BIVARIATE CHARACTERIZED MODEL BASED ON MEAN RESIDUAL LIFE PROPERTIES

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This paper uses the concept of characterized model for bivariate extensions of univariate life distributions based on mean residual life properties. Different bivariate distributions can be generated from different choices of marginal distributions. The retention of univariate IMRL, DMRL, NBUE, NWUE, HNBUE and HNWUE class properties in the bivariate setup has been ensured along with results of importance for reliability analysis. A characterization of the exponential, Lomax and finite range distributions has been obtained in this process.

Key words and phrases: Bivariate distribution, characterized model, failure rates, hazard function, survival function.

1. Introduction

According to Galambos and Kotz (1978), multivariate modeling remains a major challenging problem in the theory and analysis of multivariate probability involving dependence among given marginal probability laws. This problem is still of interest now as there is no unified approach available on this issue. Different studies reported in the statistical literature in this direction can be classified under two streams. These are known as modeling approach and characterization approach.

Roy (2002) pointed out five fundamental problems faced during the extension of univariate distributions to multivariate forms through characterization approach. The first problem arises at the time of using multivariate extensions of univariate measures. For example, multivariate concept of failure rate has been presented in the literature in different ways by Basu (1971), Johnson and Kotz (1975) and Shanbhag and Kotz (1987). The next problem arises out of non-unique multivariate generalizations of a univariate characterizing property. The third one is the problem of selection i.e. the choice of a characterizing property from amongst a number of such important properties. Different choices may lead to different multivariate forms. Problem also arises from the fact that the characterization results depend on the nature of the distribution to be extended. The last major limitation of the characterization approach lies in its unclear position about the retention of the basic life distribution class properties. Important life distribution classes are Increasing Failure Rate (IFR) class, Increasing Failure Rate Average (IFRA) class, New Better than Used (NBU) class, Decreasing Mean Residual Life (DMRL) class, New Better than Used in Expectation (NBUE) class, Harmonic New Better than Used in Expectation (HNBUE) class

and their dual classes, viz. DFR, DFRA, NWU, IMRL, NWUE and HNWUE.

Modeling approach on the other hand is a subjective method. Important models are due to Morgenstern (1956), Farlie (1960), Kelker (1970) and Roy and Mukherjee (1998). A major limitation of the modeling approach lies in its unclear position about the retention of the life distribution class properties. Only the Multivariate Extension Model of Roy and Mukherjee, (1998) can preserve the IFR and IFRA properties. Rests are having inconclusive answers on this issue.

Recently Roy (2002) has suggested an integration of these two approaches to avail the merits of both modeling and characterization approaches and eliminate their individual limitations. He has also demonstrated the ability of characterized model in retaining marginal class properties of specific nature.

Keeping these discussions in the backdrop, we like to increase the appeal of the characterized model of Roy (2002) by considering the modeling criterion as the mean residual life in place of reversed hazard rate. Mean residual life, being one of the most important measures in survival analysis, will be of much help for reliability study. In Section 2 of this paper we present a characterized model retaining mean residual life properties of the marginal distributions. In Section 3 we ensure bivariate IMRL, DMRL, NBUE, NWUE, HNBUE and HNWUE class properties given the corresponding univariate properties. Preservation of IFR, IFRA and NBU class properties has also been examined and a characterization of exponential, Lomax and finite range distributions has been presented in this process. In Section 4 we present a multivariate model.

2. Characterization of the model

Let $\boldsymbol{X} = (X_1, X_2)$ denote the life vector of a two-component system with component lives, X_1 and X_2. Let the marginal distribution function of X_i be $F_i(x_i)$, $i = 1, 2$, with the corresponding survival function, hazard function and failure rates as $S_i(x_i)$, $R_i(x_i)$ and $r_i(x_i)$ respectively. By definition

$$S_i(x_i) = 1 - F_i(x_i), \ R_i(x_i) = -\log S_i(x_i), \ r_i(x_i) = (d/dx_i)R_i(x_i).$$

