

ON BAYESIAN ANALYSIS OF BINOMIAL RELIABILITY GROWTH

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This paper introduces a new class of prior distributions for reliability growth tests with binomial data under the monotone model. The proposed prior has a conditional form, which accords well with various actual situations in reliability growth tests. The expressions of the corresponding means and variances for all stages with and without conditioning are obtained, and the relationship between the shape of the prior distributions and their parameters are discussed. These results are helpful in order to incorporate expert opinions. The posterior density and the Bayesian lower bound of the reliability at the end of the test, and a computation method for them are given. The new family of prior distributions includes the uniform prior used by Smith (1977) and the ordered Dirichlet priors presented by Mazzuchi and Soyer (1992, 1993) as special cases. Comparisons are made by two examples, which show the limitations of the later two.

Key words and phrases: Attribute reliability growth, monotone model, prior distributions, posterior density, Bayesian lower bound.

1. Introduction

During the development of every new product there is always a process of testing-modification cycles. Testing is performed in stages. At the end of each stage (except the final one), failures are examined such that modification can be made to the product. This results in improved reliability of the product. This is usually called the reliability growth test (RGT for short).

There has been much work done in the area of statistical analysis for RGT, and many reliability growth models have been proposed. (cf. Department of Defence (1981), Taneja and Safie (1992) and references there). For an RGT with attribute outcomes, the most attractive model is the monotone model, which says that in an RGT of m independent stages, due to modifications, the reliabilities of the product at all stages satisfy

$$(1.1) \quad 0 \leq R_1 \leq R_2 \cdots \leq R_m \leq 1,$$

where R_i is the reliability of the product at stage i . One wishes to make inference for the current reliability, R_m , of the product. This model was first introduced and discussed by Barlow and Scheuer (1966), and Read (1971) made additional remarks on the estimation of the model.

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Very often, the technician may have certain knowledge of the effectiveness of the modification. This additional information can be very helpful to the inference, especially in cases where the number of testing items is small. In these circumstances, the Bayesian method is useful. Smith (1977) discussed the Bayesian inference of the monotone model for binomial trials. He assumed a uniform prior on the parameter space. Fard and Dietrich (1987) made a comment on Smith's derivation of the proportionality constant of the joint posterior density. This prior is totally determined by the number, m , of the stages in RGT, and has no parameter that needs to be adjusted to an actual situation. Hence, it usually does neither describe the reliability engineers' knowledge nor provide guidances to the analysis of RGT.

Mazzuchi and Soyer (1992, 1993) considered a family of prior distributions, the ordered Dirichlet distributions, for binomial trials and derived prior and posterior expressions for joint and marginal distributions. The relevant decision aspects were discussed in van Dorp *et al.* (1997), and the associated computations were presented in Erkanli *et al.* (1998). The ordered Dirichlet family has $(m + 1)$ independent parameters, namely $\beta, \alpha_1, \dots, \alpha_m$, and includes the uniform prior as a special case. By choosing $\{\alpha_i\}$, the ordered Dirichlet prior describes well the engineer's knowledge on the reliability mean for each stage.

Still, many situations can not be well incorporated since it has only one parameter, β , to adjust the prior variances for all stages. For example, in most cases in reality, the prior density for each stage is unimodal. To meet this requirement, β has to be large, which result in small variances for all stages. This gives strong priors, and, hence, may largely dominate the inference, especially when the number of items in the test is small. Furthermore, the ordered Dirichlet prior requires that its $(m + 1)$ parameters are determined before the RGT. However, according to our experience in working with reliability engineers, their prior knowledge is usually based on the actual failures and the corresponding modification performed to correct the cause of the failures, since the corrective actions are directed against the failures. Also, for a specific failure there may be several corrective actions, and different actions will result in different increments of reliability. In another words, only after the first $(k - 1)$ stages of the test including the corresponding modifications are finished, can they elicit measures for specifying the prior parameters of the k th stage. It is generally difficult to determine all the prior parameters without any trials.

In this paper a new family of prior distributions is presented. The proposed prior is of conditional form and has two parameters for each stage. The prior parameters for the k th stage are determined according to the engineer's knowledge about the failures in $(k - 1)$ -th stage and the associated corrective actions for them, and can approximately describe various real situations. Also, our prior family includes the uniform and ordered Dirichlet prior distributions as special cases.

Section 2 presents a general account of Bayesian analysis for binomial RGT. The uniform prior, and the ordered Dirichlet prior, and their limitations are briefly discussed in section 3. Section 4 is devoted to the proposed family of

prior distributions. It is shown that the uniform prior and the ordered Dirichlet prior are both members of our family. For the new family, the prior means and variances with and without conditioning for each stage of the RGT are derived, and the shapes of stage-wise prior distributions with different parameters are discussed. These will help reliability engineers incorporate their opinions into testing cycles. The posterior density of the current reliability, R_m , of the product and its Bayesian lower bound (BLB) are given for the both cases of non-sequential and sequential RGT. The number theory method for the posterior computation is presented. In addition, comparisons of the proposed prior family with the ordered Dirichlet priors and the uniform prior are made by two examples.

2. General

Assumptions (Smith (1977)):

(i) Trials are statistically independent, and during stage i the probability of a success is R_i , $i = 1, \dots, m$.

(ii) During stage i , there are n_i trials ($n_i \geq 1$) resulting in s_i successes ($0 \leq s_i \leq n_i$).

(iii) Due to modification, $R_1 \leq R_2 \leq \dots \leq R_m$.

