Elect. Comm. in Probab. **13** (2008), 54–59

ELECTRONIC COMMUNICATIONS in PROBABILITY

A REMARK ON THE EQUIVALENCE OF GAUSSIAN PROCESSES

HARRY VAN ZANTEN[1]
Department of Mathematics, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
email: harry@cs.vu.nl

Submitted June 19, 2007, *accepted in final form* January 17, 2008

AMS 2000 Subject classification: 60G15, 60G30
Keywords: Gaussian processes with stationary increments, equivalence of laws, spectral methods

Abstract
In this note we extend a classical equivalence result for Gaussian stationary processes to the more general setting of Gaussian processes with stationary increments. This will allow us to apply it in the setting of aggregated independent fractional Brownian motions.

1 Introduction and main result

It is well known that every mean-square continuous, centered, stationary Gaussian process $X = (X_t)_{t\geq 0}$ admits a spectral representation. Indeed, by Bochner's theorem there exists a symmetric, finite Borel measure μ on the line such that

$$\mathbb{E}X_sX_t = \int_{\mathbb{R}} e^{i(t-s)\lambda}\,\mu(d\lambda).$$

The measure μ is called the spectral measure. If it admits a Lebesgue density, this is called the spectral density of the process.
A classical result in the theory of continuous-time stationary Gaussian processes gives sufficient conditions for the equivalence of the laws of two centered processes with different spectral densities, see for instance [7], or [8], Theorem 17 on p. 104. The result says that if the two densities f, g involved satisfy

$$\int_R^\infty \Big|\frac{g(\lambda)-f(\lambda)}{f(\lambda)}\Big|^2\,d\lambda < \infty \tag{1.1}$$

for some $R > 0$ then, under a regularity condition on the tail behaviour of the densities, the laws of the associated processes on $(\mathbb{R}^{[0,T]}, \mathscr{B}(\mathbb{R}^{[0,T]}))$ are equivalent for any $T > 0$. Here, as

[1]PARTIALLY FUNDED BY THE NETHERLANDS ORGANIZATION FOR SCIENTIFIC RESEARCH (NWO)

usual, $\mathbb{R}^{[0,T]}$ is the collection of all real-valued functions on $[0,T]$ and $\mathscr{B}(\mathbb{R}^{[0,T]})$ is the σ-field on $\mathbb{R}^{[0,T]}$ generated by the projections $h \mapsto h(t)$.
Unfortunately, the proof of this result, as given for instance on pp. 105–107 of [8], does not allow extension to the setting of processes with stationary increments. Processes of the latter type admit a spectral representation as well. If $X = (X_t)_{t\geq 0}$ is a mean-square continuous, centered Gaussian process with stationary increments that starts from 0, i.e. $X_0 = 0$ (we call such processes Gaussian si-processes from now on), there exists a unique symmetric Borel measure μ on the line such that $\int (1+\lambda^2)^{-1}\,\mu(d\lambda) < \infty$, and

$$\mathbb{E}X_sX_t = \int_{\mathbb{R}} \frac{(e^{i\lambda s}-1)(e^{-i\lambda t}-1)}{\lambda^2}\,\mu(d\lambda)$$

for all $s,t \geq 0$ (cf., e.g., [3]). Slightly abusing terminology we also call μ the spectral measure of the process X in this case and if it admits a Lebesgue density we call it the spectral density again. The main example is the fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$, which has spectral density

$$f_H(\lambda) = c_H|\lambda|^{1-2H}, \qquad c_H = \frac{\sin(\pi H)\Gamma(1+2H)}{2\pi} \tag{1.2}$$

(see for instance [9]).
It turns out that if we just do as if equivalence result cited above is valid for si-processes, we obtain equivalence statements that are actually correct and can be proved rigorously. Consider for instance the so-called mixed fBm as introduced in [2], which is the sum $W+X$ of a standard Brownian motion W and an independent fBm X with some Hurst index $H \in (0,1)$. The process W has spectral density f identically equal to $1/(2\pi)$ and hence the mixed fBm has spectral density $g(\lambda) = 1/(2\pi) + c_H|\lambda|^{1-2H}$. We see that condition (1.1) becomes in this case