Writing $S(x_1, x_2)$ as the survival function of $\boldsymbol{X}$ we note from Johnson and Kotz (1975) that the corresponding bivariate failure rates are defined by

$$r_i(x_1, x_2) = (\partial/\partial x_i)R(x_1, x_2), \quad i = 1, 2$$

where $R(x_1, x_2) = -\log S(x_1, x_2)$ is the underlying bivariate hazard function. Let us denote by $m_i(x_i)$, the mean residual life of X_i after the time point $X_i = x_i$, and is, by definition, $E(X_i - x_i \mid X_i \geq x_i)$, $i = 1, 2$. Now, given these marginal distributions of X_1 and X_2, which may or may not belong to the same class of life distributions, let us characterize a Dependence Model (DM) based on the following consideration.

Consideration: The basic feature of the univariate life distributions, expressed in terms of a reliability measure, must be retained in the bivariate life distribution model. One such measure is the mean residual life function. It gives rise to life distribution classes and reliability bounds (see Barlow and Proschan,

1975) and unique determination of life distribution models (see Ferguson, 1967; Sahobov and Geshev, 1974; Wang and Srivastava, 1980; Mukherjee and Roy, 1986). Then, functional forms of $m_1(x_1)$ and $m_2(x_2)$ are the two basic features that one should retain in the bivariate setup in terms of the corresponding bivariate mean residual lives

$$m_i(x_1, x_2) = E(X_i - x_i \mid X_1 \geq x_1, \ X_2 \geq x_2), \quad i = 1, 2,$$

of $X_i, \ i = 1, 2$ for $\boldsymbol{X}$. This means, in mathematical term, that for each $i = 1, 2$, $m_i(x_1, x_2)$ must be locally proportional to $m_i(x_i)$. Result 2.2 ensures that this retention of mean residual life structure characterizes a bivariate model. This characterized model will be referred as Dependence Model in our subsequent discussion. However, before we present this main result we present a lemma from Roy and Gupta (1996) which will be used for the latter derivations.

LEMMA 2.1.   *For nonnegative random variables X_1 and X_2, let the bivariate failure rates be $r_1(x_1, x_2)$ and $r_2(x_1, x_2)$ and the bivariate mean residual lives be $m_1(x_1, x_2)$ and $m_2(x_1, x_2)$. Then the following relationships hold* :

$$r_1(x_1, x_2) = \frac{1 + \dfrac{\partial}{\partial x_1} m_1(x_1, x_2)}{m_1(x_1, x_2)}, \qquad r_2(x_1, x_2) = \frac{1 + \dfrac{\partial}{\partial x_2} m_2(x_1, x_2)}{m_2(x_1, x_2)}.$$

RESULT 2.2.   *For nonnegative random variables X_1 and X_2, bivariate mean residual lives are locally proportional to the corresponding univariate mean residual lives if and only if the bivariate survival function of (X_1, X_2) is of the form*

$$(2.1) \quad S(x_1, x_2) = S_1(x_1) S_2(x_2) \exp\left[-B \int_0^{x_1} \frac{1}{m_1(u, 0)} du \int_0^{x_2} \frac{1}{m_2(0, v)} dv \right]$$

where B is a constant determined from the boundary condition.

PROOF.   (Only if) Let the bivariate mean residual lives $m_1(x_1, x_2)$ and $m_2(x_1, x_2)$ of (X_1, X_2) be locally proportional to corresponding univariate mean residual lives.

$$(2.2) \qquad m_1(x_1, x_2) = k_1(x_2) m_1(x_1, 0), \quad m_2(x_1, x_2) = k_2(x_1) m_2(0, x_2)$$

where $k_1(x_2)$ and $k_2(x_1)$ are functions of x_2 and x_1 only. Note that $k_1(0) = k_2(0) = 1$.

