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ON MINIMAXITY OF SOME ORTHOGONALLY INVARIANT ESTIMATORS OF BIVARIATE NORMAL DISPERSION MATRIX

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We consider an orthogonally invariant estimation of Σ of Wishart distribution using Stein's loss (entropy loss) or a quadratic loss. In these problems the best lower triangular matrix invariant estimators are minimax estimators. Some orthogonally invariant estimators were derived from those minimax estimators. It is conjectured that they are also minimax estimators, but some estimators have not yet been proved to be minimax. In this paper we prove the minimaxity of some estimators when the dimension is two. We also present the necessary conditions for a class of estimators to be minimax when the dimension is two.

Key words and phrases: Wishart distribution, Covariance Matrix, Minimax, Stein's loss, Quadratic loss.

1. Introduction

We consider the estimation of Σ in a multivariate normal distribution $N_p(\mu, \Sigma)$ when μ is known. This is equivalent to the estimation problem of Σ in a Wishart distribution $W_p(k, \Sigma)$ in view of sufficient statistics. Let W be distributed according to $W_p(k, \Sigma)$. We consider Stein's loss (entropy loss) and a quadratic loss, i.e.

$$L_1(\hat{\Sigma}, \Sigma) = \mathrm{tr}(\hat{\Sigma}\Sigma^{-1}) - \log|\hat{\Sigma}\Sigma^{-1}| - p,$$
$$L_2(\hat{\Sigma}, \Sigma) = \mathrm{tr}(\hat{\Sigma}\Sigma^{-1} - I)^2.$$

With respect to the transformation by a lower triangular matrix A,

$$\Sigma \to A\Sigma A', \quad W \to AWA', \quad \hat{\Sigma}(W) \to \hat{\Sigma}(AWA') = A\hat{\Sigma}(W)A',$$

the estimation problem is invariant with respect to either of the loss functions. Let

$$W = TT',$$

where T is the lower triangular matrix with positive diagonal elements. Then every estimator that is invariant under this transformation has the form

$$\hat{\Sigma} = T\Delta T' \tag{1.1}$$

with a constant diagonal matrix

$$\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_p).$$

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James and Stein (1961) derived the best estimator (say $\hat{\boldsymbol{\Sigma}}_{l1}$) w.r.t. $L_1(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$ among those which are invariant under this transformation. It is given by

$$\delta_i = \frac{1}{k-2i+p+1} \qquad i=1,\ldots,p \quad k \geq p. \tag{1.2}$$

The derivation of the best lower triangular matrix invariant estimator (say $\hat{\boldsymbol{\Sigma}}_{l2}$) for the loss $L_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$ is more complex especially when the dimension p is large. Olkin and Selliah (1977) gave the linear simultaneous equations whose solution gives the δ's of $\hat{\boldsymbol{\Sigma}}_{l2}$. It is

$$\Lambda_k \boldsymbol{\delta} = \boldsymbol{\lambda}_k, \tag{1.3}$$

where

$$\begin{aligned}
&(\Lambda_k)_{ii} = (k+p-2i+1)(k+p-2i+3),\\
&(\Lambda_k)_{ij} = (k+p-2j+1) \quad \text{if} \quad j>i,\\
&(\Lambda_k)_{ij} = (k+p-2i+1) \quad \text{if} \quad j<i,\\
&\boldsymbol{\lambda}_k = (k+p-1, k+p-3, \ldots, k-p+1)',\\
&\boldsymbol{\delta} = (\delta_1, \ldots, \delta_p)'.
\end{aligned}$$

The explicit form of δ's in the case $p=2$ is given by

$$\delta_1 = \frac{(k+1)^2-(k-1)}{(k+1)^2(k+3)-(k-1)}, \quad \delta_2 = \frac{(k+1)(k+2)}{(k+1)^2(k+3)-(k-1)}, \quad k \geq 2. \tag{1.4}$$

Note that there is a typographical error in the expression of δ_1 in Olkin and Selliah (1977). The correct description can be found in Sharma and Krishnamoorthy (1983). In the case $p=3 (k \geq 3)$, the δ's are given by

$$\begin{aligned}
\delta_1 &= \frac{k^4+2k^3+5k^2+4}{k^5+8k^4+17k^3+14k^2+4k+16},\\
\delta_2 &= \frac{k^4+4k^3+3k^2+4k+12}{k^5+8k^4+17k^3+14k^2+4k+16},\\
\delta_3 &= \frac{k^4+6k^3+11k^2+6k}{k^5+8k^4+17k^3+14k^2+4k+16}.
\end{aligned} \tag{1.5}$$

These best lower triangular matrix invariant estimators have constant risk and are minimax from Kiefer's well-known theorem. Several estimators have been proposed which are thought to dominate these best invariant estimators. Some of these are theoretically proven to be minimax but others are not. For review and classification of those estimators, see Pal (1993). In this paper we focus on orthogonally invariant estimators, especially those of the type derived by Stein (1982), Dey and Srinivasan (1985, 1986). For another type of orthogonally invariant (minimax) estimators, see Sharma and Krishnamoorthy (1983) and Takemura (1984).

Every orthogonally invariant estimator of $\boldsymbol{\Sigma}$ has the form

$$\hat{\boldsymbol{\Sigma}} = \boldsymbol{H}\boldsymbol{\Psi}\boldsymbol{H}', \qquad \boldsymbol{\Psi} = \operatorname{diag}(\psi_1(\boldsymbol{l}), \ldots, \psi_p(\boldsymbol{l})), \tag{1.6}$$

where $\boldsymbol{W} = \boldsymbol{HLH}'$ is the spectral decomposition with $\boldsymbol{H} \in \mathcal{O}(p)$ (the group of $p \times p$ orthogonal matrices) and $\boldsymbol{L} = \mathrm{diag}(l_1, \ldots, l_p)$. $\boldsymbol{l} = (l_1, \ldots, l_p)$ is the vector of eigenvalues of $\boldsymbol{W}$ with the order $0 < l_p \leq \cdots \leq l_1$. Stein (1982) proposed an orthogonally invariant estimator $\hat{\boldsymbol{\Sigma}}_{o1}$ defined by

$$\psi_i(\boldsymbol{l}) = \delta_i l_i, \qquad i = 1, \ldots, p \quad \text{with } \delta\text{'s as in (1.2)}$$

for $L_1(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$. Dey and Srinivasan (1985) proved that this estimator is minimax for arbitrary p and $k(\geq p)$. (Furthermore they obtained estimators superior to $\hat{\boldsymbol{\Sigma}}_{o1}$. For more details, see Dey and Srinivasan (1985, 1986).) The fact that $\hat{\boldsymbol{\Sigma}}_{o1}$ is a minimax estimator naturally provokes the following conjecture.