Then, for the above RGT, the likelihood function is

$$(2.1) \quad \prod_{i=1}^m \binom{n_i}{s_i} R_i^{s_i} (1 - R_i)^{f_i}, \quad \text{with } f_i = n_i - s_i, \quad i = 1, \dots, m.$$

An alternative for assumption (ii) (Mazzuchi and Soyer (1992)) is

(ii)' During stage i identical replications of the product are tested until a failure is observed.

We will call it a sequential RGT (SRGT for short). For distinction, the former (under assumption (ii)) is called a non-sequential RGT (NRGT).

Let N_i be the number of items tested at stage i ($i = 1, 2, \dots$) in a SRGT, then the likelihood function after k stages of testing for $N_i = n_i$, $i = 1, 2, \dots, k$ is

$$(2.2) \quad \prod_{i=1}^k (1 - R_i) R_i^{n_i - 1},$$

and the probability distribution of N_i for fixed R_i is

$$(2.3) \quad P_r(N_i = n_i | R_i) = (1 - R_i) R_i^{n_i - 1} \quad n_i = 1, 2, \dots$$

Denote $\tilde{R} = (R_1, \dots, R_m)$, $\tilde{s} = (s_1, \dots, s_m)$, $\tilde{r} = (r_1, r_2, \dots, r_m)$, $\Omega_k(a) = \{(x_1, \dots, x_k): 0 \leq x_1 \leq \dots \leq x_k \leq a\}$, $0 < a \leq 1$, $k = 1, 2, \dots, m$, and $\Omega = \Omega_m(1)$. Let $g(\tilde{r})$ be a prior density of \tilde{R} . For the likelihood function (2.1) of a NRGT, the joint posterior density function is

$$(2.4) \quad h(\tilde{r} | \tilde{s}) = g(\tilde{r}) \prod_{i=1}^m r_i^{s_i} (1 - r_i)^{f_i} / f(\tilde{s}),$$

where

$$(2.5) \quad f(\tilde{s}) = \int_{\Omega} g(\tilde{r}) \prod_{i=1}^m r_i^{s_i} (1 - r_i)^{f_i} d\tilde{r}.$$

The marginal posterior density function of R_m , the parameter of interest, is

$$(2.6) \quad h_m(r | \tilde{s}) = I_{[0,1]}(r) \int_{\Omega_{m-1}(r)} h(\tilde{r} | \tilde{s}) dr_1 \cdots dr_{m-1}.$$

Then, each of the following two

$$(2.7) \quad \hat{R}_m = \int_0^1 r h_m(r | \tilde{s}) dr, \quad \hat{R}_m = \text{med}\{h_m(r | \tilde{s})\}$$

is a Bayesian estimator of R_m with respect to the prior distribution, where $\text{med}\{h_m(r | \tilde{s})\}$, denoted by $M(R_m)$, is the posterior median of R_m which satisfies

$$(2.8) \quad \int_0^{M(R_m)} h_m(r | \tilde{s}) dr = 0.5.$$

For $0 < \gamma < 1$, the $(1 - \gamma)$ -th quantile, $\hat{R}_{m,L}$, of the posterior distribution which satisfies

$$(2.9) \quad \int_0^{\hat{R}_{m,L}} h_m(r | \tilde{s}) dr = 1 - \gamma$$

is Bayesian lower bound (BLB) for R_m with coverage probability γ .

For a SRGT with $N_i = n_i$, $i = 1, 2, \dots, m$, the joint posterior density of R_1, \dots, R_m can be obtained by letting $s_i = 1$, $f_i = n_i - 1$ in (2.4) and (2.5). Also, the predictive distributions for R_I and N_I after stage $I - 1$ can be easily derived, which are, respectively

$$(2.10) \quad \zeta_I(r_I | n_1, \dots, n_{I-1}) = [\eta(n_1, \dots, n_{I-1})]^{-1} \int_{\Omega_{I-1}(r_I)} g_I(r_1, \dots, r_I) \\ \cdot \prod_{i=1}^{I-1} r_i (1 - r_i)^{n_i - 1} dr_1 \cdots dr_{I-1}$$

$$(2.11) \quad P_r(N_I = n_I | n_1, \dots, n_{I-1}) = \int_0^1 r_I (1 - r_I)^{n_I - 1} \zeta_I(r_I | n_1, \dots, n_{I-1}) dr_I,$$

where

$$(2.12) \quad \eta(n_1, \dots, n_{I-1}) = \int_{\Omega} g(\tilde{r}) \prod_{i=1}^{I-1} r_i (1 - r_i)^{n_i - 1} d\tilde{r},$$

and

$$\begin{aligned}
(2.13) \quad g_I(r_1, \dots, r_I) &= \int_{r_I}^1 dr_{I+1} \cdots \int_{r_{m-1}}^1 g(\tilde{r}) dr_m \\
&= (1 - r_I)^{m-I} \int_{\Omega_{m-I}(1)} g(r_1, \dots, r_I, (1 - r_I)t_1 \\
&\quad + r_I, \dots, (1 - r_I)t_{m-I} + r_I) \\
&\quad \cdot dt_1 \cdots dt_{m-I},
\end{aligned}$$

which is the marginal prior density of R_1, \dots, R_I .