Then, from Lemma 2.1 we can write

$$r_1(x_1, x_2) = \{1 + k_1(x_2) m_1{}'(x_1, 0)\} / \{k_1(x_2) m_1(x_1, 0)\},$$
$$r_2(x_1, x_2) = \{1 + k_2(x_1) m_2{}'(0, x_2)\} / \{k_2(x_1) m_2(0, x_2)\}.$$

Following the concept of line integration it is now possible to obtain the following two alternative and equivalent expressions of $S(x_1, x_2)$:

(2.3)
$$S(x_1, x_2) = \exp\left[-\int_0^{x_1} \frac{1 + k_1(0)m_1'(u, 0)}{k_1(0)m_1(u, 0)} du - \int_0^{x_2} \frac{1 + k_2(x_1)m_2'(0, v)}{k_2(x_1)m_2(0, v)} dv\right]$$

and

(2.4)
$$S(x_1, x_2) = \exp\left[-\int_0^{x_2} \frac{1 + k_2(0)m_2'(0, v)}{k_2(0)m_2(0, v)} dv - \int_0^{x_1} \frac{1 + k_1(x_2)m_1'(u, 0)}{k_1(x_2)m_1(u, 0)} du\right].$$

Simplification of (2.3) gives rise to

$$S(x_1, x_2) = S_1(x_1)S_2(x_2)$$
$$\times \exp\left[-\frac{1 - k_1(0)}{k_1(0)} \int_0^{x_1} \frac{1}{m_1(u, 0)} du - \frac{1 - k_2(x_1)}{k_2(x_1)} \int_0^{x_2} \frac{1}{m_2(0, v)} dv\right]$$

(2.5)
$$= S_1(x_1)S_2(x_2) \exp\left[-\frac{1 - k_2(x_1)}{k_2(x_1)} \int_0^{x_2} \frac{1}{m_2(0, v)} dv\right].$$

Similarly, from (2.4) we have

(2.6)
$$S(x_1, x_2) = S_1(x_1)S_2(x_2) \exp\left[-\frac{1 - k_1(x_2)}{k_1(x_2)} \int_0^{x_1} \frac{1}{m_1(u, 0)} du\right].$$

Comparing (2.5) and (2.6) we note that

(2.7)
$$\frac{1 - k_2(x_1)}{k_2(x_1)} \int_0^{x_2} \frac{1}{m_2(0, v)} dv = \frac{1 - k_1(x_2)}{k_1(x_2)} \int_0^{x_1} \frac{1}{m_1(u, 0)} du$$

implying thereby that for some constant B

(2.8)
$$\begin{cases} \dfrac{1 - k_2(x_1)}{k_2(x_1)} = B \displaystyle\int_0^{x_1} \frac{1}{m_1(u, 0)} du \\ \dfrac{1 - k_1(x_2)}{k_1(x_2)} = B \displaystyle\int_0^{x_2} \frac{1}{m_2(0, v)} dv. \end{cases}$$

Using (2.8) we can simplify either (2.5) or (2.6) to get (2.1).

(If part) Let (2.1) be a bivariate survival function. By definition,

(2.9)
$$\begin{cases} m_1(x_1, x_2) = \displaystyle\int_0^\infty \frac{S(x_1 + t, x_2)}{S(x_1, x_2)} dt \\ m_2(x_1, x_2) = \displaystyle\int_0^\infty \frac{S(x_1, x_2 + t)}{S(x_1, x_2)} dt. \end{cases}$$

Simplifying (2.9) with (2.8) we observe that

$$m_1(x_1, x_2)$$

$$= \int_0^\infty \frac{S_1(x_1 + t)}{S_1(x_1)} \exp\left[-B \int_0^{x_2} \frac{1}{m_2(0,v)} dv \left\{\int_{x_1}^{x_1+t} \frac{1}{m_1(u,0)} du\right\}\right] dt$$

$$= \int_0^\infty \frac{m_1(x_1, 0)}{m_1(x_1 + t, 0)} \exp\left[-\left(1 + B \int_0^{x_2} \frac{1}{m_2(0,v)} dv\right) \int_{x_1}^{x_1+t} \frac{1}{m_1(u,0)} du\right] dt$$

(2.10)

$$= \left\{1 + B \int_0^{x_2} \frac{1}{m_2(0,v)} dv\right\}^{-1} m_1(x_1, 0),$$

because of the identity

$$1 = \int_0^\infty \frac{1}{m_1(x_1 + t, 0)} \exp\left[-\int_{x_1}^{x_1+t} \frac{1}{m_1(u,0)} du\right] dt.$$