$$\int_{|\lambda|>R} |\lambda|^{2-4H}\,d\lambda < \infty,$$

which is fulfilled if and only if $H > 3/4$. This would suggest that the mixed fBm is equivalent to ordinary Brownian motion if $H > 3/4$. And indeed, this is exactly what [2] proved, cf. also [1].
The main purpose of this note is to show that this example is not a coincidence, and that the classical equivalence result for stationary processes indeed extends to si-processes.
We call two processes equivalent on $[0,T]$ if the laws they induce on $(\mathbb{R}^{[0,T]}, \mathscr{B}(\mathbb{R}^{[0,T]}))$ are equivalent. Recall that an entire function φ on the complex plane is said to be of exponential type τ if

$$\limsup_{r\to\infty} \frac{1}{r} \max_{|z|=r} \log|f(z)| = \tau.$$

It is said to be of finite exponential type if it is of exponential type τ for some $\tau < \infty$. We denote by $\mathscr{L}_T^e$ the linear span of the collection of functions $\{\lambda \mapsto (\exp(i\lambda t)-1)/(i\lambda) : t \in [0,T]\}$.

Theorem 1. *Let X and Y be centered, mean-square continuous Gaussian processes with stationary increments and spectral densities f and g, respectively. Suppose there exist positive constants c_1, c_2 and an entire function φ of finite exponential type such that*

$$c_1|\varphi(\lambda)|^2 \leq f(\lambda) \leq c_2|\varphi(\lambda)|^2 \tag{1.3}$$

for all real λ large enough. For $T > 0$, suppose there exists a constant $C > 0$ such that $\|\psi\|_{L^2(f)} \leq C\|\psi\|_{L^2(g)}$ for all $\psi \in \mathscr{L}_T^e$. Then if condition (1.1) holds for some $R > 0$, the processes X and Y are equivalent on $[0,T]$.

As explained on p. 104 of [8], condition (1.3) is for instance fulfilled if for some $p \in (-\infty, 1)$ it holds that $c_1|\lambda|^p \leq f(\lambda) \leq c_2|\lambda|^p$ for $|\lambda|$ large. In the stationary process result of [8] it is assumed that g satisfies condition (1.3) as well (with the same φ). The condition on g in Theorem 1 gives somewhat more flexibility in special cases, since it is for instance satisfied as soon as $g \geq Cf$ for some constant $C > 0$.
In the next section we present the proof of Theorem 1. Then in Section 3 the result is used to extend an equivalence result for aggregated fBm's of [10].

2 Proof

The proof of the theorem exploits the fact that for Gaussian si-processes, we have a reproducing kernel Hilbert space (RKHS) structure in the frequency domain. For $T > 0$ and a spectral measure μ, let $\mathscr{L}_T(\mu)$ be the closure in $L^2(\mu)$ of the set of functions $\mathscr{L}_T^e$, which is defined as the linear span of the collection $\{\lambda \mapsto (\exp(i\lambda t) - 1)/(i\lambda) : t \in [0,T]\}$. Then $\mathscr{L}_T(\mu)$ is a RKHS of entire functions (see for instance [4], or [6]). We denote its reproducing kernel by S_T. This function has the property that $S_T(\omega, \cdot) \in \mathscr{L}_T(\mu)$ for every $\omega \in \mathbb{R}$ and for every $\psi \in \mathscr{L}_T(\mu)$ and $\omega \in \mathbb{R}$,

$$\langle \psi, S_T(\omega, \cdot)\rangle_{L^2(\mu)} = \psi(\omega),$$

where $\langle \varphi, \psi\rangle_{L^2(\mu)} = \int \varphi\bar{\psi}\, d\mu$. Below we will use the fact that every $\psi \in \mathscr{L}_T(\mu)$ has a version that can be extended to an entire function on the complex plane, that is of finite exponential type (at most T). Conversely, the restriction to the real line of an entire function ψ of exponential type at most T that satisfies

$$\int_{\mathbb{R}} |\psi(\lambda)|^2\, \mu(d\lambda) < \infty,$$

belongs to $\mathscr{L}_T(\mu)$ (cf. [4], [6]).
We shall apply the following theorem obtained in [10]. It gives sufficient conditions for equivalence of Gaussian si-processes involving spectral densities and reproducing kernels.

Theorem 2. *Let X and Y be centered, mean-square continuous Gaussian processes with stationary increments and spectral densities f and g, respectively. Fix $T > 0$ and suppose there exists a constant $C > 0$ such that $\|\psi\|_{L^2(f)} \leq C\|\psi\|_{L^2(g)}$ for all $\psi \in \mathscr{L}_T^e$. Let S_T be the reproducing kernel of $\mathscr{L}_T(f)$. Then if*

$$\int_R^\infty \Big(\frac{g(\lambda) - f(\lambda)}{f(\lambda)}\Big)^2 S_T(\lambda,\lambda) f(\lambda)\, d\lambda < \infty$$

for some $R > 0$, the processes X and Y are equivalent on $[0,T]$.