3. On the uniform prior and the ordered Dirichlet prior

Smith (1977) used the uniform prior distribution over Ω , $g(\tilde{r}) = m!I_\Omega$. Then the marginal prior density of (R_1, \dots, R_k) is

$$\begin{aligned}
(3.1) \quad m! \int_{r_k}^1 dr_{k+1} \cdots \int_{r_{m-1}}^1 dr_m I_{\Omega_k(1)} \\
= m(m-1) \cdots (m-k+1)(1-r_k)^{m-k} I_{\Omega_k(1)}, \quad k = 1, \dots, m-1,
\end{aligned}$$

and, the marginal prior density of R_k is

$$(3.2) \quad [B(k, m-k+1)]^{-1} r^{k-1} (1-r)^{m-k},$$

where $B(a, b) = [\Gamma(a)\Gamma(b)]/\Gamma(a+b)$ is the Beta function. From (3.1) it follows that the conditional density of R_k given R_1, \dots, R_{k-1} is

$$(3.3) \quad (m-k+1)(1-R_{k-1})^{k-m-1} (1-r_k)^{m-k} I_{(R_{k-1}, 1)}(r_k),$$

for $k = 2, \dots, m$. This prior is entirely determined by m , the number of stages in the test. It is usually difficult to convey the engineer's prior knowledge. For example, in real situations it is common that when a product is put into a RGT, the true value of its reliability is in between of 0.50 and 0.80, i.e., $ER_1 \in (0.5, 0.8)$. After the RGT, its reliability is usually increased up to somewhere between 0.85 and 0.98, and sometimes up to 0.999 for highly reliable products. However, with the uniform prior,

$$ER_1 = \frac{1}{m+1} = \begin{cases} 0.25 & m = 3 \\ 0.125 & m = 7 \end{cases} \quad ER_m = \frac{m}{m+1} = \begin{cases} 0.75 & m = 3 \\ 0.875 & m = 7 \end{cases}$$

it is obviously impossible to have $ER_1 \in (0.5, 0.8)$ and $ER_m \in (0.85, 0.98)$ simultaneously for any m at all.

The ordered Dirichlet prior of \tilde{R} is

$$(3.4) \quad \frac{\Gamma(\beta)}{\prod_{j=1}^{m+1} \Gamma(\beta\alpha_j)} \prod_{j=1}^{m+1} (r_j - r_{j-1})^{\beta\alpha_j - 1} I_\Omega,$$

where $\beta > 0$, $\alpha_i > 0$, $\sum_{i=1}^{m+1} \alpha_i = 1$, $r_{m+1} = 1$ and $r_0 = 0$. It is obvious that if $\beta = m+1$ and $\alpha_i = 1/(m+1)$, then the ordered Dirichlet prior becomes uniform

prior. From the identity

$$\begin{aligned}
 (3.5) \quad & \int_{\alpha}^1 x^k (x - \alpha)^{a-1} (1 - x)^{b-1} dx \\
 &= (1 - \alpha)^{a+b-1} \int_0^1 [\alpha + (1 - \alpha)y]^k y^{a-1} (1 - y)^{b-1} dy \\
 &= \sum_{i=0}^k \binom{k}{i} \alpha^{k-i} (1 - \alpha)^{a+b+i-1} \Gamma(a + i) \Gamma(b) / \Gamma(a + b + i)
 \end{aligned}$$

we obtain that the marginal prior density of R_1, \dots, R_k is

$$\begin{aligned}
 (3.6) \quad & \frac{\Gamma(\beta)}{\Gamma(\beta \sum_{i=k+1}^{m+1} \alpha_i) \prod_{i=1}^k \Gamma(\beta \alpha_i)} r_1^{\beta \alpha_1 - 1} (r_2 - r_1)^{\beta \alpha_2 - 1} \dots (r_k - r_{k-1})^{\beta \alpha_k - 1} \\
 & \cdot (1 - r_k)^{\beta \sum_{k+1}^{m+1} \alpha_i - 1} I_{\Omega_k(1)},
 \end{aligned}$$

$k = 1, 2, \dots, m - 1$; and the marginal prior density of R_k is

$$\begin{aligned}
 (3.7) \quad & \left[B \left(\beta \sum_{i=1}^k \alpha_i, \beta \sum_{j=k+1}^{m+1} \alpha_j \right) \right]^{-1} r^{\beta \sum_{i=1}^k \alpha_i - 1} (1 - r)^{\beta \sum_{j=k+1}^{m+1} \alpha_j - 1} I_{(0,1)}(r), \\
 & k = 1, 2, \dots, m,
 \end{aligned}$$

which is a Beta distribution with parameters $\beta \sum_{i=1}^k \alpha_i$ and $\beta \sum_{j=k+1}^{m+1} \alpha_j$, denoted by $Beta(\beta \sum_{i=1}^k \alpha_i, \beta \sum_{j=k+1}^{m+1} \alpha_j)$, then, the conditional prior of R_k given R_1, \dots, R_{k-1} is $Beta(\beta \alpha_k, \beta \sum_{k+1}^{m+1} \alpha_j)$ with density

$$\begin{aligned}
 (3.8) \quad & \left[B \left(\beta \alpha_k, \beta \sum_{i=k+1}^{m+1} \alpha_i \right) \right]^{-1} (1 - R_{k-1})^{1 - \beta \sum_{i=k}^{m+1} \alpha_i} (r_k - R_{k-1})^{\beta \alpha_k - 1} \\
 & \cdot (1 - r_k)^{\beta \sum_{i=k+1}^{m+1} \alpha_i - 1} I_{(R_{k-1}, 1)}(r_k),
 \end{aligned}$$

$k = 2, \dots, m$.

