

PRESERVATION OF SOME NEW PARTIAL ORDERINGS UNDER POISSON AND CUMULATIVE DAMAGE SHOCK MODELS

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Suppose each of the two devices is subjected to shocks occurring randomly as events in a Poisson process with constant intensity λ . Let \bar{P}_k denote the probability that the first device will survive the k shocks and \bar{Q}_k denote such a probability for second device. Let $\bar{F}(t)$ and $\bar{G}(t)$ denote the survival functions of the first and second device respectively. In this paper we show that some new partial ordering, namely dual (D), dual stochastic (DST), dual weak likelihood ratio ($DWLR$), increasing failure rate (IFR), dual mean residual lives ($DMRL$) and dual convex (DCX) orderings between the shock survival probabilities \bar{P}_k and \bar{Q}_k are preserved by the corresponding survival function $\bar{F}(t)$ and $\bar{G}(t)$. We also obtain sufficient condition under which the above mentioned relations between the discrete distributions are verified in some cumulative damage shock models.

Key words and phrases: Stochastic order; Dual stochastic order; Dual weak likelihood order; Increasing failure rate; Dual mean residual lives; Dual convex; Shock models.

1. Introduction

Suppose that a device is subjected to shocks occurring randomly as events in a Poisson process with constant λ . Suppose further that the device has probability \bar{P}_k of surviving the first k shocks, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \dots$. The survival function of the device is given by

$$(1.1) \quad \bar{F}(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!} \bar{P}_k.$$

Let $p_{k+1} = \bar{P}_k - \bar{P}_{k+1}$, $k = 0, 1, 2, \dots$, and let $f(t)$ be the probability density function corresponding to survival function $\bar{F}(t)$, such that

$$f(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!} \lambda p_{k+1}.$$

Esary *et al.* (1973) have shown that the positive aging classes of life distribution, namely a PF_2 density, increasing failure rate (IFR), decreasing mean residual life ($DMRL$), increasing failure rate average ($IFRA$), new better than used (NBU) and new better than used in expectation ($NBUE$) are preserved

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under the transformation (1.1) in the sense that if the shocks survival probability \bar{P}_k belongs to a discrete version of one of the above classes, then the continuous-time survival probability $\bar{F}(t)$ belongs to that class. Klefsjo (1981) has established that if \bar{P}_k belongs to the discrete version of the harmonic new better than used in expectation (*HNBUE*) class of life distributions, then $\bar{F}(t)$ belongs to the *HNBUE* class.

Consider another device which is also subjected to shocks occurring randomly as events in a Poisson process with same constant intensity λ and the device has probability \bar{Q}_k of surviving the first k shocks, where $1 = \bar{Q}_0 \geq \bar{Q}_1 \geq \dots$. The survival function of this device is given by

$$\bar{G}(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!} \bar{Q}_k.$$

Let $q_{k+1} = \bar{Q}_k - \bar{Q}_{k+1}$, $k = 0, 1, 2, \dots$, and let $g(t)$ be the probability density function corresponding to the survival function $\bar{G}(t)$ given by

$$g(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!} \lambda q_{k+1}.$$

Singh and Jain (1989) have shown that some partial orderings, namely likelihood ratio (*LR*) ordering, failure rate (*FR*) ordering, stochastic (*ST*) ordering, variable (*V*) ordering and mean residual life (*MRL*) ordering between the two shock survival probabilities \bar{P}_k and \bar{Q}_k are preserved by the corresponding survival functions $\bar{F}(t)$ and $\bar{G}(t)$ of the devices.

In this paper, we show that some new partial ordering, namely dual (*D*), dual stochastic (*DST*), dual weak likelihood ratio (*DWLR*), increasing failure rate (*IFR*), dual mean residual lives (*DMRL*) and dual convex (*DCX*) ordering, between the two shock survival probabilities \bar{P}_k and \bar{Q}_k are preserved by the corresponding survival function $\bar{F}(t)$ and $\bar{G}(t)$. We also obtain sufficient condition under which the above mentioned relations between the discrete distributions are verified in some cumulative damage shock models.