The following crucial lemma shows that under condition (1.3), we can in fact bound the reproducing kernel S_T of $\mathscr{L}_T(f)$ on the diagonal by a multiple of $1/f$. The proof of Theorem 1 then simply consists of combining this lemma with Theorem 2 above.

Lemma 3. *Suppose the spectral density f satisfies (1.3) for $|\lambda|$ large enough, with c_1, c_2 positive constants and φ an entire function of finite exponential type. Then for $T > 0$ the reproducing kernel S_T of $\mathscr{L}_T(f)$ satisfies*

$$|S_T(\omega,\lambda)|^2 \leq C\frac{S_T(\omega,\omega)}{f(\lambda)}$$

for all real ω and all real λ large enough, where C is a positive constant independent of ω and λ. In particular,

$$S_T(\lambda,\lambda) \leq \frac{C}{f(\lambda)}$$

for $|\lambda|$ large enough.

Proof. Put $f_0 = |\varphi|^2$. Then since φ is entire, f_0 is bounded near 0 and hence, by the first inequality in (1.3), f_0 is the spectral density of a Gaussian si-process. Let ψ_k be an arbitrary orthonormal basis of $\mathscr{L}_T(f_0)$. For every k the function $\psi_k\varphi$ is an entire function of finite exponential type (not depending on k), say S. Moreover, we have

$$\int |\psi_k(\lambda)\varphi(\lambda)|^2\,d\lambda = \int |\psi_k|^2 f_0 = 1 < \infty.$$

Hence, by the Paley-Wiener theorem, $\psi_k\varphi = \hat{f}_k$ for certain $f_k \in L^2[-S,S]$, where $\hat{h}$ denotes the Fourier transform of the function h. By the Parseval relation for the Fourier transform, the fact that the ψ_k are an orthonormal basis of $\mathscr{L}_T(f_0)$ implies that the f_k are orthonormal in $L^2[-S,S]$. By Bessel's inequality, it follows that

$$2\pi\sum|\psi_k(\lambda)|^2 f_0(\lambda) = \sum\Big|\int_{-S}^{S} e^{-i\lambda t} f_k(t)\,dt\Big|^2 \leq \int_{-S}^{S} |e^{i\lambda t}|^2\,dt = 2S,$$

hence $\sum|\psi_k(\lambda)|^2 \leq S/(\pi f_0(\lambda))$.
Now fix $\omega \in \mathbb{R}$ and consider $S_T(\omega,\cdot)$. This function is entire, of exponential type at most T and belongs to $\mathscr{L}_T(f)$ and hence, by the first inequality in (1.3), belongs to $\mathscr{L}_T(f_0)$ as well (cf. [4], Chapter 6). Expanding it in the basis ψ_k of the first paragraph gives

$$S_T(\omega,\lambda) = \sum\langle S_T(\omega,\cdot),\psi_k\rangle_{L^2(f_0)}\,\psi_k(\lambda).$$

By Cauchy-Schwarz, we obtain

$$|S_T(\omega,\lambda)|^2 \leq \sum|\,\langle S_T(\omega,\cdot),\psi_k\rangle_{L^2(f_0)}\,|^2\sum|\psi_k(\lambda)|^2.$$

By the first paragraph, the second factor on the right is bounded by a constant times $1/f_0(\lambda)$, which, by the second inequality of (1.3), is bounded by a constant times $1/f(\lambda)$ for $|\lambda|$ large enough. The first factor equals $\|S_T(\omega,\cdot)\|^2_{L^2(f_0)}$.
To bound this last quantity, observe that since f_0 is bounded near 0, we have for every $a > 0$ and $\psi \in \mathscr{L}_T^e$,

$$\int_{|\lambda|\leq a} |\psi(\lambda)|^2 f_0(\lambda)\,d\lambda \leq c\int_{|\lambda|\leq a} |\psi(\lambda)|^2\,d\lambda$$

for some $c > 0$. On the other hand, the Gaussian si-processes with spectral measures f and $1_{[-a,a]} + f1_{[-a,a]^c}$ are locally equivalent (see [10], Theorem 5.1), in particular the L^2-norms corresponding to the two densities are equivalent on $\mathscr{L}_T^e$ (e.g. Theorem 4.1 of [10]). It follows that

$$\int_{|\lambda|\leq a} |\psi(\lambda)|^2 f_0(\lambda)\, d\lambda \leq c' \int |\psi(\lambda)|^2 f(\lambda)\, d\lambda,$$

the constant c' not depending on ψ. Condition (1.3) implies that for a large enough we have

$$\int_{|\lambda|>a} |\psi(\lambda)|^2 f_0(\lambda)\, d\lambda \leq \frac{1}{c_1} \int_{|\lambda|>a} |\psi(\lambda)|^2 f(\lambda)\, d\lambda.$$