From (3.5) and (3.8), it follows that under the ordered Dirichlet prior the conditional mean and variance of R_k given R_1, \dots, R_{k-1} are, respectively,

$$\begin{aligned}
 (3.9) \quad & \mu_k^* = \alpha_k / \sum_{i=k}^{m+1} \alpha_i + R_{k-1} \sum_{i=k+1}^{m+1} \alpha_i / \sum_{i=k}^{m+1} \alpha_i, \\
 & v_k^* = (1 - R_{k-1})^2 \alpha_k \sum_{i=k+1}^{m+1} \alpha_i / \left[\left(\sum_{i=k}^{m+1} \alpha_i \right)^2 \left(1 + \beta \sum_{i=k}^{m+1} \alpha_i \right) \right].
 \end{aligned}$$

Mazzuchi and Soyer (1992, 1993) already calculated the expectation and variance of R_k without conditioning, which are

$$(3.10) \quad \mu_k = \sum_{i=1}^k \alpha_i, \quad v_k = \left(\sum_{i=1}^k \alpha_i \right) \left(\sum_{j=k+1}^{m+1} \alpha_j \right) / (\beta + 1).$$

Now we can see that by choosing $\{\alpha_i\}$ properly the ordered Dirichlet prior can input the engineer's perception about the reliability mean of each stage. However, it has only one parameter, β , to adjust the variance for all stages. Hence, some limitations remain to be settled. First, in reality the prior densities for product reliability are unimodal (or, at least, are bounded), which requires $\beta \sum_1^k \alpha_i > 1, \beta \sum_{k+1}^{m+1} \alpha_j > 1$ and, $\beta \alpha_k > 1$ (or, $\beta \sum_1^k \alpha_i \geq 1, \beta \sum_{k+1}^{m+1} \alpha_j \geq 1$ and $\beta \alpha_k \geq 1$) for $k = 1, 2, \dots, m$. Since, as just mentioned, usually $ER_1 \in (0.5, 0.8)$, and $ER_m \in (0.85, 0.98)$, β has to be large, which gives, by (3.9) and (3.10), small variances, and indicates strong priors for all stages.

For example, consider a RGT of 3 stages. Suppose that the engineer knows approximately the values of the 3 stages' reliability are mostly likely 0.7, 0.90 and 0.975 respectively. Then the choice for $\{\alpha_i\}$ would be

$$\alpha_1 = 0.70 \quad \alpha_2 = 0.20 \quad \alpha_3 = 0.075 \quad \alpha_4 = 0.025.$$

If he/she wants the prior densities for all stages are unimodal (or bounded), then according to (3.7), he /she must choose $\beta > 40$ (or $\beta \geq 40$). With $\beta = 44$, say, it follows from (3.10) that the prior variances of the 3 stages are, respectively

$$v_1 = 0.0047 \quad v_2 = 0.0020 \quad v_3 = 0.0005.$$

This is a very strong prior. Secondly, all the parameters of an ordered Dirichlet prior are determined simultaneously before the RGT is started (cf. Mazzuchi and Soyer (1992)). According to our experience, this is not realistic. Actually, the engineers' prior knowledge is based on the failures that have appeared during the trials and the associated modifications to correct the cause of the failures. The corrective actions are directed against the failures. Also, for a specific failure there may be several corrective actions, and different actions will result in different increments of reliability. Therefore, only after the first $(k - 1)$ stages of the test, including the corresponding modifications, are finished can they elicit a measure for specifying the prior parameters of the k th stage. However, in these situations the ordered Dirichlet priors are not quite capable. Consider the example above again, the engineer can set the prior mean and standard deviation for the first stage to be approximately 0.70 and 0.14. Then, by (3.10) $\alpha_1 = 0.7$, $\beta = 10$. In this case the ordered Dirichlet prior can be adjusted only to the second and third prior means, but not the two variances. In the example with $ER_2 = 0.9$, $ER_3 = 0.975$, the ordered Dirichlet priors for the 2nd and 3rd stages are entirely fixed, that is

$$R_2 \sim \text{Beta}(9, 1), \quad R_3 \sim \text{Beta}(9.75, 0.25) \\ v_2 = 0.0082, \quad v_3 = 0.0022.$$

Therefore, it is impossible to change anything to meet the engineer's knowledge about these two variances at all. Also, the prior density of R_3 is unbounded, which is not reasonable in practice.

4. New family of prior distributions

Motivated by the above analysis of previous work and our experiences of working with reliability engineers, a new family of prior distributions is proposed. In subsection 4.1 we introduce this family of priors and discuss its properties. Subsection 4.2 presents the associated inference and a method for the posterior computation. Finally, comparisons with ordered Dirichlet priors are made in subsection 4.3.

4.1. Definition and properties

This family of prior distributions is of conditional form, and the linearly transformed beta distributions are adopted. Specifically, let

$$(4.1) \quad g_1(r) = g_1(r | a_1, b_1) = [B(a_1, b_1)]^{-1} r^{a_1-1} (1-r)^{b_1-1} I_{(0,1)}(r),$$

$$a_1 > 0, \quad b_1 > 0,$$

$$(4.2) \quad g_k(r | R_{k-1}) = g_k(r | R_{k-1}; a_k, b_k) = [B(a_k, b_k)]^{-1} (1 - R_{k-1})^{1-a_k-b_k} \\ \cdot (r - R_{k-1})^{a_k-1} (1-r)^{b_k-1} I_{(R_{k-1},1)}(r),$$