In Section 2, we present definitions, notation, and basic facts used throughout the paper. We also introduce the definitions of the partial orderings, which are known in the literature and our new partial orderings.

In Section 3, Poisson shock models are considered, and preservation result on new partial orderings of survival function of two devices subjected to similar shocks is proposed.

In Section 4, we compare two cumulative damage shock models. We suppose that the k th shocks causes a common damage to the two devices and damage accumulates additively in each device. The first (second) device fails when the accumulated damage exceeds the threshold $M(N)$, where M and N are random variables with different distributions. Then we consider a common fixed threshold Z , and suppose that the k -th shocks causes damage X_i to the first device and damage Y_i to the second one, and that damage accumulates additively.

2. Utility notions and definitions

In this section, we present definitions, notation, and basic facts used throughout the paper. Let X be a continuous [discrete] non-negative random variable. Define $N = \{0, 1, 2, \dots\}$. Let $F(x)$ [$P_k, k \in N$] be a continuous [discrete] distribution function, $f(x)$ [p_k] probability density function, and mean μ_x , which we suppose to exist.

The function $\bar{F}(x) = 1 - F(x)$ [$\bar{P}_k = 1 - P_k$] is called the survival function of the continuous [discrete] non-negative random variable X , and the function $r_F(x) = f(x)/\bar{F}(x)$ [$r_{P_k} = p_k/\bar{P}_k$] is called its failure rate. The equilibrium distribution corresponding to the continuous [discrete] non-negative random variable X is defined as

$$E_F(x) = \frac{1}{\mu_x} \int_0^x \bar{F}(u) du \left[E_P = 1 - \left(\frac{\sum_{j=k}^{\infty} \bar{P}_j}{\sum_{j=0}^{\infty} \bar{P}_j} \right), k \in N \right].$$

Let Y be another non-negative absolutely continuous [discrete] random variable, with distribution function $G(x)$ [Q_k], density function $g(x)$ [q_k], survival function $\bar{G}(x)$ [\bar{Q}_k] and failure rate $r_G(x)$ [r_{Q_k}]. The equilibrium distribution corresponding to the continuous [discrete] non-negative random variable Y is defined as

$$E_G(x) = \frac{1}{\mu_x} \int_0^x \bar{G}(u) du \left[E_Q = 1 - \left(\frac{\sum_{j=k}^{\infty} \bar{Q}_j}{\sum_{j=0}^{\infty} \bar{Q}_j} \right), k \in N \right].$$

For the sake of completeness, we present at first definitions, notation, and basic facts used throughout the paper we also introduce the partial orderings, which are known in literature.

DEFINITION 2.1. *Let A and B be subset of the real line. A function $k(x, y)$ on $A \times B$ is said to be totally positive of order 2 (TP_2) if*

$$\begin{vmatrix} k(x_1, y_1) & k(x_1, y_2) \\ k(x_2, y_1) & k(x_2, y_2) \end{vmatrix} \geq 0 \quad \text{for all } x_1 < x_2 \text{ in } A \text{ and } y_1 < y_2 \text{ in } B.$$

DEFINITION 2.2. *A function $h(x)$ is PF_2 if*

(a) $h(x) \geq 0$ for $-\infty < x < \infty$.

$$(b) \begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \geq 0, \quad \text{for all } \begin{matrix} -\infty < x_1 < x_2 < \infty, \\ -\infty < y_1 < y_2 < \infty. \end{matrix}$$

DEFINITION 2.3. *The continuous [discrete] random variable X , or its distribution $F(x)$ [P_k], is said to be larger than Y in stochastic ordering (denoted by $X \stackrel{ST}{\geq} Y$) if*

$$\frac{\bar{F}(x)}{\bar{G}(x)} \geq \frac{\bar{F}(0)}{\bar{G}(0)}, \quad \forall x \geq 0 \quad \left[\frac{\bar{P}_k}{\bar{Q}_k} \geq \frac{\bar{P}_0}{\bar{Q}_0}, \forall k \geq 0 \right].$$