Similarly,

$$(2.11) \qquad m_2(x_1, x_2) = \left\{1 + B \int_0^{x_1} \frac{1}{m_1(u,0)} du\right\}^{-1} m_2(0, x_2).$$

From (2.10) and (2.11) we conclude that if $S(x_1, x_2)$ is of the form (2.8) then $m_1(x_1, x_2)$ and $m_2(x_1, x_2)$ are locally proportional to $m_2(x_1, 0)$ and $m_2(0, x_2)$ respectively. Hence follows the result. □

The following is an example where a bivariate exponential distribution is derived under the assumption that bivariate mean residual lives are locally constants, being proportional to constant mean residual lives of two marginal exponential distributions. The bivariate distribution so derived is of the second form of Gumbel's bivariate exponential distributions introduced in 1960. While the work of Gumbel (1960) does not clearly spell out the underlying motivation and presents three alternative forms in a heuristic way our approach is a systematic and motivational one.

Example 2.3. Let X_1 and X_2 be two exponential variates with means μ_1 and μ_2. It is well known from univariate results that the corresponding mean residual lives will be μ_1 and μ_2. Under the proposed modeling approach we have from (2.2)

$$m_1(x_1, x_2) = k_1(x_2)\mu_1$$
$$m_2(x_1, x_2) = k_2(x_1)\mu_2.$$

In other words, mean residual lives are locally constants. Following (2.1) we obtain the corresponding survival function as

$$(2.12) \qquad S(x_1, x_2) = \exp(-x_1/\mu_1) \exp(-x_2/\mu_2) \exp[-B(x_1/\mu_1)(x_2/\mu_2)].$$

It is easy to note that $0 \leq B \leq 1$ to make (2.12) a proper survival function.

Example 2.4. Let X_1 and X_2 be two Lomax variates with survival functions as

$$S_1(x_1) = (1 + x_1/a_1)^{-\alpha_1}, \quad S_2(x_2) = (1 + x_2/a_2)^{-\alpha_2},$$
$$a_1 > 0, \ a_2 > 0, \ \alpha_1 > 1, \ \alpha_2 > 1.$$

It is well known from univariate results that the corresponding mean residual lives will be $m_1(x_1) = (a_1 + x_1)/(\alpha_1 - 1)$, and $m_2(x_1) = (a_2 + x_2)/(\alpha_2 - 1)$. Under the proposed modeling approach we have from (2.2)

$$m_1(x_1, x_2) = k_1(x_2)(a_1 + x_1)/(\alpha_1 - 1),$$
$$m_2(x_1, x_2) = k_2(x_1)(a_2 + x_2)/(\alpha_2 - 1).$$

In other words, mean residual lives are locally linear. Following (2.1) we obtain the corresponding survival function as

$$S(x_1, x_2) = (1 + x_1/a_1)^{-\alpha_1}(1 + x_2/a_2)^{-\alpha_2}$$
$$\times \exp[-B(\alpha_1 - 1)(\alpha_2 - 1)\{\log(1 + x_1/a_1)\}\{\log(1 + x_2/a_2)\}].$$

It is easy to note that $0 \leq B \leq \alpha_1\alpha_2/\{(\alpha_1 - 1)(\alpha_2 - 1)\}$ to make the above a proper survival function. This distribution is markedly different from the bivariate Lomax distribution studied in Lindley and Singpurwalla (1986). It is also different from those proposed in Arnold (1983) and in Roy and Mukherjee (1991).

From the above examples it becomes clear that for $S(x_1, x_2)$, given by (2.1), to be a proper survival function there may be some restriction on the parameter B. However, one thing is assured that for $B = 0$ we obtain the distribution of two independent variables and hence for at least one choice of B, (2.1) represents a proper survival function. To examine the other possible choices of B we consider the nature of

$$\frac{\partial^2 S(x_1, x_2)}{\partial x_1 \partial x_2},$$

which, after simplification, reduces to the following form:

$$\frac{\partial^2 S(x_1, x_2)}{\partial x_1 \partial x_2} = \left[\exp\left\{ -B \int_0^{x_1} \frac{1}{m_1(u, 0)} du \int_0^{x_2} \frac{1}{m_2(0, v)} dv \right\} \right] S_1(x_1) S_2(x_2)$$
$$\times [m_1(x_1, 0) m_2(0, x_2)]^{-1}$$
$$\times \left[\left\{ r_1(x_1, 0) m_1(x_1, 0) + B \int_0^{x_2} \frac{1}{m_2(0, v)} dv \right\} \right.$$
(2.13)
$$\left. \times \left\{ r_2(0, x_2) m_2(0, x_2) + B \int_0^{x_1} \frac{1}{m_1(u, 0)} du \right\} - B \right].$$