$$a_k > 0, \quad b_k > 0, \quad k = 2, \dots, m.$$

Note that if we set $R_0 = 0$, then g_1 can be rewritten as

$$g_1(r) = [B(a_1, b_1)]^{-1} (1 - R_0)^{1-a_1-b_1} (r - R_0)^{a_1-1} (1-r)^{b_1-1} I_{(R_0,1)}(r) \hat{=} g_1(r | R_0)$$

which is of the same form as g_k in (4.2). We will refer to it hereafter. Then, the joint prior of (R_1, \dots, R_m) is

$$(4.3) \quad g(\tilde{r}) = \prod_{i=1}^m g_i(r_i | r_{i-1}) I_{\Omega}.$$

From (3.5) it follows that for given R_0, R_1, \dots, R_{k-1} , the conditional mean and variance of R_k are, respectively,

$$(4.4) \quad \mu_k^* = E^* R_k = a_k / (a_k + b_k) + b_k R_{k-1} / (a_k + b_k),$$

$$(4.5) \quad v_k^* = \text{Var}^* R_k = (1 - R_{k-1})^2 a_k b_k / \{(a_k + b_k)^2 (a_k + b_k + 1)\},$$

where E^* and Var^* stand for the conditional expectation and variance. If $a_k > 1$, $b_k > 1$, then the conditional mode of R_k is

$$(4.6) \quad m_k^* = [(a_k - 1) + (b_k - 1)R_{k-1}] / (a_k + b_k - 2).$$

The expectation and variance without conditioning can be given recursively. It is clear that

$$(4.7) \quad \mu_1 = ER_1 = a_1 / (a_1 + b_1)$$

$$(4.8) \quad \mu_k = a_k / (a_k + b_k) + b_k \mu_{k-1} / (a_k + b_k) = 1 - \prod_{i=1}^k \frac{b_i}{a_i + b_i}, \quad k = 2, \dots, m.$$

Because $\text{Var}(X) = E[\text{Var}(X | Y)] + \text{Var}[E(X | Y)]$, by simple calculation, we have

$$(4.9) \quad v_1 = \text{Var} R_1 = a_1 b_1 / [(a_1 + b_1)^2 (a_1 + b_1 + 1)],$$

$$(4.10) \quad v_k = \{a_k b_k / [(a_k + b_k)^2 (a_k + b_k + 1)]\} \prod_{i=1}^{k-1} [b_i^2 / (a_i + b_i)^2] \\ + v_{k-1} b_k (b_k + 1) / [(a_k + b_k)(a_k + b_k + 1)].$$

The expressions (4.4)–(4.10) and the following facts will help for the incorporation of the expert opinions.

(i) If $a_k = b_k = 1$, then g_k is the uniform distribution over $(R_{k-1}, 1)$, and is called the non-informative prior for stage k .

(ii) If $a_k > 1, b_k > 1$, then $g_k(R_{k-1} | R_{k-1}) = g_k(1 | R_{k-1}) = 0$, $g_k(\cdot | R_{k-1})$ is unimodal with mode given in (4.6). In detail, there are three cases:

(ii.1) $a_k = b_k > 1$ implies that $g_k(\cdot | R_{k-1})$ is symmetric about $\frac{1}{2}(1 + R_{k-1})$ with $\mu_k^* = m_k^* = \frac{1}{2}(1 + R_{k-1})$, $v_k^* = (1 - R_{k-1})^2 / [4(1 + 2a_k)]$.

(ii.2) $a_k > b_k > 1$ implies that $\frac{1}{2}(1 + R_{k-1}) < \mu_k^* < m_k^* < 1$.

(ii.3) $b_k > a_k > 1$ implies that $R_{k-1} < m_k^* < \mu_k^* < \frac{1}{2}(1 + R_{k-1})$.

(iii) If $a_k = 1 < b_k$, then $g_k(\cdot | R_{k-1})$ is strictly decreasing, $g_k(R_{k-1} | R_{k-1}) = b_k / (1 - R_{k-1})$ and $g_k(1 | R_{k-1}) = 0$.

(iv) If $b_k = 1 < a_k$, then $g_k(\cdot | R_{k-1})$ is strictly increasing, $g_k(R_{k-1} | R_{k-1}) = 0$ and $g_k(1 | R_{k-1}) = a_k / (1 - R_{k-1})$.

(v) If $a_k = 1 > b_k$, then $g_k(\cdot | R_{k-1})$ is strictly increasing, $g_k(R_{k-1} | R_{k-1}) = b_k / (1 - R_{k-1})$ and $g_k(1 | R_{k-1}) = \infty$.

(vi) If $b_k = 1 > a_k$, then $g_k(\cdot | R_{k-1})$ is strictly decreasing, $g_k(R_{k-1} | R_{k-1}) = \infty$ and $g_k(1 | R_{k-1}) = a_k / (1 - R_{k-1})$.

(vii) If $a_k > 1, b_k < 1$, then $g_k(\cdot | R_{k-1})$ is increasing, $g_k(R_{k-1} | R_{k-1}) = 0$ and $g_k(1 | R_{k-1}) = \infty$.

(viii) If $a_k < 1, b_k > 1$, then $g_k(\cdot | R_{k-1})$ is decreasing, $g_k(R_{k-1} | R_{k-1}) = \infty$ and $g_k(1 | R_{k-1}) = 0$.

In the last four cases, (v) ~ (viii), the prior densities are unbounded, and, therefore, are seldom adopted in practice. The case of $a_k < 1$ and $b_k < 1$, which implies $g_k(\cdot | R_{k-1})$ is convex with $g_k(R_{k-1} | R_{k-1}) = \infty$ and $g_k(1 | R_{k-1}) = \infty$, is clearly not practical and, hence, is not included in above.