- The orthogonally invariant estimator (say $\hat{\boldsymbol{\Sigma}}_{o2}$) of $\boldsymbol{\Sigma}$ defined by

$$\psi_i(\boldsymbol{l}) = \delta_i l_i, \qquad i = 1, \ldots, p, \qquad \text{with } \delta\text{'s as the solution of (1.3)}$$

is minimax with respect to the loss $L_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$.
More general conjecture including this conjecture is stated in Krishnamoorthy and Gupta (1989). See also Perron (1997) for this conjecture. In Section 2 of this paper, we prove that this conjecture holds true in the case $p = 2$. We are also interested in the following question.

- What is the necessary condition on constant δ's for the estimator defined by

$$\psi_i(\boldsymbol{l}) = \delta_i l_i \qquad i = 1, \ldots, p \tag{1.7}$$

to be minimax ?
We prove that the δ's in (1.2) and (1.4) are the only values that make the estimator (1.7) minimax for $L_1(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$ and $L_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$, respectively. In Section 2, we prove it for $L_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$ in the wake of the proof for the minimaxity of $\hat{\boldsymbol{\Sigma}}_{o2}$. The proof for $L_1(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$ is presented in Section 3.

Note that the estimation problems considered here are invariant with respect to the orthogonal matrix transformation, and we can assume without loss of generality that

$$\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1^2, \sigma_2^2), \qquad \sigma_1^2 \geq \sigma_2^2 > 0.$$

2. Case of L_2

In this section, we consider the estimation of $\boldsymbol{\Sigma}$ using the loss function $L_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})$.

THEOREM 1. *Suppose $p = 2$ and $k \geq 2$. Then $\hat{\boldsymbol{\Sigma}}_{o2}$ dominates $\hat{\boldsymbol{\Sigma}}_{l2}$ and hence is a minimax estimator.*

PROOF. The density of $\boldsymbol{l}$ and $\boldsymbol{H}$ with respect to the product measure of the Lebesgue measure and the invariant probability measure μ on $\mathcal{O}(p)$ is given by

$$\begin{aligned} &K \prod_{i=1}^{p} l_i^{(k-p-1)/2} \prod_{i<j} (l_i - l_j) \exp\left(-\frac{1}{2}\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{HLH}')\right) \\ &= K \prod_{i=1}^{p} l_i^{(k-p-1)/2} \prod_{i<j} (l_i - l_j) \exp\left(-\frac{1}{2}\sum_{j=1}^{p} a_{jj} l_j\right), \end{aligned} \tag{2.1}$$

where

$$K = \frac{\pi^{p^2/2}}{2^{kp/2}|\boldsymbol{\Sigma}|^{k/2}\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{k}{2}\right)},$$

and $\boldsymbol{A} = (a_{ij}) = \boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}$ (See for example Th.3.2.18 Muirhead (1982)). In the case when $p = 2$, the density function is given by

$$(2.2) \qquad K l_1^{(k-3)/2} l_2^{(k-3)/2}(l_1 - l_2)\exp\left(-\frac{1}{2}(a_{11}l_1 + a_{22}l_2)\right)$$

with

$$(2.3) \qquad \begin{aligned} K &= \frac{\pi^2}{2^k|\boldsymbol{\Sigma}|^{k/2}\Gamma_2\left(\frac{k}{2}\right)\Gamma_2(1)} \\ &= \frac{\sqrt{\pi}S_2^{k/2}}{2^k\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k-1}{2}\right)} \qquad (S_2 \equiv |\boldsymbol{\Sigma}^{-1}|) \\ &= \frac{S_2^{k/2}}{4(k-2)!}, \end{aligned}$$

since $\Gamma(k/2)\Gamma((k-1)/2) = \Gamma(k-1)\sqrt{\pi}2^{-(k-2)}$. We will use the notation $G(\boldsymbol{l})$ hereafter which is defined by

$$G(\boldsymbol{l}) = \int_{O(2)} \exp\left(-\frac{1}{2}(a_{11}l_1 + a_{22}l_2)\right) d\mu(\boldsymbol{H}).$$

We consider general estimators of the form (1.7) without specifying the δ's. Let $R_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma}) = \mathrm{E}[L_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma})]$ denote the risk of $\hat{\boldsymbol{\Sigma}}$ given by (1.7). Straightforward calculation shows that

$$R_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma}) = \mathrm{E}[\delta_1^2 l_1^2 a_{11}^2 + \delta_2^2 l_2^2 a_{22}^2 + 2\delta_1\delta_2 l_1 l_2 a_{12}^2] - 2\mathrm{E}[\delta_1 l_1 a_{11} + \delta_2 l_2 a_{22}] + 2.$$

From (2.2), we have

$$\begin{aligned} &\mathrm{E}[\delta_1^2 l_1^2 a_{11}^2] \\ &= K\delta_1^2 \int_{\mathcal{L}} (l_1 - l_2) l_1^{(k+1)/2} l_2^{(k-3)/2} \int_{O(2)} a_{11}^2 \exp\left(-\frac{1}{2}(a_{11}l_1 + a_{22}l_2)\right) d\mu(\boldsymbol{H}) d\boldsymbol{l}, \end{aligned}$$

where $\mathcal{L} = \{\boldsymbol{l} \mid l_1 > l_2 > 0\}$. Using integration by parts, we have