DEFINITION 2.4. *The continuous [discrete] random variable X , or its distribution $F(x) [P_k]$, is said to be larger than Y in likelihood ratio ordering (denoted by $X \overset{\text{LR}}{\geq} Y$) if*

$$\frac{f(x)}{g(x)} \left[\frac{p_{k+1}}{q_{k+1}} \right] \text{ is non-decreasing in } x \geq 0 [k \geq 0].$$

DEFINITION 2.5. *The continuous [discrete] random variable X , or its distribution $F(x) [P_k]$, is said to be larger than Y in weak likelihood ratio ordering (denoted by $X \overset{\text{WLR}}{\geq} Y$) if*

$$\frac{f(x)}{g(x)} \geq \frac{f(0)}{g(0)}, \quad \forall x \geq 0 \quad \left[\frac{p_{k+1}}{q_{k+1}} \geq \frac{p_0}{q_0}, \forall k \geq 0 \right].$$

DEFINITION 2.6. *The continuous [discrete] random variable X , or its distribution $F(x) [P_k]$, is said to be larger than Y in failure rate ordering (denoted by $X \overset{\text{FR}}{\geq} Y$) if*

$$\frac{\bar{F}(x)}{\bar{G}(x)} \left[\frac{\bar{P}_k}{\bar{Q}_k} \right] \text{ is non-decreasing in } x \geq 0 [k \geq 0].$$

DEFINITION 2.7. *The continuous [discrete] random variable X , or its distribution $F(x) [P_k]$, is said to be larger than Y in mean residual life ordering (denoted by $X \overset{\text{mrl}}{\geq} Y$) if*

$$\frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(u) du \geq \frac{1}{\bar{G}(t)} \int_t^\infty \bar{G}(u) du \quad \forall t \geq 0$$

$$\left[\frac{1}{\bar{P}_j} \sum_{k=j}^\infty \bar{P}_k \geq \frac{1}{\bar{Q}_j} \sum_{k=j}^\infty \bar{Q}_k, j = 0, 1, \dots \right].$$

DEFINITION 2.8. *The continuous [discrete] random variable X , or its distribution $F(x) [P_k]$, is said to be larger than Y in variance residual life ordering (denoted by $X \overset{\text{VRL}}{\geq} Y$) if*

$$\frac{\int_x^\infty \bar{E}_F(s) ds}{\int_x^\infty \bar{E}_G(s) ds} \left[\frac{1}{\bar{P}_j} \sum_{l=j}^\infty \sum_{k=j}^\infty \bar{P}_k / \frac{1}{\bar{Q}_j} \sum_{l=j}^\infty \sum_{k=j}^\infty \bar{Q}_k \right] \text{ is non-decreasing in } x \geq 0 [l \geq 0].$$

DEFINITION 2.9. *The continuous [discrete] random variable X , or its distribution $F(x)$ $[P_k]$, is said to be larger than Y in convex ordering (denoted by $X \overset{\text{CX}}{\geq} Y$) if*

$$\int_x^\infty \bar{F}(s)ds \geq \int_x^\infty \bar{G}(s)ds, \quad \forall x \geq 0 \quad \left[\sum_{k=j}^\infty \bar{P}_k \geq \sum_{k=j}^\infty \bar{Q}_k, \forall j = 0, 1, 2, \dots \right].$$

DEFINITION 2.10. *The continuous [discrete] random variable X , or its distribution $F(x)$ $[P_k]$, is said to be larger than Y in concave ordering (denoted by $X \overset{\text{CV}}{\geq} Y$) if*

$$\int_0^x \bar{F}(s)ds \geq \int_0^x \bar{G}(s)ds, \quad \forall x \geq 0 \quad \left[\sum_{k=0}^j \bar{P}_k \geq \sum_{k=0}^j \bar{Q}_k, \forall j = 0, 1, 2, \dots \right].$$

In the following we state some properties of increasing failure rate (*IFR*) and decreasing mean residual life (*DMRL*) orderings in reliability theory. For any random variable Z and an event A we denote by $[Z | A]$ any random variable that has as its distribution the conditional distribution of Z given A .

