; \theta)$ and $\phi_L(c^*; \theta)$ for the POI and LBI tests and x^* denotes the 95th percentage point for each test. Note that, since the POI test is the most powerful test against the single alternative $\rho^* = \rho$ ($c^* = c$), the power envelope can be calculated using (3.1) with $\phi(c, c^*; \theta) = \phi_P(c^*, c^*; \theta)$. We also note that we have to use the different critical point x^* for each value of c^* to calculate the power envelope, because the distribution of the POI test depends on the value of c and the power envelope is calculated using $\phi_P(c, c^*; \theta)$ with $c = c^*$.

To calculate the power of the POI test, we have to pre-specify the value of c . One of the candidates for the practical value of c is the estimate of c , such as the maximum likelihood estimate. However, as far as I know, the limiting behavior of the local parameter estimator is unknown, even it might not be consistent. Instead of using the estimate of c , several rules of selecting the value of c have been proposed in the literature. For example, King (1983) proposed to select the alternative point so that the power function of the test is tangent to the power envelope at a power of 25%, 50% or 75%, while Tanaka (1996) considered the POI test whose power function is tangent to the envelope at a power of 50%. We adopted the latter strategy to select the value of c . In this example, c is 15.75, 24.25, 12.00 and 11.75 in each case and the power function using this rule corresponds to “POI(1)” in each figure.

We also consider another selection rule that is proposed by Cox and Hinkley (1974, p.102). Their strategy is to select the alternative point that maximizes the weighted average of power:

$$(3.2) \quad \max \int_{c_0}^\infty P(Y \in w_\alpha; c) dk(c)$$

for suitable $k(\cdot)$ when the alternatives are $c > c_0$, where Y is a test statistic and w_α is a critical region of size α . In this paper we used $c_0 = 0$ and $dk(c) = dc$ for $c \leq \bar{c}$ and $dk(c) = 0$ for $c > \bar{c}$, where \bar{c} is selected so that the power envelope at $c^* = \bar{c}$ is greater than 0.99, and it is 70, 100, 40 and 40 for each case. The integral in (3.2) is approximated by summation with c changing from 0 to \bar{c} step by 0.25. According to this rule, the value of c in the POI test is selected as 24.5, 37.00, 16.75 and 16.25 for each case. The power function using this rule corresponds to “POI(2)” in each figure.

Each figure shows that the powers of both POI tests are very close to the power envelope, whereas that of the LBI test is far below the envelope when c^* is away from zero. The LBI test is more powerful than the POI tests when c^* is very close to zero, but the difference is only slight. As a whole, the POI tests are equivalent to or more powerful than the LBI test in these examples.

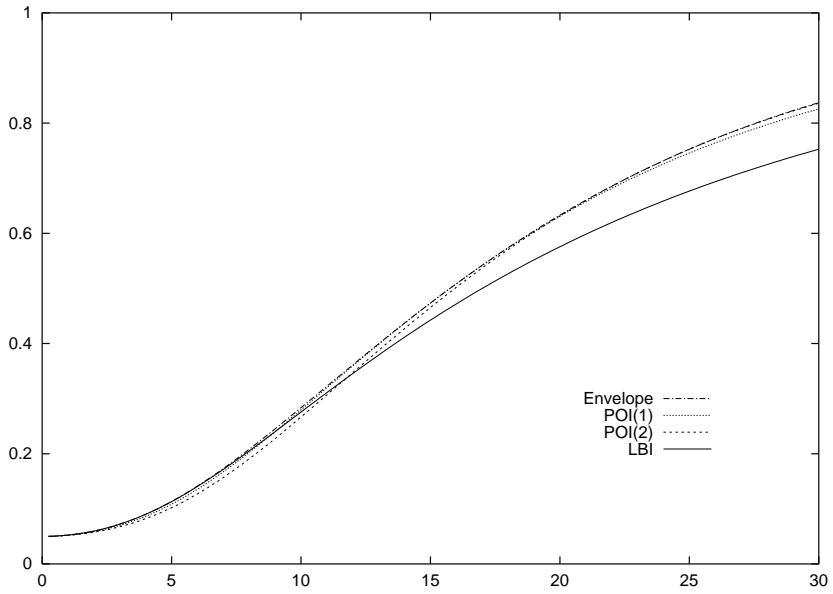


Figure 1(a). The Limiting powers and the power envelope (Case (A), $m = 1$).