$$\begin{aligned} &\mathrm{E}[\delta_1^2 l_1^2 a_{11}^2] \\ &= 4K\delta_1^2 \int_{\mathcal{L}} (l_1 - l_2) l_1^{(k+1)/2} l_2^{(k-3)/2} \frac{\partial^2 G(\boldsymbol{l})}{\partial l_1^2} d\boldsymbol{l} \\ &= -4K\delta_1^2 \int_0^\infty \int_{l_2}^\infty \frac{\partial}{\partial l_1}((l_1 - l_2) l_1^{(k+1)/2} l_2^{(k-3)/2}) \frac{\partial G(\boldsymbol{l})}{\partial l_1} dl_1 dl_2 \\ &= -2K\delta_1^2 \int_0^\infty \int_{l_2}^\infty \{(k+3) l_1^{(k+1)/2} l_2^{(k-3)/2} - (k+1) l_1^{(k-1)/2} l_2^{(k-1)/2}\} \frac{\partial G(\boldsymbol{l})}{\partial l_1} dl_1 dl_2 \end{aligned}$$

$$= 2K\delta_1^2 \int_0^\infty \int_{l_2}^\infty \frac{\partial}{\partial l_1}\{(k+3)l_1^{(k+1)/2}l_2^{(k-3)/2} - (k+1)l_1^{(k-1)/2}l_2^{(k-1)/2}\}G(\boldsymbol{l})dl_1dl_2$$

$$-2K\delta_1^2 \int_0^\infty [\{(k+3)l_1^{(k+1)/2}l_2^{(k-3)/2} - (k+1)l_1^{(k-1)/2}l_2^{(k-1)/2}\}G(\boldsymbol{l})]_{l_1=l_2}^{l_1=\infty}dl_2$$

$$= K\delta_1^2 \int_{\mathcal{L}}\{(k+1)(k+3)l_1^{(k-1)/2}l_2^{(k-3)/2} - (k-1)(k+1)l_1^{(k-3)/2}l_2^{(k-1)/2}\}G(\boldsymbol{l})d\boldsymbol{l}$$

$$+4K\delta_1^2 \int_0^\infty l^{k-1}\exp\left(-\frac{l}{2}S_1\right)dl,$$

where $S_1 = a_{11} + a_{22} = \mathrm{tr}\boldsymbol{\Sigma}^{-1}$.
Similarly, we have

$$\mathrm{E}[\delta_2^2 l_2^2 a_{22}^2]$$

$$= K\delta_2^2 \int_{\mathcal{L}}\{(k-1)(k+1)l_1^{(k-1)/2}l_2^{(k-3)/2} - (k+1)(k+3)l_1^{(k-3)/2}l_2^{(k-1)/2}\}G(\boldsymbol{l})d\boldsymbol{l}$$

$$+4K\delta_2^2 \int_0^\infty l^{k-1}\exp\left(-\frac{l}{2}S_1\right)dl.$$

Therefore, we have

(2.4) $\mathrm{E}[\delta_1^2 l_1^2 a_{11}^2 + \delta_2^2 l_2^2 a_{22}^2]$

$$= \{\delta_1^2(k+1)(k+3) + \delta_2^2(k-1)(k+1)\}K\int_{\mathcal{L}} l_1^{(k-1)/2}l_2^{(k-3)/2}G(\boldsymbol{l})d\boldsymbol{l}$$

$$-\{\delta_1^2(k-1)(k+1) + \delta_2^2(k+1)(k+3)\}K\int_{\mathcal{L}} l_1^{(k-3)/2}l_2^{(k-1)/2}G(\boldsymbol{l})d\boldsymbol{l}$$

$$+4K(\delta_1^2+\delta_2^2)\Gamma(k)\left(\frac{2}{S_1}\right)^k$$

$$= (\delta_1^2+\delta_2^2)(k+1)^2K\int_{\mathcal{L}}(l_1-l_2)l_1^{(k-3)/2}l_2^{(k-3)/2}G(\boldsymbol{l})d\boldsymbol{l}$$

$$+2(\delta_1^2-\delta_2^2)(k+1)K\int_{\mathcal{L}}(l_1^{(k-1)/2}l_2^{(k-3)/2} + l_1^{(k-3)/2}l_2^{(k-1)/2})G(\boldsymbol{l})d\boldsymbol{l}$$

$$+4K(\delta_1^2+\delta_2^2)\Gamma(k)\left(\frac{2}{S_1}\right)^k$$

$$= (\delta_1^2+\delta_2^2)(k+1)^2 + 4K(\delta_1^2+\delta_2^2)\Gamma(k)\left(\frac{2}{S_1}\right)^k$$

$$+2K(\delta_1^2-\delta_2^2)(k+1)I\left(\frac{k-3}{2}\right),$$

where

$$I(\alpha) = \int_{\mathcal{L}}(l_1^{\alpha+1}l_2^{\alpha} + l_1^{\alpha}l_2^{\alpha+1})G(\boldsymbol{l})d\boldsymbol{l} \qquad \alpha \geq -\frac{1}{2}.$$

Besides, we have

$$\mathrm{E}[2\delta_1\delta_2 l_1 l_2 a_{12}^2]$$

$$= 2K\delta_1\delta_2\int_{\mathcal{L}}(l_1-l_2)l_1^{(k-1)/2}l_2^{(k-1)/2}$$

$$\cdot\int_{\mathcal{O}(2)} a_{12}^2\exp\left(-\frac{1}{2}(l_1a_{11}+l_2a_{22})\right)d\mu(\boldsymbol{H})d\boldsymbol{l}.$$