The random variable X is *IFR* if, one of the following equivalent conditions holds (when the support of the distribution function of X is bounded):

- (a) $[X - t | X > t] \geq_{\text{FR}} [X - t' | X > t']$ whenever $t \leq t'$.
- (b) $X \geq_{\text{FR}} [X - t | X > t]$ for all $t \geq 0$ (when X is a nonnegative random variable).
- (c) $X + t \leq_{\text{FR}} X + t'$ whenever $t \leq t'$.

The random variable X is *DMRL* if, and only if, one of the following equivalents conditions hold:

- (a) $[X - t | X > t] \geq_{\text{mrl}} [X - t' | X > t']$ whenever $t \leq t'$.
- (b) $X \geq_{\text{mrl}} [X - t | X > t]$ for all $t \geq 0$ (when X is a nonnegative random variables).
- (c) $X + t \leq_{\text{mrl}} X + t'$ whenever $t \leq t'$.

Note that if X is the lifetime of a device, then $[X - t | X > t]$ is the residual life of such a device with age t . Alternative properties of *IFR* and *DMRL* distribution and all other previously orderings may be found in Shaked and Shanthikumar (1994).

Now we introduce our new partial orderings.

DEFINITION 2.11. *The continuous [discrete] random variable X , or its distribution $F(x)$ $[P_k]$, is said to be larger than Y in dual ordering (denoted by $X \overset{\text{D}}{\geq} Y$) if*

$$\bar{F}(x) \leq R\bar{G}(x), \quad x \geq 0, \quad R \geq 1[\bar{P}_k \leq R\bar{Q}_k, K = 1, 2, \dots].$$

DEFINITION 2.12. *The continuous [discrete] random variable X , or its distribution $F(x)$ [P_k], is said to be larger than Y in dual stochastic ordering (denoted by $X \stackrel{\text{DST}}{\geq} Y$) if*

$$\frac{\bar{F}(x)}{\bar{G}(x)} \leq \lim_{z \rightarrow \infty} \frac{\bar{F}(z)}{\bar{G}(z)} = \lim_{z \rightarrow \infty} \frac{f(z)}{g(z)} \left[\frac{\bar{P}_k}{\bar{Q}_k} \leq \lim_{j \rightarrow \infty} \frac{\bar{P}_j}{\bar{Q}_j} = \lim_{j \rightarrow \infty} \frac{p_j}{q_j} \right].$$

DEFINITION 2.13. *The continuous [discrete] random variable X , or its distribution $F(x)$ [P_k], is said to be larger than Y in dual weak likelihood ratio ordering (denoted by $X \stackrel{\text{DWLR}}{\geq} Y$) if*

$$\frac{f(x)}{g(x)} \leq \lim_{z \rightarrow \infty} \frac{f(z)}{g(z)} \left[\frac{p_k}{q_k} \leq \lim_{j \rightarrow \infty} \frac{p_j}{q_j} \right].$$

DEFINITION 2.14. *X is said to be larger than Y in increasing failure rate ordering (denoted by $X \stackrel{\text{IFR}}{\geq} Y$) if*

$$\frac{r_X(t)}{r_Y(t)} \left[\frac{p_k}{\bar{P}_k} / \frac{q_k}{\bar{Q}_k} \right] \text{ is non-decreasing in } t \forall t \geq 0 [\text{in } k \in N].$$

DEFINITION 2.15. *The continuous [discrete] random variable X , or its distribution $F(x)$ [P_k], is said to be larger than Y in dual mean residual life ordering (denoted by $X \stackrel{\text{DMRL}}{\geq} Y$) if*

$$\frac{\int_t^\infty \bar{F}(x) dx}{\int_t^\infty \bar{G}(x) dx} \leq \lim_{z \rightarrow \infty} \frac{\bar{F}(z)}{\bar{G}(z)} = \lim_{z \rightarrow \infty} \frac{f(z)}{g(z)} \left[\frac{\sum_{k=j}^\infty \bar{P}_k}{\sum_{k=j}^\infty \bar{Q}_k} \leq \lim_{j \rightarrow \infty} \frac{\bar{P}_j}{\bar{Q}_j} = \lim_{j \rightarrow \infty} \frac{p_j}{q_j} \right].$$