4.2. Inference and computation

Under the prior of (4.1) and (4.2), the posterior density function of R_m for a NRG T is

$$(4.11) \quad h_m(r | \tilde{s}) = [f(\tilde{s})]^{-1} \int_{\Omega_{m-1}(r)} \prod_{i=1}^m r_i^{s_i} (r_i - r_{i-1})^{a_i - 1} (1 - r_i)^{b_i + f_i - a_{i+1} - b_{i+1}} \\ \cdot dr_1 dr_2 \cdots dr_{m-1},$$

where $r_0 = 0, a_{m+1} = 0, b_{m+1} = 1$, and

$$(4.12) \quad f(\tilde{s}) = \int_{\Omega} \prod_{i=1}^m r_i^{s_i} (r_i - r_{i-1})^{a_i-1} (1 - r_i)^{b_i+f_i-a_{i+1}-b_{i+1}} d\tilde{r}.$$

For a SRGT, the posterior density of R_m is of the same form as in (4.11) but with $s_i = n_i - 1, f_i = 1$. After stage $I - 1$ with $N_i = n_i, i = 1, \dots, I - 1, (I < m)$, the predictive distributions of R_I and N_I are, respectively, given by (2.10) and (2.11) with $g(\tilde{r})$ being expressed in (4.3).

With the posterior density of $R_m, h_m(\cdot | \tilde{s})$, being known, the point estimation of R_m can be given by (2.7), and the BLB of R_m will be obtained by (2.9).

Erkanli *et al.* (1998) presented the MCMC method for the computation of the Bayesian analysis under ordered Dirichlet priors. However, the sampling by MCMC method is relatively complicated. Here, we adopt a simple method, the number theory method to calculate the integrals in (4.11), (4.12) and (2.7)–(2.13) of the Bayesian inference. Note that the calculations of (2.8) and (2.9) are the same. If we can calculate the integrals in the left hands of (2.8) and (2.9), which are monotonously increasing in $M(R_m)$ and $\hat{R}_{m,L}$ respectively, then the solutions of the two equations will be easily obtained by iteration. Also, it is obvious that

$$\int_{\Omega_k(a)} f(r_1, \dots, r_k) dr_1 \cdots dr_k = \int_{\Omega_k(1)} a^k f(at_1, at_2, \dots, at_k) dt_1 \cdots dt_k.$$

Therefore the integrals in (4.11), (4.12), (2.7)–(2.9), (2.12) and (2.13) are all of the form

$$(4.13) \quad \int_{\Omega_k(1)} f(r_1, r_2, \dots, r_k) dr_1 dr_2 \cdots dr_k.$$

Now look at the two integrals in (2.10) and (2.11). Denote $f_1(\tilde{r}) = g(\tilde{r}) \prod_{i=1}^{I-1} r_i(1 - r_i)^{n_i-1}$ and $f_2(\tilde{r}) = g(\tilde{r}) \prod_{i=1}^I r_i(1 - r_i)^{n_i-1}$. It is obvious that these two integrals can be rewritten as

$$\begin{aligned} & \zeta_I(r_I | n_1, \dots, n_{I-1}) \\ &= \eta^{-1} r_I^{I-1} (1 - r_I)^{m-I} \int_{\Omega_{I-1}(1)} dr_1 \cdots dr_{I-1} \int_{\Omega_{m-I}(1)} \\ & \quad \cdot f_1(r_I r_1, r_I r_2, \dots, r_I r_{I-1}, r_I, r_I + (1 - r_I)t_1, \dots, r_I + (1 - r_I)t_{m-I}) \\ & \quad \cdot dt_1 \cdots dt_{m-I}, \\ & P_r(N_I = n_I | n_1, \dots, n_{I-1}) \\ &= \eta^{-1} \int_{\Omega_I(1)} dr_1 \cdots dr_I \int_{\Omega_{m-I}(1)} (1 - r_I)^{m-I} \\ & \quad \cdot f_2(r_1, r_2, \dots, r_I, r_I + (1 - r_I)t_1, \dots, r_I + (1 - r_I)t_{m-I}) dt_1 \cdots dt_{m-I}. \end{aligned}$$

Thus, these two integrals are of form

$$(4.14) \quad \int_{\Omega_I(1)} dr_1 \cdots dr_I \int_{\Omega_k(1)} f(r_1, r_2, \dots, r_I, t_1, t_2, \dots, t_k) dt_1 \cdots dt_k.$$

We suggest to use the Number Theory method (NTM for short) in Hua and Wang (1981) or Fang and Wang (1994)(see also Niederreiter (1992)) to calculate the integrals in (4.13) and (4.14). To do so, we need only to construct good represent points (called NT-net) over the integral regions $\Omega_k(1)$ and $\Omega_I(1) \times \Omega_k(1)$, denoted by $\mathcal{P}_N = \{\tilde{x}_i = (x_{i1}, \dots, x_{ik}), i = 1, \dots, N\}$ and $\mathcal{Q}_M = \{(\tilde{y}_i, \tilde{z}_i) = (y_{i1}, y_{i2}, \dots, y_{iI}, z_{i1}, \dots, z_{ik}), i = 1, \dots, M\}$ respectively, where N and M are both prime numbers. Then the integrals (4.13) and (4.14) can be calculated approximately as

$$\frac{1}{Nk!} \sum_{i=1}^N f(\tilde{x}_i), \quad \text{and} \quad \frac{1}{MI!k!} \sum_{i=1}^M f((\tilde{y}_i, \tilde{z}_i))$$

respectively.

