Generating functions and special functions

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Abstract. In this paper we introduce the general form of generating functions. By using the generating function we obtain the terms of different polynomials. Also we calculate the n-th term a_n of the polynomial.

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1 Introduction

In solving many problems of theoretical and mathematical physics one is led to use various special functions. Such problems arise, for example, in connection with heat conduction, the interaction between radiation and matter, the propagation of electromagnetic or acoustic waves, the theory of nuclear reactors, and the internal structure of stars. In practice, special functions usually arise as solutions of the following differential equations [1-5],

$$U''(x) + \frac{\tau(x)}{\sigma(x)}U'(x) + \frac{\lambda(x)}{\sigma^2(x)}U(x) = 0 \tag{1.1}$$

where $\sigma(x)$ and $\lambda(x)$ are polynomials of the degree at the most 2, and $\tau(x)$ is polynomial of degree at the most 1.

Among the solutions of equations of the form (1.1), there are several classes of special functions: the classical orthogonal polynomials (Jacobi, Laguerre, Hermite), Spherical harmonics, Bessel and Hypergeometric functions. These are often refferred to as the special functions in mathematical physics. So the natural approach for mathematical physics is to deduce the properties of the functions directly from the differential equations. But here we will try to deduce the properties of the special functions from Generating function. For this reason we have developed a method which makes it possible to present the theory of special functions , starting from general form of Generating functions.

2 Generating functions

Here we try to introduce the general form of generating function. Also we obtain the polynomial functions and their coeffients which is important in structure of special

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functions.

$$G(x,t)=\sum_{n=0}^{\infty}F_n(x)t^n, \tag{2.1}$$

where $G(x,t)$ and $F_n(x)$ are generating and polynomial functions respectively.

From Tylor expansion we have following expression for $G(x,t)$

$$G(x,t)=\sum_{n=0}\frac{1}{n!}t^n\frac{\partial^n G(x,t)}{\partial t^n}. \tag{2.2}$$

In order to obtain the $F_n(x)$ using equation (2.2) in (2.1) so we arrive at

$$F_n(x)=\frac{1}{n!}\frac{\partial^n G(x,t)}{\partial t^n}|t=0 \tag{2.3}$$

where equation (2.3) is general form for obtainning the different polynomial.

The next step here is to obtain the coefficents of a_n, which are important in special function. We rewrite the polynomial $F_n(x)$ as follows:

$$F_n(x)=\sum_{n=0}^{\infty}a_n x^n \tag{2.4}$$

also we have

$$F_n(x)=\sum_{n=0}\frac{x^n}{n!}\frac{\partial^n F_n(x)}{\partial x^n}. \tag{2.5}$$

Finally from equation (2.2) and (2.4) we obtain following expression;

$$a_n=\frac{1}{n!^2}\frac{\partial^{2n}G_n}{\partial x^n\partial t^n}_{(x=0,t=0)}=\frac{1}{n!}\frac{\partial^n F_n(x)}{\partial x^n}_{(x=0)} \tag{2.6}$$

3 Various special functions

As mentioned in Refs. [1,4], the generating function corresponding to the Legendre polynomial is,

$$G(x,t)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum P_n(x)t^n, \tag{3.1}$$

where $P_n(x)$ is Legendrre polynomial.

Now in order to extract the Legendre polynomial, we use the equation (2.3) and we get:

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= \frac{\partial G}{\partial t}|_{t=0}=x\\
P_2(x) &= \frac{1}{2!}\frac{\partial^2 G}{\partial t^2}|_{t=0}=\frac{1}{2}(3x^2-1)\\
P_3(x) &= \frac{1}{3!}\frac{\partial^3 G}{\partial t^3}|_{t=0}=\frac{1}{3!}(15x^3-9x)
\end{aligned}$$

$$\begin{aligned}
P_4(x) &= \frac{1}{4!}\frac{\partial^4 G}{\partial t^4}|_{t=0} = \frac{1}{4!}(105x^4 - 90x^2 + 9) \\
P_5(x) &= \frac{1}{5!}\frac{\partial^5 G}{\partial t^5}|_{t=0} = \frac{1}{5!}(945x^5 - 1050x^3 + 225)
\end{aligned} \tag{3.2}$$

And also in order to obtain the general coeffients of Legender polynomial we have to use the equation (2.6) to have:

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= \frac{1}{(1!)^2}\frac{\partial^2 G}{\partial x \partial t} = 1 \\
a_2 &= \frac{1}{(2!)^2}\frac{\partial^4 G}{\partial x^2 \partial t^2} = \frac{3}{2} \\
a_3 &= \frac{1}{(3!)^2}\frac{\partial^6 G}{\partial x^3 \partial t^3} = \frac{5}{2} \\
a_4 &= \frac{1}{(4!)^2}\frac{\partial^8 G}{\partial x^4 \partial t^4} = \frac{35}{8} \\
a_5 &= \frac{1}{(5!)^2}\frac{\partial^{10} G}{\partial x^5 \partial t^5} = \frac{63}{10}
\end{aligned} \tag{3.3}$$