DEFINITION 2.16. *The continuous [discrete] random variable X , or its distribution $F(x)$ [P_k], is said to be larger than Y in dual convex ordering (denoted by $X \stackrel{\text{DCX}}{\geq} Y$) if*

$$\frac{\int_x^\infty \bar{F}(u) du}{\int_x^\infty \bar{G}(u) du} \leq \lim_{z \rightarrow \infty} \frac{\bar{F}(z)}{\bar{G}(z)} = \lim_{z \rightarrow \infty} \frac{f(z)}{g(z)} \left[\frac{\sum_{k=j}^\infty \bar{P}_k}{\sum_{k=j}^\infty \bar{Q}_k} \leq \lim_{j \rightarrow \infty} \frac{\bar{P}_j}{\bar{Q}_j} = \lim_{j \rightarrow \infty} \frac{p_j}{q_j} \right].$$

3. Preservation of new partial orderings under Poisson shock model

In the derivation of the result, we used methods of total positively and particular the variation diminishing property of the total positive (TP) function

$$Z_k(t) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}, \quad t \geq 0, \quad k = 0, 1, 2, \dots \quad (\text{see Karlin 1968, p. 21}).$$

The results are summarized as follows.

THEOREM 3.1. *Let $\bar{P}_k \stackrel{*}{\leq} \bar{Q}_k$, then $\bar{F}(t) \leq \bar{G}(t)$ where $(*)$ is one of the following:*

- (a) *Dual ordering (D).*
- (b) *Dual stochastic ordering (DST).*
- (c) *Dual weak likelihood ratio (DWLR).*
- (d) *Increasing failure rate ordering (IFR).*
- (e) *Dual mean residual live (DMRL).*
- (f) *Dual convex ordering (DCX).*

PROOF. (i) Since $\bar{P}_k \stackrel{D}{\leq} \bar{Q}_k$, then $\bar{P}_k \geq R\bar{Q}_k$, $R \geq 1$; that is

$$\bar{P}_k - R\bar{Q}_k \geq 0, \quad \forall k \in N.$$

By variation diminishing property of $Z_k(t)$, it follows that

$$\sum_{k=0}^{\infty} \{\bar{P}_k - R\bar{Q}_k\} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \geq 0.$$

Thus also

$$\sum_{k=0}^{\infty} \bar{P}_k \left[\frac{(\lambda t)^k}{k!} \right] e^{-\lambda t} \geq R \sum_{k=0}^{\infty} \bar{Q}_k \left[\frac{(\lambda t)^k}{k!} \right] e^{-\lambda t}, \quad \forall t \geq 0,$$

which implies that $\bar{F}(t) \stackrel{D}{\leq} \bar{G}(t)$.

(ii) Since $\bar{P}_k \stackrel{DST}{\leq} \bar{Q}_k$, this implies that $\bar{P}_k/\bar{Q}_k \geq \lim_{j \rightarrow \infty} \bar{P}_j/\lim_{j \rightarrow \infty} \bar{Q}_j$, $\forall k \in N$; that is,

$$\left\{ \bar{P}_k \left[\lim_{j \rightarrow \infty} \bar{Q}_j \right] - \bar{Q}_k \left[\lim_{j \rightarrow \infty} \bar{P}_j \right] \right\} \geq 0, \quad \forall k \in N;$$

that is by variation diminishing property of $Z_k(t)$, it follows that

$$\sum_{k=0}^{\infty} \left[\bar{P}_k \left[\lim_{j \rightarrow \infty} \bar{Q}_j \right] - \bar{Q}_k \left[\lim_{j \rightarrow \infty} \bar{P}_j \right] \right] \frac{(\lambda t)^k}{k!} e^{-\lambda t} \geq 0, \quad \forall t \geq 0.$$