Before the construction of \mathcal{P}_N and \mathcal{Q}_M , we first explain how to construct the NT-net of N points over $(0, 1)^m$. According to Hua and Wang (1981), given the number of dimension, m , and the prime number N , there exists a suitable generating vector $\tilde{w} = \{w_1, w_2, \dots, w_m\}$ such that the NT-net over $(0, 1)^m$, denoted by $\{\tilde{c}_i = (c_{i1}, \dots, c_{im}), i = 1, \dots, N\}$, can be obtained as follows: Define

$$(4.15) \quad q_{ij} = \begin{cases} iw_j, & \text{if } iw_j < N \\ (iw_j) \bmod(N), & \text{if } iw_j > N \text{ and } (iw_j) \bmod(N) \neq 0 \\ N, & \text{others} \end{cases}$$

where \bmod stands for the congruent operator (e.g. $(7) \bmod(5) = 2$).

Then $c_{ij} = (2q_{ij} - 1)/2N$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, m$. For $m = 2, 3, \dots, 18$ and many different N of each m , the corresponding generating vectors \tilde{w} were listed in Hua and Wang (1981).

Now we can construct the above two NT-nets \mathcal{P}_N and \mathcal{Q}_M . First, construct two NT-nets over $(0, 1)^k$ and $(0, 1)^{I+k}$, denoted by $\{\tilde{c}_i = (c_{i1}, \dots, c_{ik}), i = 1, \dots, N\}$ and $\{\tilde{d}_j = (d_{j1}, \dots, d_{jI+k}), j = 1, 2, \dots, M\}$ respectively. Then \mathcal{P}_N and \mathcal{Q}_M are, respectively:

$$(4.16) \quad \left\{ \begin{array}{l} x_{i1} = c_{i1}c_{i2}^{1/2} \cdots c_{ik}^{1/k}, \\ x_{i2} = c_{i2}^{1/2} \cdots c_{ik}^{1/k}, \\ \dots \\ x_{ik} = c_{ik}^{1/k}, \\ i = 1, 2, \dots, N \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} y_{j1} = d_{j1}d_{j2}^{1/2} \cdots d_{jI}^{1/I} \\ y_{j2} = d_{j2}^{1/2} \cdots d_{jI}^{1/I} \\ \dots \\ y_{jI} = d_{jI}^{1/I} \\ z_{j1} = d_{jI+1}d_{jI+2}^{1/2} \cdots d_{jI+k}^{1/k} \\ z_{j2} = d_{jI+2}^{1/2} \cdots d_{jI+k}^{1/k} \\ \dots \\ z_{jk} = d_{jI+k}^{1/k} \\ j = 1, \dots, M. \end{array} \right.$$

The prime number $N(M)$ is determined according to the required computation accuracy. Fang and Wang (1994) showed that under mild conditions, for a given integral dimension, the computation error of the NTM is of order $O(N^{-1+\epsilon})$ for any $\epsilon > 0$. It is well known that the computation error of random sampling method (including MCMC) is of the order $O_p(N^{-1/2})$ in probability.

Table 1.

| Testing stages j | 1 | 8 | 11 |
|--------------------|-------|-------|-------|
| μ_j -exact | 0.360 | 0.951 | 0.988 |
| μ_j -NTM | 0.360 | 0.952 | 0.989 |

The following example shows the NTM's accuracy numerically. Van Dorp *et al.* (1997) assumed the reliability growth model of (1.1) for developing optimal stopping rules during a system-development phase, and adopt the ordered Dirichlet prior with $\alpha_1 = 0.36$, $\alpha_2 = 0.34$, $\alpha_3 = 0.102$, $\alpha_4 = 0.0985$, $\alpha_5 = 0.0128$, $\alpha_6 = 0.0127$, $\alpha_7 = 0.0126$, $\alpha_8 = 0.0125$, $\alpha_9 = 0.0124$, $\alpha_{10} = 0.0123$, $\alpha_{11} = 0.0122$, $\alpha_{12} = 0.0120$ and $\beta = 50$. By (3.10), the prior mean, μ_i , of each stage without conditioning can be exactly calculated and denoted by μ_j -exact. We use NTM to calculate μ_i and compare them with the exact results. Here the above mentioned prime number is 698047. Part of the results by these two methods is given in Table 1. It is seen that the results using NTM are almost the same as the exact results.

4.3. Comparisons

Compare (3.8) with (3.3) and (4.2) we see that the ordered Dirichlet priors are members of the proposed family with $a_k = \beta\alpha_k$ and $b_k = \beta \sum_{k+1}^{m+1} \alpha_j$ and the uniform prior is the one with $a_k = 1$, $b_k = m - k + 1$, $k = 1, 2, \dots, m$.

The main advantage of the proposed family of priors over the Dirichlet priors is that it has $2m$ free parameters, 2 for each stage of the RGT. Given the prior mean of each stage, there is still one freedom of parameters to adjust for its variance, where as the Dirichlet prior has only one free parameter to adjust the variances for all m stages.

As pointed out before, the uniform prior depends only on the number of stages, m . In this subsection we present two examples and show the differences of the three priors in incorporating the experts' knowledge and the resulted inference.