For $S(x_1, x_2)$ to be proper survival function we need to ensure that the right hand side of (2.13) is nonnegative for all choices of $x_1, x_2(\geq 0)$.

A sufficient condition for the same is

$$(2.14) \qquad 0 \leq B \leq \left\{ \min_{x_1} r_1(x_1,0)m_1(x_1,0) \right\} \left\{ \min_{x_2} r_2(0,x_2)m_2(0,x_2) \right\}.$$

For example, for IMRL marginals

$$r_1(x_1,0)m_1(x_1,0) \geq 1, \ r_2(0,x_2)m_2(0,x_2) \geq 1$$

and hence $0 \leq B \leq 1$ offers a sufficient parametric choice for the feasibility (2.1) as a survival function. For exponential marginals

$$r_1(x_1,0)m_1(x_1,0) = r_2(0,x_2)m_2(0,x_2) = 1$$

and hence $0 \leq B \leq 1$ describes the complete parametric choice of B. Following Mukherjee and Roy (1986), we have for Lomax and finite range distributions

$$r_1(x_1,0)m_1(x_1,0) = r_2(0,x_2)m_2(0,x_2) = K$$

where K is a constant other than 1 (greater than 1 for Lomax distribution and less than 1 for finite-range distribution). Corresponding bivariate distribution will admit the parametric choice of B as $0 \leq B \leq K^2$.

3. Some properties

Let us present some important properties of the proposed model. The first set of properties will deal with general retention of univariate class properties in the bivariate setup. The second set will cover characterizations of exponential, Lomax and a finite range distribution. For the purpose of bivariate classifications of life distributions, we refer to the complete classification system proposed by Roy (1994) retaining the univariate chain of implications.

PROPERTY 3.1. *If both the marginal distributions of X_1 and X_2 belong to IMRL/DMRL/NBUE/NWUE/HNBUE/HNWUE class of life distributions then the joint distribution of (X_1, X_2) under (2.1) also belongs to the corresponding bivariate class of life distribution.*

PROOF. Under (2.1) we have $m_1(x_1, x_2)$ locally proportional to $m_1(x_1, 0)$ and $m_2(x_1, x_2)$ locally proportional to $m_2(0, x_2)$. Thus, if X_1 and X_2 belong to IMRL (or DMRL) class then $m_1(x_1, 0)$ and $m_2(0, x_2)$ will be increasing (or decreasing) in x_1 and x_2 respectively and hence $m_1(x_1, x_2)$ will be increasing (or decreasing) in x_1 for all choices of x_2 and $m_2(x_1, x_2)$ will be increasing (or decreasing) in x_2 for all choices of x_1. Combining these observations, we have, from the definition of Roy (1994), BIMRL (or BDMRL) class property for (X_1, X_2).

Further, from property (2.2)

$$(3.1) \qquad m_1(x_1, x_2) - m_1(0, x_2) = k_1(x_2)\{m_1(x_1, 0) - m_1(0, 0)\}$$

$$(3.2) \qquad m_2(x_1, x_2) - m_2(x_1, 0) = k_2(x_1)\{m_2(0, x_1) - m_2(0, 0)\}$$

where $k_1(x_2)$ and $k_2(x_1)$ are necessarily positive. From the NBUE (or NWUE) class property of X_1 and X_2 we have the right hand sides of (3.1) and (3.2) to be negative (or positive) and hence the left hand sides of (3.1) and (3.2) must be negative (or positive) because $k_1(x_2)$ and $k_2(x_1)$ are positive. Thus, $m_1(x_1, x_2) \leq (\text{or} \geq) m_1(0, x_2)$ for all $x_1 (\geq 0)$ for every given values of x_2 and $m_2(x_1, x_2) \leq (\text{or} \geq) m_2(x_1, 0)$ for all $x_2 (\geq 0)$ for every given values of x_1. Hence from the definition of Roy (1994) we have BNBUE (or BNWUE) class property for (X_1, X_2).