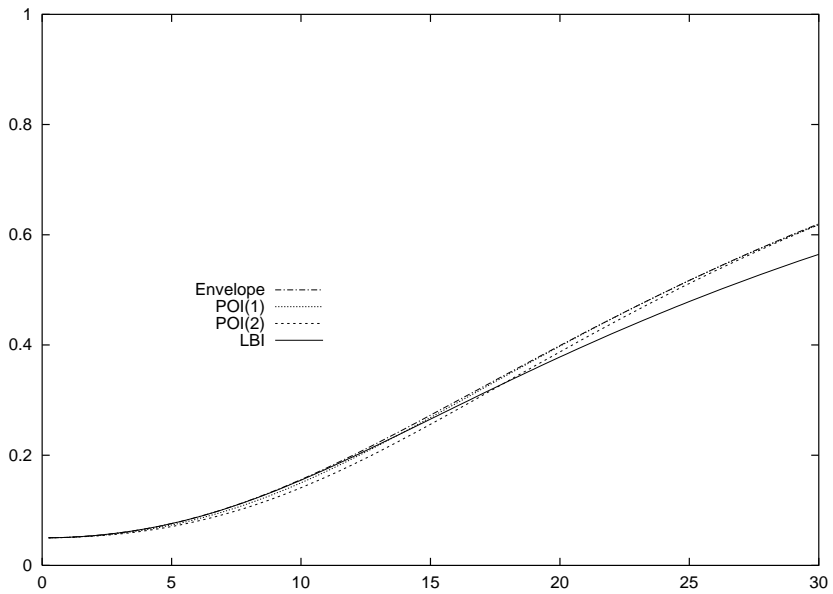


Figure 1(b). The Limiting powers and the power envelope (Case (A), $m = 2$).

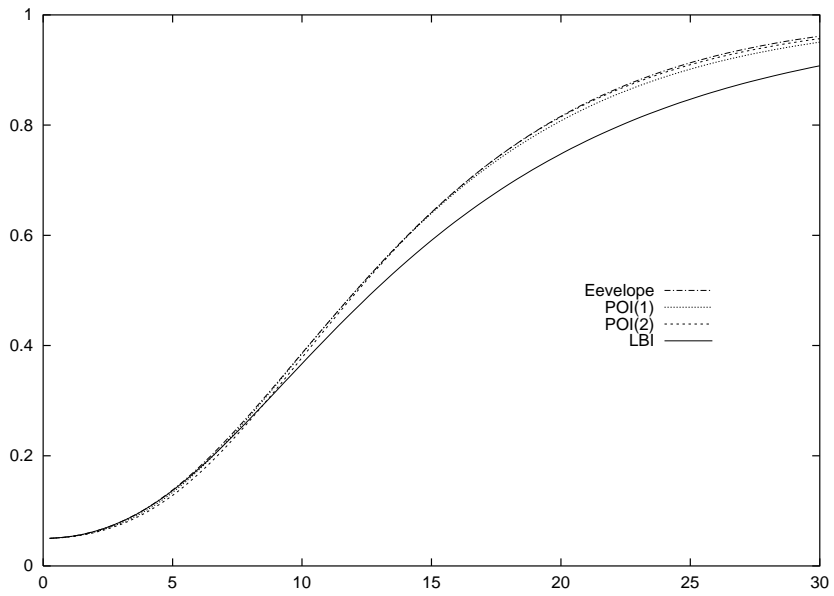


Figure 1(c). The Limiting powers and the power envelope (Case (B), $m = 1$).