Example 1. Consider the example in section 3 again. Keep the prior means of the three stages to be 0.7, 0.9 and 0.975 respectively, for a weaker prior, we can choose $a_1 = 2.8$, $b_1 = 1.2$, $a_2 = 2.2$, $b_2 = 1.2$, $a_3 = 3.3$ and $b_3 = 1.1$. This prior is certainly not an ordered Dirichlet prior. Let us call it NF for simplicity. By (4.10), it follows easily that the prior (marginal) variances of the three stages are, respectively, 0.0420, 0.0115 and 0.0015, as given in Table 2. These values are much larger than the corresponding values given by the ordered Dirichlet prior discussed in section 3, which has the same three prior marginal means and the same a_3 and b_3 (Let's call it OD). This ordered Dirichlet prior is quite strong and will largely dominate the inference if the sample sizes of the stages are small.

In the above example, suppose that the data set that resulted from this three staged RGT is $n_1 = 3$, $s_1 = 2$, $n_2 = 5$, $s_2 = 4$, $n_3 = s_3 = 5$. Using the two priors NF and OD, the resultant BLB for the reliability of the third stage with coverage probability 0.90 and 0.95 are listed in the last two columns of Table 2.

Table 2.

| Prior | v_1 | v_2 | v_3 | $\hat{R}_{3,L}(0.90)$ | $\hat{R}_{3,L}(0.95)$ |
|-------|--------|--------|--------|-----------------------|-----------------------|
| NF | 0.0420 | 0.0115 | 0.0015 | 0.934 | 0.911 |
| OD | 0.0047 | 0.0020 | 0.0005 | 0.945 | 0.931 |

Note: (1) NF belongs to the proposed prior family with $a_1 = 2.8$, $b_1 = 1.2$, $a_2 = 2.2$, $b_2 = 1.1$, $a_3 = 3.3$, $b_3 = 1.1$. OD is an ordered Dirichlet prior with $\alpha_1 = 0.7$, $\alpha_2 = 0.2$, $\alpha_3 = 0.075$, $\alpha_4 = 0.025$, $\beta = 44$, also a member of the proposed family with $a_1 = 30.8$, $b_1 = 13.2$, $a_2 = 8.8$, $b_2 = 4.4$, $a_3 = 3.3$, $b_3 = 1.1$.

(2) The Bayesian lower bound of the last stage is calculated for a RGT of three stages with trial data $n_1 = 3$, $s_1 = 2$, $n_2 = 5$, $s_4 = 4$, $n_3 = s_3 = 5$.

It is obvious that the two BLB from OD is higher than those from NF. The reason is that the sample size is small and the OD is much stronger than NF.

Example 2. This example is taken from a real RGT with a small modification made on the requests of the engineers who conducted the tests. The RGT was the first group of the tests for this kind of product. The target of the RGT was to achieve $R_m \geq 0.92$ with a confidence level 0.90. Since the products are expensive, the engineers wanted to apply the Bayesian method. Thus two statisticians were involved. However, they all agreed that the priors should approximately reflect the engineers' knowledge, as well as be simple and on the conservative side.

For the first stage, the engineers had little prior knowledge for the products reliability, though they thought the reliability was somewhere between 0.50 and 0.65. They agreed to adopt the non-informative prior, that is $a_1 = b_1 = 1$, which gives, by (4.7)–(4.9), the prior mean and standard deviation of R_1 to be $ER_1 = 0.5$, $Sd(R_1) = 0.289$. The outcome was $n_1 = 9$, $s_1 = 6$. Then the lower bound of R_1 with coverage probability 0.8 is $\hat{R}_{1L}(0.8) = 0.516$. The engineers knew that the corrective action after the first stage was strongly effective and the resulted reliability was $0.80 \sim 0.88$. Thus, they decided to take $a_2 = 4$, $b_2 = 2$, from which we have $ER_2 = 0.833$, $Sd(R_2) = 0.141$, the data of the second stage was $n_2 = 10$, $s_2 = 9$. The corresponding lower bound was $\hat{R}_{2L}(0.8) = 0.840$.

After the second stage, some improvement was made and the reliability would be, according to the engineers, about 0.90. They took $a_3 = 1$, $b_3 = 2$, then $ER_3 = 0.889$, $Sd(R_3) = 0.107$. The observation was $n_3 = 12$, $s_3 = 11$. The lower bound at this time was $\hat{R}_{3L}(0.8) = 0.890$. Then, a modification was made, by which the engineers were pretty sure that the reliability would meet the target, and the average reliability was close to 0.95. After a discussion they decided to take $a_4 = b_4 = 2$, and $n_4 = 10$. Then it follows that $ER_4 = 0.944$ and $Sd(R_4) = 0.030$. If there was no failure in the fourth stage, they would have $\hat{R}_{4L}(0.9) = 0.938$. The outcome of this stage was indeed $s_4 = n_4 = 10$. Therefore the RGT was successfully finished.

Remark. (1) If the uniform prior is adopted, then $\hat{R}_{4L}(0.9) = 0.906$, which is very conservative. (2) Obviously, the ordered Dirichlet priors can not accord this RGT. For example, to make $a_1 = b_1 = 1$, we need $\alpha_1 = 0.5$, $\sum_{i=2}^5 \alpha_i = 0.5$,

$\beta = 2$. Then, $a_k \leq 1$, $b_k \leq 1$ for $k = 2, 3, 4$. If we want $a_1 = 1$, $a_2 = 4$, $a_3 = 1$, $a_4 = b_4 = 2$, then $\alpha_1 = 0.1$, $\alpha_2 = 0.4$, $\alpha_3 = 0.1$, $\alpha_4 = 0.2$, $\alpha_5 = 0.2$, $\beta = 10$, which results $b_1 = 9$, $b_2 = 5$, $b_3 = 4$. These are very different from what they needed.

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