Thus also

$$\frac{\sum_{k=0}^{\infty} \bar{P}_k [(\lambda t)^k/k!] e^{-\lambda t}}{\sum_{k=0}^{\infty} \bar{Q}_k [(\lambda t)^k/k!] e^{-\lambda t}} \geq \frac{\lim_{j \rightarrow \infty} \left\{ \sum_{j=k}^{\infty} \bar{P}_j [(\lambda t)^j/j!] e^{-\lambda t} \right\}}{\lim_{j \rightarrow \infty} \left\{ \sum_{j=k}^{\infty} \bar{Q}_j [(\lambda t)^j/j!] e^{-\lambda t} \right\}}, \quad k \in N, \quad \forall t \geq 0,$$

or, equivalently,

$$\frac{\bar{F}(t)}{\bar{G}(t)} \geq \lim_{z \rightarrow \infty} \frac{\bar{F}(z)}{\bar{G}(z)},$$

which implies that $\bar{F}(t) \stackrel{DST}{\leq} \bar{G}(t)$

(iii) Since $\bar{P}_k \stackrel{DWLR}{\leq} \bar{Q}_k$, this implies that $p_k/q_k \geq \lim_{j \rightarrow \infty} p_j/q_j$, and the proof is similar as (ii).

(iv) For any real c ,

$$\bar{F}(t) - c\bar{G}(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} (\bar{P}_k - c\bar{Q}_k).$$

The assumption $\bar{P}_k \stackrel{\text{IFR}}{\leq} \bar{Q}_k, \forall k \in N$, implies that \bar{P}_k/\bar{Q}_k is a non-decreasing function in $k = 1, 2, \dots$, $\bar{P}_k - c\bar{Q}_k$ has at most one change of sign, and if one change occur, it occurs from + to -. By variation diminishing property of $Z_k(t) = e^{-\lambda t}(\lambda t)^k/k!, t \geq 0, k \in N$ it follows that the same statement may be made for $\bar{F}(t) - c\bar{G}(t)$ as a function of t , it is follows that $\bar{F}(t)/\bar{G}(t)$ is non-decreasing function of t implying that

$$f(t)/\bar{F}(t) \leq g(t)/\bar{G}(t) \quad \forall t \geq 0 \quad (\text{cf. Ross (1983), p. 281}),$$

which implies that $\bar{F}(t) \stackrel{\text{IFR}}{\leq} \bar{G}(t)$.

(v) Since $\bar{P}_k \stackrel{\text{DMRL}}{\leq} \bar{Q}_k$, this implies that

$$\frac{\sum_{k=j}^{\infty} \bar{P}_k}{\sum_{k=j}^{\infty} \bar{Q}_k} \geq \lim_{j \rightarrow \infty} \frac{\bar{P}_j}{\bar{Q}_j} \Rightarrow \frac{\sum_{k=j}^{\infty} \bar{P}_k}{\sum_{k=j}^{\infty} \bar{Q}_k} \text{ is a non-decreasing function in } k = 1, 2, \dots$$

Now for any real c , we have

$$(3.1) \quad \int_t^{\infty} \bar{F}(u)du - c \int_t^{\infty} \bar{G}(u)du = \frac{1}{\lambda} \sum_{j=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \left\{ \sum_{k=j}^{\infty} \bar{P}_k - c \sum_{k=j}^{\infty} \bar{Q}_k \right\}.$$

By using arguments similar to (iv), it is follows from (3.1) that

$$\int_t^{\infty} \bar{F}(u)du / \int_t^{\infty} \bar{G}(u)du \text{ is a non-decreasing function of } t \geq 0,$$

implying that

$$\frac{1}{\bar{F}(t)} \int_t^{\infty} \bar{F}(u)du \geq \frac{1}{\bar{G}(t)} \int_t^{\infty} \bar{G}(u)du, \quad \forall t \geq 0.$$

See Deshpand *et al.* (1988).

The proof of (vi) is similar to (v).