Using Theorem 5.1 from Sheena (1995), we have

$$\begin{aligned}
&\int_{\mathcal{O}(2)} a_{12}^2 \exp\left(-\frac{1}{2}(l_1 a_{11} + l_2 a_{22})\right) d\mu(\boldsymbol{H}) \\
&= \frac{1}{l_1 - l_2} \int_{\mathcal{O}(2)} (a_{22} - a_{11}) \exp\left(-\frac{1}{2}(l_1 a_{11} + l_2 a_{22})\right) d\mu(\boldsymbol{H}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(2.5) \qquad & \mathrm{E}[2\delta_1\delta_2 l_1 l_2 a_{12}^2] \\
&= 2K\delta_1\delta_2 \int_{\mathcal{L}} l_1^{(k-1)/2} l_2^{(k-1)/2} \int_{\mathcal{O}(2)} (a_{22} - a_{11}) \\
&\quad \cdot \exp\left(-\frac{1}{2}(l_1 a_{11} + l_2 a_{22})\right) d\mu(\boldsymbol{H}) d\boldsymbol{l} \\
&= -4K\delta_1\delta_2 \int_{\mathcal{L}} l_1^{(k-1)/2} l_2^{(k-1)/2} \left(\frac{\partial G(\boldsymbol{l})}{\partial l_2} - \frac{\partial G(\boldsymbol{l})}{\partial l_1}\right) d\boldsymbol{l} \\
&= 4K\delta_1\delta_2 \int_0^\infty \int_0^{l_1} \frac{\partial}{\partial l_2}(l_1^{(k-1)/2} l_2^{(k-1)/2}) G(\boldsymbol{l}) dl_2 dl_1 \\
&\quad -4K\delta_1\delta_2 \int_0^\infty [l_1^{(k-1)/2} l_2^{(k-1)/2} G(\boldsymbol{l})]_{l_2=0}^{l_2=l_1} dl_1 \\
&\quad -4K\delta_1\delta_2 \int_0^\infty \int_{l_2}^\infty \frac{\partial}{\partial l_1}(l_1^{(k-1)/2} l_2^{(k-1)/2}) G(\boldsymbol{l}) dl_1 dl_2 \\
&\quad +4K\delta_1\delta_2 \int_0^\infty [l_1^{(k-1)/2} l_2^{(k-1)/2} G(\boldsymbol{l})]_{l_1=l_2}^{l_1=\infty} dl_2 \\
&= 2K\delta_1\delta_2 \int_{\mathcal{L}} (k-1) l_1^{(k-1)/2} l_2^{(k-3)/2} G(\boldsymbol{l}) d\boldsymbol{l} \\
&\quad -4K\delta_1\delta_2 \int_0^\infty l^{k-1} \exp\left(-\frac{l}{2}S_1\right) dl \\
&\quad -2K\delta_1\delta_2 \int_{\mathcal{L}} (k-1) l_1^{(k-3)/2} l_2^{(k-1)/2} G(\boldsymbol{l}) d\boldsymbol{l} \\
&\quad -4K\delta_1\delta_2 \int_0^\infty l^{k-1} \exp\left(-\frac{l}{2}S_1\right) dl \\
&= 2K\delta_1\delta_2(k-1) \int_{\mathcal{L}} (l_1 - l_2) l_1^{(k-3)/2} l_2^{(k-3)/2} G(\boldsymbol{l}) d\boldsymbol{l} \\
&\quad -8K\delta_1\delta_2 \Gamma(k) \left(\frac{2}{S_1}\right)^k \\
&= 2\delta_1\delta_2(k-1) - 8K\delta_1\delta_2\Gamma(k)\left(\frac{2}{S_1}\right)^k.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathrm{E}[\delta_1 l_1 a_{11}] \\
&= -2K\delta_1 \int_{\mathcal{L}} (l_1 - l_2) l_1^{(k-1)/2} l_2^{(k-3)/2} \frac{\partial G(\boldsymbol{l})}{\partial l_1} d\boldsymbol{l} \\
&= 2K\delta_1 \int_0^\infty \int_{l_2}^\infty \frac{\partial}{\partial l_1}((l_1 - l_2) l_1^{(k-1)/2} l_2^{(k-3)/2}) G(\boldsymbol{l}) dl_1 dl_2
\end{aligned}$$

$$= K\delta_1 \int_{\mathcal{L}} ((k+1) l_1^{(k-1)/2} l_2^{(k-3)/2} - (k-1) l_1^{(k-3)/2} l_2^{(k-1)/2}) G(\boldsymbol{l}) d\boldsymbol{l}.$$

Similarly, we have

$$\mathrm{E}[\delta_2 l_2 a_{22}] = K\delta_2 \int_{\mathcal{L}} ((k-1) l_1^{(k-1)/2} l_2^{(k-3)/2} - (k+1) l_1^{(k-3)/2} l_2^{(k-1)/2}) G(\boldsymbol{l}) d\boldsymbol{l}.$$

Therefore, we have

$$\begin{aligned}(2.6)\quad & \mathrm{E}[\delta_1 l_1 a_{11} + \delta_2 l_2 a_{22}] \\ &= (\delta_1(k+1) + \delta_2(k-1)) K \int_{\mathcal{L}} l_1^{(k-1)/2} l_2^{(k-3)/2} G(\boldsymbol{l}) d\boldsymbol{l} \\ &\quad -(\delta_1(k-1) + \delta_2(k+1)) K \int_{\mathcal{L}} l_1^{(k-3)/2} l_2^{(k-1)/2} G(\boldsymbol{l}) d\boldsymbol{l} \\ &= (\delta_1 + \delta_2) k K \int_{\mathcal{L}} (l_1 - l_2) l_1^{(k-3)/2} l_2^{(k-3)/2} G(\boldsymbol{l}) d\boldsymbol{l} \\ &\quad +(\delta_1 - \delta_2) K \int_{\mathcal{L}} (l_1^{(k-1)/2} l_2^{(k-3)/2} + l_1^{(k-3)/2} l_2^{(k-1)/2}) G(\boldsymbol{l}) d\boldsymbol{l} \\ &= (\delta_1 + \delta_2) k + (\delta_1 - \delta_2) K I\left(\frac{k-3}{2}\right).\end{aligned}$$

Using results (2.3) to (2.6), we have

$$\begin{aligned}(2.7)\quad R_2(\hat{\boldsymbol{\Sigma}}, \boldsymbol{\Sigma}) &= (\delta_1^2 + \delta_2^2)(k+1)^2 + 2\delta_1\delta_2(k-1) - 2(\delta_1 + \delta_2)k + 2 \\ &\quad +2^k(\delta_1 - \delta_2)^2(k-1)\left(\frac{S_2^{1/2}}{S_1}\right)^k \\ &\quad +\frac{(\delta_1 - \delta_2)\{(\delta_1 + \delta_2)(k+1) - 1\}}{2(k-2)!} S_2^{k/2} I\left(\frac{k-3}{2}\right).\end{aligned}$$

Now consider the integral $I(\frac{k-3}{2})$. From the definition of $I(\alpha)$, we have

$$(2.8)\quad I(\alpha) = \int_{\mathcal{O}(2)} \int_{\mathcal{L}} (l_1^\alpha l_2^{\alpha+1} + l_1^{\alpha+1} l_2^\alpha) \exp\left(-\frac{1}{2}(a_{11} l_1 + a_{22} l_2)\right) d\boldsymbol{l} d\mu(\boldsymbol{H}).$$