Lastly, from the relation (2.2) we have

$$(3.3) \qquad \frac{1}{x_1} \int_0^{x_1} \frac{1}{m_1(u, x_2)} du = \frac{1}{k_1(x_2)} \frac{1}{x_1} \int_0^{x_1} \frac{1}{m_1(u, 0)} du$$

$$(3.4) \qquad \frac{1}{x_2} \int_0^{x_2} \frac{1}{m_2(x_1, v)} dv = \frac{1}{k_2(x_1)} \frac{1}{x_2} \int_0^{x_2} \frac{1}{m_2(0, v)} dv$$

Further, from (3.3) we have

$$(3.5)$$
$$\frac{1}{x_1} \int_0^{x_1} \frac{1}{m_1(u, x_2)} du - \frac{1}{m_1(0, x_2)} = \frac{1}{k_1(x_2)} \left\{ \frac{1}{x_1} \int_0^{x_1} \frac{1}{m_1(u, 0)} du - \frac{1}{m_1(0, 0)} \right\}$$

and from (3.4) we have

$$(3.6)$$
$$\frac{1}{x_2} \int_0^{x_2} \frac{1}{m_2(x_1, v)} dv - \frac{1}{m_2(x_1, 0)} = \frac{1}{k_2(x_1)} \left\{ \frac{1}{x_2} \int_0^{x_2} \frac{1}{m_2(0, v)} dv - \frac{1}{m_2(0, 0)} \right\}.$$

From HNBUE (or HNWUE) property of X_1 and X_2 we have the right hand side of (3.5) and (3.6) to be positive (or negative) for all $x_1 (\geq 0)$ and all $x_2 (\geq 0)$. Thus, the left hand side of (3.5) is positive (or negative) for all choices of $x_1 (\geq 0)$ for every given value of x_2 and the left hand side of (3.6) is positive (or negative) for all choices of $x_2 (\geq 0)$ for every given value of x_1. Hence from the definition of Roy (1994) we have BHNBUE (or BHNWUE) class property for (X_1, X_2). $\square$

Thus, we notice from Property 3.1 that the proposed characterized model can retain some important marginal class properties (common for both X_1 and X_2) in the bivariate set up also. This retention of class properties (IMRL, DMRL, NBUE, NWUE, HNBUE, and HNWUE) can be of much help in bivariate modeling of life distributions. We, however, did not take into consideration IFR, DFR, IFRA, DFRA, NBU, NWU properties. In our next set of properties we propose to deal with them and arrive at a characterization result for exponential, Lomax and finite range distributions.

PROPERTY 3.2. *Let the joint distribution of non-negative random variables X_1 and X_2 be given by the survival function of the form* (2.1). *Then, the bivariate*

failure rates are locally proportional to corresponding marginal failure rates if and only if the distributions of X_1 and X_2 are exponential or Lomax or finite range type.

PROOF. From Lemma 2.1 we get under (2.1)

$$(3.7) \qquad r_1(x_1, x_2) = \frac{1 - k_1(x_2)}{k_1(x_2)} \frac{1}{m_1(x_1, 0)} + r_1(x_1, 0)$$

$$(3.8) \qquad r_2(x_1, x_2) = \frac{1 - k_2(x_1)}{k_2(x_1)} \frac{1}{m_2(0, x_2)} + r_2(0, x_2).$$