4. Preservation of new partial ordering under cumulative damage shock models

Suppose that k -th shocks causes a common damage to the two devices and damage accumulates additively in each device. The first (second) device fails when the accumulated damage exceeds the threshold $M(N)$, Where M and N are random variables with different distributions. Then we consider a common fixed threshold Z , and we suppose that the k -th shocks causes damage accumulates additively. We assume that the damage (for each device) is independent of the underlying counting process.

Let $Z_i, i \in \{1, 2, \dots\} = N_+$, be independent random variables, and let W_i and w_i be their distribution and density, respectively. Write

$$W^{(k)} = W_1^* \dots^* W_k$$

and

$$w^{(k)} = w_1^* \dots^* w_k,$$

where the asterisk denotes convolution. We use the following result.

LEMMA 4.1. *If the random variables Z_i are such that they:*

- (a) *have PF_2 densities, then $w^{(k)}(t)$ is TP_2 in $(k, t) \in N_+ \times R$;*
- (b) *have the IFR property, then $\bar{W}^{(k)}(t)$ is TP_2 in $(k, t) \in N_+ \times R_j$.*
- (c) *have the DMRL property, then $\int_t^\infty \bar{W}^{(k)}(u)du$ is TP_2 in $(k, t) \in N_+ \times R$.*

The proof of Lemma 4.1 (a) can be found in Karlin and Proschan (1960) and the proof of (b) and (c) can be obtained by using similar arguments.

Let $Z_i, i \in N_+$, be the interarrival times of the counting process. Then the survival functions $\bar{F}(t)$ and $\bar{G}(t)$ can be expressed by

$$(4.1) \quad \bar{F}(t) = \sum_{k=1}^{\infty} p_k \bar{W}^{(k)}(t) \quad \text{and} \quad \bar{G}(t) = \sum_{k=1}^{\infty} q_k \bar{W}^{(k)}(t),$$

where $p_k = \bar{P}_{k-1} - \bar{P}_k$ and $q_k = \bar{Q}_{k-1} - \bar{Q}_k$ are the probabilities of failure during the k -th shocks for the first and the second device, respectively (we assume $p_0 = q_0 = 0$). The density function $f(t)$ and $g(t)$ corresponding to $F(t)$ and $G(t)$ are given by

$$f(t) = \sum_{k=0}^{\infty} p_k w^{(k)}(t) \quad \text{and} \quad g(t) = \sum_{k=0}^{\infty} q_k w^{(k)}(t).$$

Let the interarrival times Z_i have mean $\mu_i \leq +\infty$ for all $i \in N_+$. The survival equilibrium functions \bar{E}_1 and \bar{E}_2 corresponding F and G are given by

$$\bar{E}_1(t) \sum_{k=0}^{\infty} \frac{p_k}{\mu_p} \int_t^\infty \bar{W}^{(k)}(u)du \quad \text{and} \quad \bar{E}_2(t) = \sum_{k=0}^{\infty} \frac{q_k}{\mu_Q} \int_t^\infty \bar{W}^{(k)}(u)du,$$

where $\mu_p = \sum_{k=0}^{\infty} \mu_k p_k$ and $\mu_Q = \sum_{k=0}^{\infty} \mu_k q_k$ (we assume the convergence of the two sums). We also use the following results.

LEMMA 4.2. *Let $C(t, i) = \sum_{k=0}^{\infty} A(t, k)b(k, i)$ and $B(k, i) = \sum_{j=k}^{\infty} b(j, i)$, where $t \in R$ and $i \in I \subseteq N$. If $A(t, k)$ and $B(k, i)$ are TP_2 in $(t, k) \in R \times N$; and $(k, i) \in N \times I$, respectively, then $C(t, i)$ is TP_2 in $(t, i) \in R \times N$.*

This lemma is a special case of a result by Shanthikumar (1988). It must be pointed out that Lemma 4.2 is not the same as the well-known basic composition formula of Karlin (1968) (see Pellerey (1993)).