Generally using the relation (7), concludes the following expression,

$$a_n = \frac{(2n)!}{2^n (n!)^2} \tag{3.4}$$

In the second example we discuss the Chebyshev polynomial. In this case we have two types of polynomials, where the first one is

$$G_I(x,t) = \frac{t(x-t)}{1 - 2xt + t^2} \tag{3.5}$$

Similiar to the previous case, here also use the equation (2.3) to have:

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= \frac{1}{1!}\frac{\partial G_I}{\partial t}|_{t=0} = x \\
T_2(x) &= \frac{1}{2!}\frac{\partial^2 G_I}{\partial t^2}|_{t=0} = 2x^2 - 1 \\
T_3(x) &= \frac{1}{3!}\frac{\partial^3 G_I}{\partial t^3}|_{t=0} = 4x^3 - 3x \\
T_4(x) &= \frac{1}{4!}\frac{\partial^4 G_I}{\partial t^4}|_{t=0} = 8x^4 - 8x^2 + 1
\end{aligned} \tag{3.6}$$

Now using the equation (2.6) for this Chebyshev polynomial we shall obtain the following expressions;

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= \frac{1}{(1!)^2}\frac{\partial^2 G_I}{\partial x \partial t} = 1
\end{aligned}$$

$$\begin{aligned}
a_2 &= \frac{1}{(2!)^2}\frac{\partial^4 G_I}{\partial x^2 \partial t^2} = 2\\
a_3 &= \frac{1}{(3!)^2}\frac{\partial^6 G_I}{\partial x^3 \partial t^3} = 4\\
a_4 &= \frac{1}{(4!)^2}\frac{\partial^8 G_I}{\partial x^4 \partial t^4} = 8\\
a_5 &= \frac{1}{(5!)^2}\frac{\partial^{10} G_I}{\partial x^5 \partial t^5} = 16
\end{aligned} \tag{3.7}$$

These results imply that the coefficents a_n for the first type of Chebyshev polynomial become as:

$$a_n = 2^{n-1} \qquad n \geq 1$$

and

$$a_n = 1 \qquad n = 0$$

The generating function for the second type of Chebyshev polynomial is

$$G_{II}(x,t) = \frac{1}{1 - 2xt + t^2} \tag{3.8}$$

in this case also we have;

$$\begin{aligned}
U_0(x) &= 1\\
U_1(x) &= \frac{1}{1!}\frac{\partial G_{II}}{\partial t}|_{t=0} = 2x\\
U_2(x) &= \frac{1}{2!}\frac{\partial^2 G_{II}}{\partial t^2}|_{t=0} = 4x^2 - 1\\
U_3(x) &= \frac{1}{3!}\frac{\partial^3 G_{II}}{\partial t^3}|_{t=0} = 8x^3 - 4x\\
U_4(x) &= \frac{1}{4!}\frac{\partial^4 G_{II}}{\partial t^4}|_{t=0} = 16x^4 - 12x^2 + 1
\end{aligned} \tag{3.9}$$

and the coefficents are:

$$\begin{aligned}
a_0 &= 1\\
a_1 &= \frac{1}{(1!)^2}\frac{\partial^2 G_{II}}{\partial x \partial t} = 2\\
a_2 &= \frac{1}{(2!)^2}\frac{\partial^4 G_{II}}{\partial x^2 \partial t^2} = 4\\
a_3 &= \frac{1}{(3!)^2}\frac{\partial^6 G_{II}}{\partial x^3 \partial t^3} = 8\\
a_4 &= \frac{1}{(4!)^2}\frac{\partial^8 G_{II}}{\partial x^4 \partial t^4} = 16\\
a_5 &= \frac{1}{(5!)^2}\frac{\partial^{10} G_{II}}{\partial x^5 \partial t^5} = 32,
\end{aligned} \tag{3.10}$$

here also the general coeffiecent can be drawn easily as $a_n = 2^n$.

The third example is Laguerre polynomial, where its generating function is;

$$G(x,t) = \frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n \tag{3.11}$$

Here also we follow the same procedure, i.e we use equations (2.3) and (2.6),

$$\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= \frac{1}{1!}\frac{\partial G}{\partial t}|_{t=0} = 1 - x \\
L_2(x) &= \frac{1}{2!}\frac{\partial^2 G}{\partial t^2}|_{t=0} = \frac{1}{2}(x^2 - 4x + 2) \\
L_3(x) &= \frac{1}{3!}\frac{\partial^3 G}{\partial t^3}|_{t=0} = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6) \\
L_4(x) &= \frac{1}{4!}\frac{\partial^4 G}{\partial t^4}|_{t=0} = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24) \\
L_5(x) &= \frac{1}{5!}\frac{\partial^5 G}{\partial t^5}|_{t=0} = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= \frac{1}{(1!)^2}\frac{\partial^2 G}{\partial x \partial t} = -1 \\
a_2 &= \frac{1}{(2!)^2}\frac{\partial^4 G}{\partial x^2 \partial t^2} = 1 \\
a_3 &= \frac{1}{(3!)^2}\frac{\partial^6 G}{\partial x^3 \partial t^3} = -1 \\
a_4 &= \frac{1}{(4!)^2}\frac{\partial^8 G}{\partial x^4 \partial t^4} = 1 \\
a_5 &= \frac{1}{(5!)^2}\frac{\partial^{10} G}{\partial x^5 \partial t^5} = -1
\end{aligned} \tag{3.13}$$