Through change of variable $(l_1, l_2) \to (l_2, l_1)$, we have

$$I(\alpha) = \int_{\mathcal{O}(2)} \int_{\mathcal{L}^*} (l_1^\alpha l_2^{\alpha+1} + l_1^{\alpha+1} l_2^\alpha) \exp\left(-\frac{1}{2}(a_{11} l_2 + a_{22} l_1)\right) d\boldsymbol{l} d\mu(\boldsymbol{H}),$$

where $\mathcal{L}^* = \{\boldsymbol{l} \mid 0 < l_1 < l_2\}$. Since μ is invariant with respect to the exchange of the columns of $\boldsymbol{H}$, we have

$$\begin{aligned}(2.9)\quad & I(\alpha) \\ &= \int_{\mathcal{O}(2)} \int_{\mathcal{L}^*} (l_1^\alpha l_2^{\alpha+1} + l_1^{\alpha+1} l_2^\alpha) \exp\left(-\frac{1}{2}(a_{11} l_1 + a_{22} l_2)\right) d\boldsymbol{l} d\mu(\boldsymbol{H}) \\ &= \int_{\mathcal{L}^*} (l_1^\alpha l_2^{\alpha+1} + l_1^{\alpha+1} l_2^\alpha) G(\boldsymbol{l}) d\boldsymbol{l}.\end{aligned}$$

From (2.8) and (2.9), we have

$$\begin{aligned}
I(\alpha) &= \frac{1}{2}\int_0^\infty\int_0^\infty (l_1^\alpha l_2^{\alpha+1} + l_1^{\alpha+1} l_2^\alpha) G(\boldsymbol{l}) dl_1 dl_2 \\
&= \frac{1}{2}\int_{\mathcal{O}(2)} \left\{ \int_0^\infty l_1^\alpha \exp\left(-\frac{l_1}{2}a_{11}\right) dl_1 \int_0^\infty l_2^{\alpha+1} \exp\left(-\frac{l_2}{2}a_{22}\right) dl_2 \right. \\
&\qquad \left. + \int_0^\infty l_1^{\alpha+1} \exp\left(-\frac{l_1}{2}a_{11}\right) dl_1 \int_0^\infty l_2^\alpha \exp\left(-\frac{l_2}{2}a_{22}\right) dl_2 \right\} d\mu(\boldsymbol{H}) \\
&= 2^{2\alpha+2}\Gamma(\alpha+1)\Gamma(\alpha+2) \\
&\quad \times \int_{\mathcal{O}(2)} \{a_{11}^{-(\alpha+1)} a_{22}^{-(\alpha+2)} + a_{11}^{-(\alpha+2)} a_{22}^{-(\alpha+1)}\} d\mu(\boldsymbol{H}) \\
&= 2^{2\alpha+2}\Gamma(\alpha+1)\Gamma(\alpha+2) S_1 \int_{\mathcal{O}(2)} (a_{11}a_{22})^{-(\alpha+2)} d\mu(\boldsymbol{H}).
\end{aligned}$$

For the case $p = 2$, $\boldsymbol{H}$ can be simply expressed as

$$\boldsymbol{H} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \qquad \epsilon = 1 \quad \text{or} \quad -1.$$

If θ is uniformly distributed on $[0\, 2\pi]$ and $P(\epsilon = 1) = P(\epsilon = -1) = \frac{1}{2}$, then the distribution of $\boldsymbol{H}$ equals the invariant distribution, μ. See Tumura (1965) or Takemura (1991). Hence

$$\begin{aligned}
&S_2^{\alpha+(3/2)} I(\alpha) \\
&= 2^{2\alpha+2}\Gamma(\alpha+1)\Gamma(\alpha+2) S_1 S_2^{\alpha+(3/2)} \\
&\quad \times \frac{1}{2\pi}\int_0^{2\pi} \{(\sigma_1^{-2}\cos^2\theta + \sigma_2^{-2}\sin^2\theta)(\sigma_1^{-2}\sin^2\theta + \sigma_2^{-2}\cos^2\theta)\}^{-(\alpha+2)} d\theta \\
&= 2^{2\alpha+2}\Gamma(\alpha+1)\Gamma(\alpha+2) S_1 S_2^{\alpha+(3/2)} \\
&\quad \times \frac{1}{2\pi}\int_0^{2\pi} \left\{ \left(\frac{\sigma_1^{-2}+\sigma_2^{-2}}{2}\right)^2 - \left(\frac{\sigma_1^{-2}-\sigma_2^{-2}}{2}\right)^2 \cos^2 2\theta \right\}^{-(\alpha+2)} d\theta \\
&= 2^{5\alpha+8}\Gamma(\alpha+1)\Gamma(\alpha+2) S_1 S_2^{\alpha+(3/2)} \\
&\quad \times \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\{(S_1^2+4S_2) - (S_1^2-4S_2)\cos 4\theta\}^{\alpha+2}} d\theta \\
&= 2^{5\alpha+8}\Gamma(\alpha+1)\Gamma(\alpha+2) S_1 S_2^{\alpha+(3/2)} \\
&\quad \times \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\{(S_1^2+4S_2) - (S_1^2-4S_2)\cos\theta\}^{\alpha+2}} d\theta.
\end{aligned}$$

Applying Lemma 1 from the Appendix to the last integral, we have

$$\text{(2.10)} \quad S_2^{\alpha+(3/2)} I(\alpha)$$
$$= 2^{4\alpha+6}\Gamma(\alpha+1)\Gamma(\alpha+2) \left(\frac{S_2}{S_1^2}\right)^{\alpha+(3/2)} {}_2F_1\left(\frac{1}{2}, \alpha+2; 1; 1 - 4\frac{S_2}{S_1^2}\right)$$

$$= 2^{2\alpha+3}\Gamma(\alpha+1)\Gamma(\alpha+2)(1-y)^{\alpha+(3/2)}{}_2F_1\left(\frac{1}{2},\alpha+2;1;y\right)$$

$$\left(y \equiv 1 - 4\frac{S_2}{S_1^2}\right).$$

Consequently, from (2.7), (2.10) and the well-known formula

$$\Gamma\left(z+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2z-1}}\frac{\Gamma(2z)}{\Gamma(z)}, \tag{2.11}$$

we have

$$\begin{aligned} R_2(\hat{\mathbf{\Sigma}},\mathbf{\Sigma}) &= A(\delta_1,\delta_2) + (\delta_1-\delta_2)^2(k-1)(1-y)^{k/2} \\ &\quad +2\sqrt{\pi}(\delta_1-\delta_2)\{(\delta_1+\delta_2)(k+1)-1\} \\ &\quad \times\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}(1-y)^{k/2}{}_2F_1\left(\frac{1}{2},\frac{k+1}{2};1;y\right), \end{aligned} \tag{2.12}$$

where

$$A(\delta_1,\delta_2) = (\delta_1^2+\delta_2^2)(k+1)^2 + 2\delta_1\delta_2(k-1) - 2(\delta_1+\delta_2)k + 2.$$