Let U and V be two discrete random variables with survival probabilities \bar{P}_k and \bar{Q}_k , respectively. Let also T_1 and T_2 be the lifetimes of the first and

the second device, respectively. In Singh and Jain (1989) it is assumed that the underlying counting process is a homogeneous Poisson process, and it is shown that if $U \leq^* V$ then $T_1 \leq^* T_2$ where $(*)$ can be the likelihood ratio ordering (LR), the failure rate ordering (FR) or the mean residual life ordering (MRL).

We can now state our main result

THEOREM 4.3. *Let $\bar{F}(t)$ and $\bar{G}(t)$ be defined by (4.1). If the interarrivals $Z_i, i \in N_+$, have PF_2 densities (IFR property, $DMRL$ property) and if $U \leq^{FR} V$ then*

- (a) $T_1 \stackrel{DWLR}{\leq} T_2.$
- (b) $T_1 \stackrel{IFR}{\leq} T_2.$
- (c) $T_1 \stackrel{DMRL}{\leq} T_2.$

PROOF. To prove (i), let us define

$$\varphi(k, i) = \begin{cases} p_k & \text{if } i = 1 \\ q_k & \text{if } i = 2. \end{cases}$$

By assumption,

$$\phi(k, i) = \sum_{j=k}^{\infty} \varphi(j, i) = \begin{cases} \bar{P}_k & \text{if } i = 1 \\ \bar{Q}_k & \text{if } i = 2, \end{cases}$$

is TP_2 in k and i . Since $w^{(k)}(t)$ is TP_2 in t and k , as follows from Lemma 4.1 (a), also the function

$$\psi(t, i) = \sum_{k=0}^{\infty} \varphi(k, i)w^{(k)}(t) = \begin{cases} f(t) & \text{if } i = 1 \\ g(t) & \text{if } i = 2, \end{cases}$$

is TP_2 in t and i by Lemma 4.2 . It follows that, whenever $t, s \geq 0$, one has

$$f(t)g(t + s) - f(t + s)g(t) \geq 0$$

that is $T_1 \stackrel{DWLR}{\leq} T_2.$

By analogous arguments one can prove the result for the increasing failure rate (IFR) and dual mean residual life ($DMRL$) cases, defining

$$\varphi(k, i) = \begin{cases} \frac{p_k}{\mu_p} & \text{if } i = 1 \\ \frac{q_k}{\mu_q} & \text{if } i = 2, \end{cases}$$

and

$$\psi(t, i) = \sum_{k=0}^{\infty} \varphi(k, i)\bar{W}^{(k)}(t),$$

or

$$\psi(t, i) = \sum_{k=0}^{\infty} \varphi(k, i) \int_t^{\infty} \bar{W}^{(k)}(u)du,$$

respectively.

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REFERENCES

- Deshpand, J. V., Singh, H., Bagai, I. and Jain, K. (1988). Some partial orders describing positive aging, *Naval Res. Logist. Quart.*, **33**, 65–72
- Esary, J. D., Marshall, A. W. and Proschan, F. (1973). Shock models and wear processes, *Ann. Prob.*, **1**, 627–649.
- Karlin, S. (1968). Total Positivity, Vol. I, Stanford University Press.
- Karlin, S. and Proschan, F. (1960). Polya distributions of convolutions, *Ann. Math. Statist.*, **31**, 721–736.
- Klefsjo, B. (1981). HNBUE survival under some shock models. *Scand. J. Statist.*, **8**, 39–47.
- Pellerey, F. (1993). Partial orderings under cumulative damage shock models, *Adv. Appl. Prob.*, **25**, 939–946.
- Ross, S. M. (1983). Stochastic Processes, Wiley, New York.
- Shaked, M. and Shantikumar, J. G. (1994). Stochastic Orders and their Applications, Academic Press, New York.
- Shanthikumar, J. G. (1988). DFR property of first-passage times and its preservation under geometric compounding, *Ann. Prob.*, **24**, 894–914.
- Singh, H. and Jain, K. (1989). Preservation of some partial orderings under Poisson shock models, *J. Appl. Probab.*, **21**, 713–716.