These results also imply that

$$a_n = (-1)^n$$

Also for the case of associated Laguerre, we have

$$G(x,t) = \frac{e^{-\frac{xt}{1-t}}}{(1-t)^k + 1} = \sum_{n=0}^{\infty} L_n^k(x)t^n. \tag{3.14}$$

We use also equations (2.3) and (2.6) one can obtain

$$
\begin{aligned}
L_0^k(x) &= 1 \\
L_1(x)^k &= \frac{1}{1!}\frac{\partial G}{\partial t}|_{t=0} = -x + k + 1 \\
L_2^k(x)^k &= \frac{1}{2!}\frac{\partial^2 G}{\partial t^2}|_{t=0} = \frac{1}{2}x^2 - (k+2)x + \frac{1}{2}(k+1)(k+2) \\
L_3^k(x)^k &= \frac{1}{3!}\frac{\partial^3 G}{\partial t^3}|_{t=0} = \frac{1}{6}[-x^3 + (3k+9)x^2 - (3(k+1)^2 + 9k + 15)x \\
& +2k + 2 + (k+1)^3 + 3(k+1)^2] \\
L_4^k(x)^k &= \frac{1}{4!}\frac{\partial^4 G}{\partial t^4}|_{t=0} = \frac{1}{4!}[x^4 - (16+4k)x^3 + (30k + 66 + 6(k+1)^2)x^2 \\
& -(4(k+1)^3 + 24(k+1)^2 + 44k + 68)x \\
& +(k+1)^4 + 6(k+1)^3 + 6k + 6 + 11(k+1)^2]
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= \frac{1}{(1!)^2}\frac{\partial^2 G}{\partial x \partial t} = -1 \\
a_2 &= \frac{1}{(2!)^2}\frac{\partial^4 G}{\partial x^2 \partial t^2} = \frac{1}{2!} \\
a_3 &= \frac{1}{(3!)^2}\frac{\partial^6 G}{\partial x^3 \partial t^3} = -\frac{1}{3!} \\
a_4 &= \frac{1}{(4!)^2}\frac{\partial^8 G}{\partial x^4 \partial t^4} = \frac{1}{4!} \\
a_5 &= \frac{1}{(5!)^2}\frac{\partial^{10} G}{\partial x^5 \partial t^5} = \frac{-1}{5!}
\end{aligned}
\tag{3.16}
$$

so we have $a_n = \frac{(-1)^n}{n!}$.

The fourth example is Hermite Polynomial where its generating function will be:

$$G(x,t) = e^{2tx - t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{3.17}$$

Here the same proccess is applied, and also we have

$$
\begin{aligned}
&H_0(x) = 1, \\
&H_1(x) = \tfrac{1}{1!}\tfrac{\partial G}{\partial t}|_{t=0} = 2x, \\
&H_2(x) = \tfrac{1}{2!}\tfrac{\partial^2 G}{\partial t^2}|_{t=0} = 4x^2 - 2, \\
&H_3(x) = \tfrac{1}{3!}\tfrac{\partial^3 G}{\partial t^3}|_{t=0} = 8x^3 - 12x, \\
&H_4(x) = \tfrac{1}{4!}\tfrac{\partial^4 G}{\partial t^4}|_{t=0} = 16x^4 - 48x^2 + 12, \\
&H_5(x) = \tfrac{1}{5!}\tfrac{\partial^5 G}{\partial t^5}|_{t=0} = 32x^5 - 160x^3 + 120x
\end{aligned}
\tag{3.18}
$$

and

$$(3.19)\qquad \begin{array}{ll} a_0 = 1, & a_1 = \frac{1}{(1!)^2}\frac{\partial^2 G}{\partial x \partial t} = 2 \\ a_2 = \frac{1}{(2!)^2}\frac{\partial^4 G}{\partial x^2 \partial t^2} = 4, & a_3 = \frac{1}{(3!)^2}\frac{\partial^6 G}{\partial x^3 \partial t^3} = 8 \\ a_4 = \frac{1}{(4!)^2}\frac{\partial^8 G}{\partial x^4 \partial t^4} = 16, & a_5 = \frac{1}{(5!)^2}\frac{\partial^{10} G}{\partial x^5 \partial t^5} = 32, \end{array}$$

and finally we have $a_n = 2^n$.

Note that the proccess may be easily applied for the case of Jacobi and Gegenbauer (Ultra Spherical) polynomials, though the application for the Bessel polynomial may be cumbersome.

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