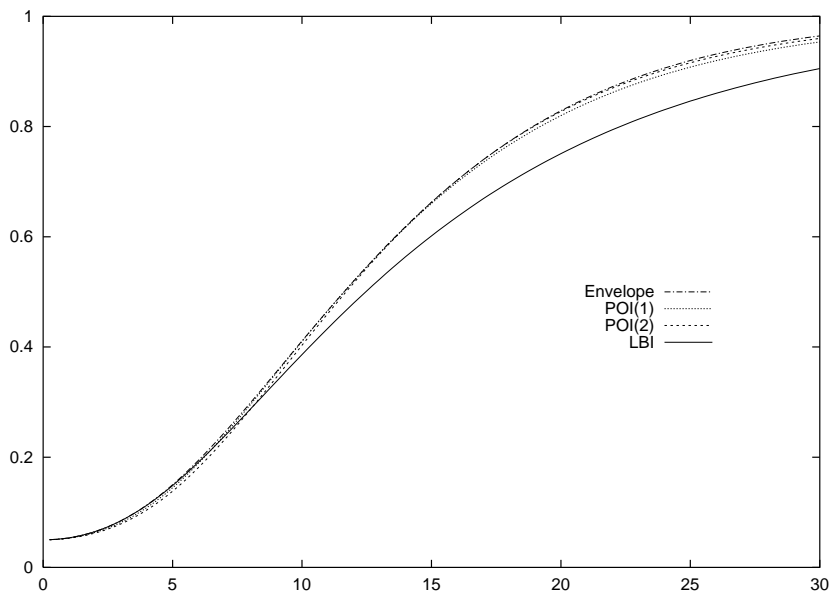


Figure 1(d). The Limiting powers and the power envelope (Case (B), $m = 2$).

Table 1. The size and power of the POI and LBI tests.

	Case (A) ($m = 1$)			Case (A) ($m = 2$)		
c^*	POI(1)	POI(2)	LBI	POI(1)	POI(2)	LBI
0	0.048	0.047	0.051	0.052	0.047	0.054
5	0.106	0.096	0.114	0.075	0.070	0.085
10	0.285	0.261	0.286	0.153	0.135	0.168
15	0.473	0.447	0.453	0.272	0.251	0.282
20	0.623	0.605	0.574	0.402	0.376	0.392
25	0.737	0.728	0.668	0.516	0.490	0.483
30	0.807	0.816	0.730	0.611	0.592	0.562
35	0.861	0.870	0.780	0.691	0.683	0.629
40	0.900	0.905	0.822	0.754	0.750	0.684
	Case (B) ($m = 1$)			Case (B) ($m = 2$)		
c^*	POI(1)	POI(2)	LBI	POI(1)	POI(2)	LBI
0	0.043	0.045	0.050	0.044	0.046	0.052
5	0.117	0.109	0.125	0.131	0.116	0.139
10	0.357	0.331	0.352	0.389	0.357	0.372
15	0.602	0.579	0.564	0.620	0.595	0.568
20	0.760	0.762	0.707	0.778	0.776	0.710
25	0.857	0.864	0.795	0.868	0.875	0.801
30	0.919	0.924	0.859	0.922	0.930	0.859
35	0.954	0.959	0.905	0.955	0.959	0.899
40	0.972	0.979	0.934	0.973	0.977	0.927

We also examine the finite sample performance of the POI and LBI tests. In the simulations, we set $\alpha_0 = 0$ and $\beta = 0$ because both tests are invariant to these parameters, and the number of iterations is 5,000. Critical values for 5% significance level are 6.4938, 6.5782, 8.0513 and 7.9572 for POI(1), 9.6976, 9.6854, 11.3682 and 11.0890 for POI(2), and 0.0958, 0.0402, 0.1479 and 0.1642 for LBI in each case. Table 1 gives the empirical size and power when $T = 100$. For each case, we can see that the empirical size is close to the nominal size, 0.05. In regard to the power of the tests, the LBI test is slightly more powerful than the POI tests when c^* is close to zero. The reason is that the LBI test is designed to be most powerful when the alternative is close to the null. On the other hand, the POI tests become more powerful when the alternative diverges from the null and in the wide range of c^* . As a whole, the finite sample performance is similar to the local asymptotics.

4. Conclusion

In this paper we considered the test for the constancy of parameters and showed that the limiting distribution of the POI test statistic exists when that of the LBI test statistic can be derived. We found that the limiting characteristic function of the POI test statistic is expressed using that of the LBI test statistic. The derivation of the characteristic function is important and useful because it is

used to calculate the percentage point and power, which are found by inverting the characteristic function.