For $r_1(x_1, x_2)$ to be locally proportional to $r_1(x_1, 0)$ we have from (3.7)

$$(3.9) \qquad m_1(x_1, 0) r_1(x_1, 0) = c_1(x_2)$$

and for $r_2(x_1, x_2)$ to be locally proportional to $r_2(0, x_2)$ we have from (3.8)

$$(3.10) \qquad m_2(0, x_2) r_2(0, x_2) = c_2(x_1)$$

where $c_1(x_2)$ and $c_2(x_1)$ are proportionality constants. Since the left hand sides of (3.9) and (3.10) are independent of x_2 and x_1 respectively we have for the right hand sides of (3.9) and (3.10) $c_1(x_2) = c_1$ and $c_2(x_1) = c_2$ independently of x_2 and x_1 respectively. Thus, $m_1(x_1, 0) r_1(x_1, 0) = c_1$ and $m_2(0, x_2) r_2(0, x_2) = c_2$, which are characterizing properties of exponential, Lomax and finite range distributions as given in Mukherjee and Roy (1986). In other words, bivariate failure rates are locally proportional to marginal failure rates if and only if X_1 and X_2 follow either exponential or Lomax or finite range distribution. □

It is easy to conclude from the above characterizing results that bivariate modeling via (2.1) for exponential Lomax and finite range distributions will retain Bivariate IFR (or Bivariate DFR) class properties. Similarly, Bivariate IFRA (or Bivariate DFRA) and Bivariate NBU (or Bivariate NWU) class properties can be retained by these three distributions.

4. Multivariate model

An extension of the Dependence Model to p-dimensional set up is given below.

$$
\begin{aligned}
S(x_1, \dots, x_p) \\
= S_1(x_1) \dots S_p(x_p) \\
(4.1) \qquad \times \exp\Bigg[- \sum_{i<j;\ i,j=1}^{p} B_{ij} \int_0^{x_i} \frac{1}{m_i(0, \dots, u_i, \dots, 0)} du_i \\
\times \int_0^{x_j} \frac{1}{m_j(0, \dots, u_j, \dots, 0)} du_j \Bigg]
\end{aligned}
$$

where a sufficient condition for $S(x_1, \ldots, x_p)$ to be a proper survival function is

$$0 \le B_{ij}$$
$$\le \left\{ \min_{x_i} r_i(0, \ldots, x_i, \ldots, 0) m_i(0, \ldots, x_i, \ldots, 0) \right\}$$
$$\times \left\{ \min_{x_j} r_j(0, \ldots, x_j, \ldots, 0) m_j(0, \ldots, x_j, \ldots, 0) \right\},$$

for $i, \ j = 1, \ \ldots, p$, where $r_i(0, \ldots, x_i, \ldots, 0)$ is the marginal failure rate of X_i.

It is easy to note that under this model, $m_i(x_1, \ldots, x_p) = E(X_i - x_i \mid X_1 > x_1, \ldots, X_i > x_i, \ldots, X_p > x_p)$, the multivariate mean residual life of X_i, will be

$$\begin{aligned}
& m_i(x_1, \ldots, x_p) \\
& = \int_0^\infty \{ S_i(x_i + t)/S_i(x_i) \} \\
& \qquad \times \exp \left[- \int_{x_i}^{x_i+t} \frac{1}{m_i(0, \ldots, u_i, \ldots, 0)} du_i \right. \\
& \qquad\qquad \left. \times \sum_{j=1, \ j \ne i}^p B_{ij} \int_0^{x_j} \frac{1}{m_j(0, \ldots, u_j, \ldots, 0)} du_j \right] dt \\
& = \left\{ 1 + \sum_{j=1, \ j \ne i}^p B_{ij} \int_0^{x_j} \frac{1}{m_j(0, \ldots, u_j, \ldots, 0)} du_j \right\}^{-1} \\
& \qquad \times m_i(0, \ldots, x_i, \ldots, 0).
\end{aligned}$$
(4.2)

Thus, $m_i(x_1, \ldots, x_p)$ is locally proportional to $m_i(0, \ldots, x_i, \ldots, 0)$ and this is true for all $i = 1, \ldots, p$. Now, making use of (4.2) and following the proof of Property 3.1 we can establish the following property.

PROPERTY 4.1. *If the marginal distributions of X_i, $i = 1, \ldots, p$ belong to IMRL/DMRL/NBUE/NWUE/HNBUE/HNWUE class of life distributions then the joint distribution of $(X_1, \ldots, X_p)$ under (4.1) also belongs to the corresponding multivariate class of life distribution.*

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