We use the following formulas in the calculation of $\sup_{0\le y<1} R_2(\hat{\mathbf{\Sigma}},\mathbf{\Sigma})$ noting that $0 \le y < 1$.

$$\begin{aligned} &\frac{d}{dy}\{(1-y)^{a+b-c}{}_2F_1(a,b;c;y)\} \\ &\quad = \frac{(c-a)(c-b)}{c}(1-y)^{a+b-c-1}{}_2F_1(a,b;c+1;y) \end{aligned} \tag{2.13}$$

$$(1-y)^{a+b-c}{}_2F_1(a,b;c;y) = {}_2F_1(c-a,c-b;c;y) \tag{2.14}$$

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.15}$$

See e.g. p. 45 (10), p. 67 (2) and p. 99 (1) in Luke (1969).

Using formula (2.13), we have

$$\begin{aligned} \frac{d}{dy}R_2(\hat{\mathbf{\Sigma}},\mathbf{\Sigma}) &= -2^{-1}(\delta_1-\delta_2)(k-1)(1-y)^{(k/2)-1} \\ &\quad \times\left[(\delta_1-\delta_2)k + \sqrt{\pi}\{(\delta_1+\delta_2)(k+1)-1\}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right. \\ &\quad \left.\times{}_2F_1\left(\frac{1}{2},\frac{k+1}{2};2;y\right)\right]. \end{aligned}$$

Now suppose δ_1 and δ_2 are given by (1.4). Then, we have

$$\delta_1 - \delta_2 = \frac{-2k}{k^3+5k^2+6k+4} < 0, \tag{2.16}$$

$$(\delta_1+\delta_2)(k+1) - 1 = \frac{k^3+k^2+2k}{k^3+5k^2+6k+4} > 0. \tag{2.17}$$

Since

$$\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} > 1, \qquad \forall k \geq 2,$$

and

$${}_2F_1\left(\frac{1}{2}, \frac{k+1}{2}; 2; y\right) > 1, \qquad 0 \leq \forall y < 1,$$

we have

$$\begin{aligned}
&\frac{d}{dy} R_2(\hat{\mathbf{\Sigma}}_{o2}, \mathbf{\Sigma}) \\
&\quad > -2^{-1}(\delta_1-\delta_2)(k-1)(1-y)^{(k/2)-1}\{(\delta_1-\delta_2)k + (\delta_1+\delta_2)(k+1)-1\} \\
&\quad = -2^{-1}(\delta_1-\delta_2)(k-1)(1-y)^{(k/2)-1}\frac{k(k^2-k+2)}{k^3+5k^2+6k+4} \\
&\quad > 0 \quad \forall k \geq 2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sup_{0 \leq y < 1} R_2(\hat{\mathbf{\Sigma}}_{o2}, \mathbf{\Sigma}) \\
&\qquad = \lim_{y \to 1} R_2(\hat{\mathbf{\Sigma}}_{o2}, \mathbf{\Sigma}).
\end{aligned}$$

From (2.12), the general estimators given by (1.7) have the limiting risk value given by

$$\begin{aligned}
&\lim_{y\to 1} R_2(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) \\
&\quad = A(\delta_1,\delta_2) + 2\sqrt{\pi}(\delta_1-\delta_2)\{(\delta_1+\delta_2)(k+1)-1\} \\
&\qquad \times \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \lim_{y\to 1}(1-y)^{k/2}{}_2F_1\left(\frac{1}{2}, \frac{k+1}{2}; 1; y\right) \\
&\quad = A(\delta_1,\delta_2) + 2\sqrt{\pi}(\delta_1-\delta_2)\{(\delta_1+\delta_2)(k+1)-1\} \\
&\qquad \times \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \lim_{y\to 1} {}_2F_1\left(\frac{1}{2}, \frac{-k+1}{2}; 1; y\right) \\
&\quad = A(\delta_1,\delta_2) + 2(\delta_1-\delta_2)\{(\delta_1+\delta_2)(k+1)-1\}.
\end{aligned} \tag{2.18}$$

The second and the third equalities are derived using the formulas (2.14) and (2.15), respectively. For the specific case of (1.4), we have

$$\text{(2.19)} \qquad \begin{aligned} \lim_{y\to 1} R_2(\hat{\mathbf{\Sigma}}_{o2}, \mathbf{\Sigma}) &= \frac{2(3k^2+5k+4)}{k^3+5k^2+6k+4} \\ &= R_2(\hat{\mathbf{\Sigma}}_{l2}, \mathbf{\Sigma}). \end{aligned}$$

For the value of $R_2(\hat{\mathbf{\Sigma}}_{l2}, \mathbf{\Sigma})$, see p. 25 of Sharma and Krishnamoorthy (1983). □

Next we prove the uniqueness of $\hat{\mathbf{\Sigma}}_{o2}$ being minimax among the estimators given by (1.7).

THEOREM 2. *Suppose* $p = 2$. $\hat{\mathbf{\Sigma}}_{o2}$ *is the only minimax estimator w.r.t.* $L_2(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma})$ *among the estimators given by* (1.7).