Together we find that for some constant $c > 0$, it holds that $\|\psi\|_{L^2(f_0)} \leq c\|\psi\|_{L^2(f)}$ for all $\psi \in \mathscr{L}_T^e$. By passing to the limit we see that the bound holds in fact for all $\psi \in \mathscr{L}_T(f)$. Applying this with $\psi = S_T(\omega, \cdot)$ and using the reproducing property yields

$$\|S_T(\omega,\cdot)\|^2_{L^2(f_0)} \leq c^2 \|S_T(\omega,\cdot)\|^2_{L^2(f)} = c^2 S_T(\omega,\omega),$$

completing the proof of the lemma. □

3 Application

One of the main motivations for the present paper is the equivalence result for aggregated fBm's given in [10]. Consider a linear combination $X = \sum a_k X^k$ of independent fBm's $X^1, \ldots, X^n$ with increasing Hurst indices $H_1 < \cdots < H_n$, for some nonzero constants $a_1, \ldots, a_n$. It is proved in [10] that X is equivalent to $a_1 X^1$ on every interval $[0, T]$ if $H_2 - H_1 > 1/4$.
Morally speaking, such an equivalence result should be true under conditions that only restrict the tails of the spectral densities of the processes involved. The proof of the result presented in [10] however relies on the explicit form of the frequency domain reproducing kernel of the fBm (cf. [5]). Using Theorem 1 we can now immediately obtain the following generalization, which shows that indeed, only conditions on the tails of the spectra are needed.

Theorem 4. *Let X and Y be Gaussian si-process with spectral densities f and g, respectively. Suppose that for $p \in (-\infty, 1)$ and positive constants c_1, c_2 we have*

$$c_1 |\lambda|^p \leq f(\lambda) \leq c_2 |\lambda|^p$$

for $|\lambda|$ large. Then if

$$\int_R^\infty \frac{|g(\lambda)|^2}{|\lambda|^{2p}} < \infty$$

for some $R > 0$, the processes X and $X + Y$ are equivalent on every interval $[0, T]$.

Observe that we recover the cited result of [10] if we apply the theorem with (using the same notations as above)

$$f = a_1 f_{H_1}, \qquad g = \sum_{k=2}^n a_k f_{H_k},$$

where f_H is the spectral density of the fBm given by (1.2).

References

[1] Baudoin, F. and Nualart, D. (2003). Equivalence of Volterra processes. *Stochastic Process. Appl.* **107**(2), 327–350. MR1999794

[2] Cheridito, P. (2001). Mixed fractional Brownian motion. *Bernoulli* **7**(6), 913–934. MR1873835

[3] Doob, J.L. (1953). *Stochastic processes*. John Wiley & Sons Inc., New York. MR0058896

[4] Dym, H. and McKean, H.P. (1976). *Gaussian processes, function theory, and the inverse spectral problem*. Academic Press, New York. MR0448523

[5] Dzhaparidze, K. and Van Zanten, J.H. (2005). Krein's spectral theory and the Paley-Wiener expansion for fractional Brownian motion. *Ann. Probab.* **33**(2), 620–644. MR2123205

[6] Dzhaparidze, K., Van Zanten, J.H. and Zareba, P. (2005). Representations of fractional Brownian motion using vibrating strings. *Stochastic Process. Appl.* **115**(12), 1928–1953. MR2178502

[7] Gihman, I.I. and Skorohod, A.V. (1980). *The theory of stochastic processes. I*. Springer-Verlag, Berlin. MR0636254

[8] Ibragimov, I.A. and Rozanov, Y.A. (1978). *Gaussian random processes*. Springer-Verlag, New York. MR0543837

[9] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable non-Gaussian random processes*. Chapman & Hall, New York. MR1280932

[10] Van Zanten, J.H. (2007). When is a linear combination of independent fBm's equivalent to a single fBm? *Stochastic Process. Appl.* **117**(1), 57–70. MR2287103