Appendix

Derivation of (2.1) and (2.2): Using the relations $\bar{M}M = \bar{M}$, $\bar{M}'\Sigma(\rho)^{-1} = \Sigma(\rho)^{-1}\bar{M}$ and $\bar{M}\bar{M} = \bar{M}$, we have

$$\begin{aligned}\bar{M}'\Sigma(\rho)^{-1}\bar{M} &= M'\bar{M}'\Sigma(\rho)^{-1}\bar{M}M \\ &= M\Sigma(\rho)^{-1}\bar{M}\bar{M}M \\ &= M(\Sigma(\rho)^{-1}\bar{M})M.\end{aligned}$$

and then $\tilde{P}_T(c, c^*)$ is expressed as

$$\tilde{P}_T(c, c^*) = \frac{1}{\sigma_\varepsilon^2} y' \{M - M(\Sigma(\rho)^{-1}\bar{M})M\} y.$$

Following Patterson and Thompson (1971), we can find a $T \times T'$ matrix P , where $T' = T - p - 1$, such that $P'P = I_{T'}$, $PP' = M$ and $P'\Sigma(\rho)P$ becomes a diagonal matrix, which we denote as

$$P'\Sigma(\rho)P = I_{T'} + \rho P'DLL'DP = I_{T'} + \rho\Lambda,$$

where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{T'}\}$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{T'}$ are non-zero eigenvalues of $MDLL'DM$. Since both $\Sigma(\rho)^{-1}\bar{M}$ and $P(I_{T'} + \rho\Lambda)^{-1}P'$ are the generalized inverse of $M\Sigma(\rho)M$, we can see that $y'M(\Sigma(\rho)^{-1}\bar{M})My = y'MP(I_{T'} + \rho\Lambda)^{-1}P'My$ by Lemma 2.2.4 (ii) in Rao and Mitra (1971). Then, using the relations $M = PP'$ and $P'P = I_{T'}$, we have

$$\begin{aligned}\tilde{P}_T(c, c^*) &= \frac{1}{\sigma_\varepsilon^2} y' \{M - MP(I_{T'} + \rho\Lambda)^{-1}P'M\} y \\ (4.1) \quad &= \frac{1}{\sigma_\varepsilon^2} y' P \{I_{T'} - (I_{T'} + \rho\Lambda)^{-1}\} P' y.\end{aligned}$$

Next, we consider the expression of the typical element of $P'y$. Since $P'X = P'PP'X = P'MX = 0$, we have $P'y = P'\varepsilon + P'DLu$. In addition, $MDL = P\Lambda^{1/2}Q'$ by the singular value decomposition, where Q is a $T \times T'$ matrix such that $Q'Q = I_{T'}$. Using these relations and noting that $P'DL = P'PP'DL = P'MDL$, we have $P'y = P'\varepsilon + \Lambda^{1/2}Q'u$. Then, by denoting (i, j) -th element of P and Q by p_{ij} and q_{ij} respectively, the j -th element of $P'y$ is given by

$$\begin{aligned}P'y(j) &= \sum_{i=1}^T p_{ij}\varepsilon_i + \lambda_j^{1/2} \sum_{i=1}^T q_{ij}u_i \\ (4.2) \quad &= \sigma_\varepsilon \left\{ \sum_{i=1}^T p_{ij}\varepsilon_i^* + (c^{*2}\lambda_j/s(T))^{1/2} \sum_{i=1}^T q_{ij}u_i^* \right\},\end{aligned}$$

where $\varepsilon_i^* = \varepsilon_i/\sigma_\varepsilon$, $u_i^* = u_i/\sigma_u$, and we used the relation $\rho^* = \sigma_u^2/\sigma_\varepsilon^2 = c^{*2}/s(T)$.

From (4.1) and (4.2), the POI test statistic is expressed as

$$\tilde{P}_T(c, c^*) = \sum_{j=1}^{T'} \left(1 - \frac{1}{1 + c^2 \tilde{\lambda}_j} \right) \left\{ \sum_{i=1}^T p_{ij} \varepsilon_i^* + (c^{*2} \tilde{\lambda}_j)^{1/2} \sum_{i=1}^T q_{ij} u_i^* \right\}^2,$$

where $\tilde{\lambda}_j = \lambda_j/s(T)$. Letting $\tilde{\varepsilon}_j = \sum_{i=1}^T p_{ij} \varepsilon_i^*$ and $\tilde{u}_j = \sum_{i=1}^T q_{ij} u_i^*$, we obtain the expression (2.1). Independence is established since $P'P = Q'Q = I_{T'}$.

In exactly the same way, we have the expression (2.2).

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