PROOF. From (2.18) in the proof of Theorem 1, we have

$$\lim_{y\to 1} R_2(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) = g_2(\delta_1, \delta_2),$$

where

$$\text{(2.20)} \qquad \begin{aligned} g_2(\delta_1,\delta_2) = {} & (k+1)(k+3)\delta_1^2 + (k+1)(k-1)\delta_2^2 + 2\delta_1\delta_2(k-1) \\ & -2\delta_1(k+1) - 2\delta_2(k-1) + 2. \end{aligned}$$

We calculate the minimum value of $g_2(\delta_1, \delta_2)$. Generally speaking, the quadratic function defined by

$$\boldsymbol{\delta}'\boldsymbol{B}\boldsymbol{\delta} + 2\boldsymbol{c}'\boldsymbol{\delta} + d$$

with $\boldsymbol{\delta}' = (\delta_1, \delta_2)$, $\boldsymbol{c}' = (c_1, c_2)$, $\boldsymbol{B} = (b_{ij}) 1 \le i,\, j \le 2$ attains its minimum value

$$\frac{1}{b_{11}b_{22} - b_{12}^2}(-b_{22}c_1^2 + 2b_{12}c_1c_2 - b_{11}c_2^2) + d$$

at

$$\begin{aligned} \delta_1^* &= \frac{1}{b_{11}b_{22} - b_{12}^2}(-b_{22}c_1 + b_{12}c_2), \\ \delta_2^* &= \frac{1}{b_{11}b_{22} - b_{12}^2}(-b_{11}c_2 + b_{12}c_1), \end{aligned}$$

if $b_{11} > 0$ and $b_{11}b_{22} - b_{12}^2 > 0$. As for the specific $g_2(\delta_1, \delta_2)$ given by (2.20), we have

$$b_{11} = (k+1)(k+3) > 0,$$

and

$$b_{11}b_{22} - b_{12}^2 = (k-1)(k^3+5k^2+6k+4)(\equiv M(k)) > 0.$$

Hence, we have

$$\min g_2(\delta_1, \delta_2)$$

$$= \frac{1}{M(k)}\{-(k-1)(k+1)^3 - (k-1)^2(k+1)(k+3) \\ +2(k-1)^2(k+1)\} + 2$$
$$= \frac{2(3k^2+5k+4)}{k^3+5k^2+6k+4}$$
$$= R(\hat{\mathbf{\Sigma}}_{l2}, \mathbf{\Sigma}).$$

δ_1^* and δ_2^* are given by

$$\delta_1^* = \frac{1}{M(k)}\{(k+1)^2(k-1) - (k-1)^2\},$$
$$\delta_2^* = \frac{1}{M(k)}\{(k-1)(k+1)(k+3) - (k-1)(k+1)\},$$

which turn out to be the δ's in (1.4) as is obvious from Theorem 1. Consequently, for any $\hat{\mathbf{\Sigma}}$ other than $\hat{\mathbf{\Sigma}}_{o2}$,

$$\lim_{y\to 1} R(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) > R(\hat{\mathbf{\Sigma}}_{l2}, \mathbf{\Sigma}).$$ □

3. Case of L_1

In this section, we consider the estimation of $\mathbf{\Sigma}$ using $L_1(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma})$. The minimaxity of the estimator $\hat{\mathbf{\Sigma}}_{o1}$ was proved in Dey and Srinivasan (1985) for general p. We prove that a similar result to Theorem 2 also holds true for $L_1(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma})$.

THEOREM 3. *Suppose $p = 2$. $\hat{\mathbf{\Sigma}}_{o1}$ is the only minimax estimator w.r.t. L_1 among the estimators given by* (1.7).

PROOF. We use the same notations as in the proof of Theorem 1. First we calculate $\lim_{y\to 1} R_1(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) = \lim_{y\to 1} \mathrm{E}[L_1(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma})]$. We use the result of Stein (1977) and Haff (1977, 1979) on the unbiased estimator of the risk. Let

$$R^*(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) = (k-p-1)\sum_{i=1}^{p}\frac{\psi_i}{l_i} + 2\sum_{i=1}^{p}\sum_{j\neq i}\frac{\psi_i}{l_i - l_j} + 2\sum_{i=1}^{p}\frac{\partial \psi_i}{\partial l_i} \\ - \sum_{i=1}^{p}\log\psi_i + \log|\mathbf{\Sigma}| - p.$$

Then, we have

$$\mathrm{E}[R^*(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma})] = R_1(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) \quad \forall \mathbf{\Sigma}.$$

In the case when $p = 2$ and $\psi_i = \delta_i l_i$, we have

(3.1) $R^*(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma})$
$$= 2\frac{\delta_1 l_1 - \delta_2 l_2}{l_1 - l_2} + (k-1)(\delta_1+\delta_2) - \log l_1 l_2 - \log \delta_1\delta_2 + \log|\mathbf{\Sigma}| - 2$$
$$= k(\delta_1+\delta_2) - \log\delta_1\delta_2 - 2 \\ + \frac{l_1+l_2}{l_1-l_2}(\delta_1 - \delta_2) - \log|\mathbf{\Sigma}^{-(1/2)}\mathbf{W}\mathbf{\Sigma}^{-(1/2)}|.$$

From (2.2), (2.3), (2.10), (2.11) and (2.14), we have

$$
\begin{aligned}
\mathrm{E}\left[\frac{l_1+l_2}{l_1-l_2}\right]
&= K\int_{\mathcal{L}}(l_1+l_2)l_1^{(k-3)/2}l_2^{(k-3)/2}G(l)dl \\
&= \frac{1}{4(k-2)!}S_2^{k/2}I\left(\frac{k-3}{2}\right) \\
&= \frac{2^{k-2}}{(k-2)!}\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)(1-y)^{k/2}{}_2F_1\left(\frac{1}{2},\frac{k+1}{2};1;y\right) \\
&= \frac{2^{k-2}}{(k-2)!}\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{k+1}{2}\right){}_2F_1\left(\frac{1}{2},\frac{1-k}{2};1;y\right) \\
&= \sqrt{\pi}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}{}_2F_1\left(\frac{1}{2},\frac{1-k}{2};1;y\right).
\end{aligned}
$$

Using (2.15), we have

$$
\begin{aligned}
\lim_{y\to 1}\mathrm{E}\left[\frac{l_1+l_2}{l_1-l_2}\right] &= \sqrt{\pi}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\lim_{y\to 1}{}_2F_1\left(\frac{1}{2},\frac{1-k}{2};1;y\right) \\
&= 1.
\end{aligned}
\tag{3.2}
$$

From (3.1) and (3.2), we have

$$
\begin{aligned}
\lim_{y\to 1}R(\hat{\mathbf{\Sigma}},\mathbf{\Sigma}) &= (k+1)\delta_1+(k-1)\delta_2-\log\delta_1\delta_2-\mathrm{E}[\log|\boldsymbol{W}|\mid\mathbf{\Sigma}=I_p]-2 \\
&\quad(\equiv g_1(\delta_1,\delta_2)).
\end{aligned}
$$

$g_1(\delta_1,\delta_2)$ attains its minimum value when

$$\delta_1=\frac{1}{k+1},\qquad \delta_2=\frac{1}{k-1}.$$

These are the δ's given by (1.2). The attained minimum value is

$$\log(k+1)+\log(k-1)-\mathrm{E}[\log|\boldsymbol{W}|\mid\mathbf{\Sigma}=I_p],$$

which is equal to $R_1(\hat{\mathbf{\Sigma}}_{l1},\mathbf{\Sigma})$. (See p. 377 of James and Stein (1961).)

Consequently, we have

$$\lim_{y\to 1}R_1(\hat{\mathbf{\Sigma}},\mathbf{\Sigma}) > R_1(\hat{\mathbf{\Sigma}}_{l1},\mathbf{\Sigma})$$

for any $\hat{\mathbf{\Sigma}}$ of the form (1.7) other than $\hat{\mathbf{\Sigma}}_{o1}$. □

Appendix

LEMMA 1.

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{(a-b\cos\theta)^{\alpha+2}}d\theta=\frac{1}{(a+b)^{\alpha+2}}{}_2F_1\left(\frac{1}{2},\alpha+2;1;\frac{2b}{a+b}\right),$$

where ${}_2F_1$ is the hypergeometric function.

PROOF.

$$\begin{aligned}
&(2\pi)^{-1}\int_0^{2\pi}(a-b\cos\theta)^{-\alpha-2}d\theta\\
&=(2\pi)^{-1}\int_0^{2\pi}(a+b\cos\theta)^{-\alpha-2}d\theta\\
&=\pi^{-1}\int_0^{\pi}(a+b\cos\theta)^{-\alpha-2}d\theta\\
&=\pi^{-1}\int_0^{\infty}\left(a+b\frac{1-t^2}{1+t^2}\right)^{-\alpha-2}\frac{2}{1+t^2}dt \qquad (t=\tan(\theta/2))\\
&=2\pi^{-1}\int_0^{\infty}\{(a-b)t^2+a+b\}^{-\alpha-2}(1+t^2)^{\alpha+1}dt\\
&=\pi^{-1}\int_0^{\infty}\{(a-b)x+a+b\}^{-\alpha-2}(1+x)^{\alpha+1}x^{-1/2}dx \qquad (x=t^2)\\
&=\frac{1}{\pi(a+b)^{\alpha+2}}\int_0^{\infty}\left(1+\frac{a-b}{a+b}x\right)^{-\alpha-2}(1+x)^{\alpha+1}x^{-1/2}dx\\
&=(a+b)^{-\alpha-2}{}_2F_1\left(\frac{1}{2},\alpha+2;1;\frac{2b}{a+b}\right).
\end{aligned}$$

For the last equation, we use the integral representation of the hypergeometric function,

$${}_2F_1(a,b;a+b+1-c;1-z)=\frac{\Gamma(a+b+c-1)}{\Gamma(a)\Gamma(b+1-c)}\int_0^{\infty}t^{a-1}(1+t)^{c-a-1}(1+zt)^{-b}dt.$$

See e.g. p. 57 (3) in Luke (1969). □

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REFERENCES

Dey, D. K. and Srinivasan, C. (1985). Estimation of a covariance matrix under Stein's loss, *Ann. Statist.*, **13**, 1581–1591.

Dey, D. K. and Srinivasan, C. (1986). Trimmed minimax estimator of a covariance matrix, *Ann. Inst. Statist. Math.*, **38**, 101–108.

Haff, L. R. (1977). Minimax estimators for a multinormal precision matrix, *J. Multivariate Anal.*, **7**, 374–385.

Haff, L. R. (1979). An identity for the Wishart distribution with applications, *J. Multivariate Anal.*, **9**, 531–542.

James, W. and Stein, C. (1961). Estimation with quadratic loss, *Proc. 4th Berkeley Symp. Math. Statist. Prob.*, (ed. Neyman, J.), **1**, 361–379, University of California Press, Berkeley.

Krishnamoorthy, K. and Gupta, A. K. (1989). Improved minimax estimation of a normal precision matrix, *Canad. J. Statist.*, **17**, 91–102.

Luke, L. Y. (1969). *The Special Functions and Their Approximations*, Vol. 1, Academic Press, New York.

Muirhead, R. J. (1982). *Aspects of multivariate statistical theory*, John Wiley and Sons, New York.

Olkin, I. and Selliah, J. B. (1977). Estimating covariances in a multivariate normal distribution, *Statistical Decision Theory and Related Topics.*, (ed. Gupta, S. S.), **2**, 313–326.

Pal, N. (1993). Estimating the normal dispersion matrix and the precision matrix from a decision-theoretic point of view: a review, *Statist. Papers*, **34**, 1–26.

Perron, F. (1997). On a conjecture of Krishnamoorthy and Gupta, *J. Multivariate Anal.*, **62**, 110–120.

Sharma, D. and Krishnamoorthy, K. (1983). Orthogonal equivariant minimax estimators of bivariate normal covariance matrix and precision matrix, *Calcutta Statist. Ass. Bull.*, **32**, 23–45.

Sheena, Y. (1995). Unbiased estimator of risk for an orthogonally invariant estimator of a covariance matrix, *J. Japan Statist. Soc.*, **25**, 35–48.

Stein, C. (1977). Unpublished notes on estimating the covariance matrix.

Stein, C. (1982). Lectures at the University of Washington, Seattle.

Takemura, A. (1984). An orthogonally invariant minimax estimator of the covariance matrix of a multivariate normal population, *Tsukuba J. Math.*, **8**, 367–376.

Takemura, A. (1991). *Tahenryo Suisoku Toukei no Kiso* (in Japanese), Kyoritsu Shuppan, Tokyo.

Tumura, Y. (1965). The distributions of latent roots and vectors, *TRU Mathematics*, Science University of Tokyo, **1**, 1–16.