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MOMENT ESTIMATES FOR LÉVY PROCESSES

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Abstract

For real Lévy processes $(X_t)_{t\geq 0}$ having no Brownian component with Blumenthal-Getoor index β, the estimate $\mathbb{E}\sup_{s\leq t}|X_s - a_p s|^p \leq C_p t$ for every $t\in[0,1]$ and suitable $a_p\in\mathbb{R}$ has been established by Millar [6] for $\beta < p \leq 2$ provided $X_1 \in L^p$. We derive extensions of these estimates to the cases $p > 2$ and $p \leq \beta$.

1 Introduction and results

We investigate the L^p-norm (or quasi-norm) of the maximum process of real Lévy processes having no Brownian component. A (càdlàg) Lévy process $X = (X_t)_{t\geq 0}$ is characterized by its so-called local characteristics in the Lévy-Khintchine formula. They depend on the way the "big" jumps are truncated. We will adopt in the following the convention that the truncation occurs at size 1. So that

$$\mathbb{E}\, e^{iuX_t} = e^{-t\Psi(u)} \text{ with } \Psi(u) = -iua + \frac{1}{2}\sigma^2u^2 - \int (e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}})d\nu(x) \tag{1.1}$$

where $u,\, a\in\mathbb{R}, \sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int x^2\wedge 1 d\nu(x) < +\infty$. The measure ν is called the Lévy measure of X and the quantities (a,σ^2,ν) are referred to as the characteristics of X. One shows that for $p > 0, \mathbb{E}\,|X_1|^p < +\infty$ if and only if $\mathbb{E}\,|X_t|^p < +\infty$ for every $t \geq 0$ and this in turn is equivalent to $\mathbb{E}\sup_{s\leq t}|X_s|^p < +\infty$ for every $t\geq 0$. Furthermore,

$$\mathbb{E}\,|X_1|^p < +\infty \text{ if and only if } \int_{\{|x|>1\}} |x|^p d\nu(x) < +\infty \tag{1.2}$$

(see [7]). The index β of the process X introduced in [2] is defined by

$$\beta = \inf\{p > 0 : \int_{\{|x|\leq 1\}} |x|^p d\nu(x) < +\infty\}. \tag{1.3}$$

Necessarily, $\beta \in [0,2]$. This index is often called Blumenthal-Getoor index of X.
In the sequel we will assume that $\sigma^2 = 0$, *i.e.* that X has no Brownian component. Then the Lévy-Itô decomposition of X reads

$$X_t = at + \int_0^t \int_{\{|x|\leq 1\}} x(\mu - \lambda \otimes \nu)(ds, dx) + \int_0^t \int_{\{|x|>1\}} x\mu(ds, dx) \tag{1.4}$$

where λ denotes the Lebesgue measure and μ is the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ associated with the jumps of X by

$$\mu = \sum_{t\geq 0} \varepsilon_{(t,\triangle X_t)} \mathbf{1}_{\{\triangle X_t \neq 0\}},$$

$\triangle X_t = X_t - X_{t-}, \triangle X_0 = 0$ and where ε_z denotes the Dirac measure at z (see [4], [7]).

Theorem 1. *Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristics $(a, 0, \nu)$ and Blumenthal-Getoor index β. Assume either*
– $p \in (\beta, \infty)$ such that $\mathbb{E}|X_1|^p < +\infty$
or
– $p = \beta$ provided $\beta > 0$ and $\int_{\{|x|\leq 1\}} |x|^\beta d\nu(x) < +\infty$. Then, for every $t \geq 0$

$$\begin{aligned} \mathbb{E} \sup_{s\leq t} |Y_s|^p &\leq C_p t \qquad \text{if } p < 1, \\ \mathbb{E} \sup_{s\leq t} |X_s - s\,\mathbb{E}\,X_1|^p &\leq C_p t \qquad \text{if } 1 \leq p \leq 2 \end{aligned}$$

for a finite real constant C_p, where $Y_t = X_t - t\left(a - \int_{\{|x|\leq 1\}} x d\nu(x)\right)$. Furthermore, for every $p > 2$,

$$\mathbb{E} \sup_{s\leq t} |X_s|^p = O(t) \qquad \text{as} \qquad t \to 0.$$

If X_1 is symmetric one observes that $Y = X$ since the symmetry of X_1 implies $a = 0$ and the symmetry of ν (see [7]). We emphasize that in view of the Kolmogorov criterion for continuous modifications, the above bounds are best possible as concerns powers of t. In case $p > \beta$ and $p \leq 2$, these estimates are due to Millar [6]. However, the Laplace-transform approach in [6] does not work for $p > 2$. Our proof is based on the Burkholder-Davis-Gundy inequality.
For the case $p < \beta$ we need some assumptions on X. Recall that a measurable function $\varphi : (0, c] \to (0, \infty)$ $(c > 0)$ is said to be regularly varying at zero with index $b \in \mathbb{R}$ if, for every $t > 0$,

$$\lim_{x\to 0} \frac{\varphi(tx)}{\varphi(x)} = t^b.$$

This means that $\varphi(1/x)$ is regularly varying at infinity with index $-b$. Slow variation corresponds to $b = 0$. One defines on $(0, \infty)$ the tail function $\underline{\nu}$ of the Lévy measure ν by $\underline{\nu}(x) := \nu([-x, x]^c)$.

Theorem 2. *Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristics $(a,0,\nu)$ and index β such that $\beta>0$ and $\mathbb{E}\,|X_1|^p<+\infty$ for some $p\in(0,\beta)$. Assume that the tail function of the Lévy measure satisfies*

$$\exists\, c\in(0,1], \qquad \underline{\nu}\leq\varphi \ \text{ on } \ (0,c] \tag{1.5}$$

where $\varphi:(0,c]\to(0,\infty)$ is a regularly varying function at zero of index $-\beta$. Let $l(x)=x^\beta\varphi(x)$ and assume that $l(1/x), x\geq 1/c$ is locally bounded. Let $\underline{l}(x)=\underline{l}_\beta(x)=l(x^{1/\beta})$.

(a) Assume $\beta>1$. Then as $t\to 0$, for every $r\in(\beta,2]$, $q\in[p\vee 1,\beta)$,

$$\mathbb{E}\sup_{s\leq t}|X_s|^p=O(t^{p/\beta}[\underline{l}(t)^{p/r}+\underline{l}(t)^{p/q}]) \quad \text{if} \quad \beta<2,$$

$$\mathbb{E}\sup_{s\leq t}|X_s|^p=O(t^{p/\beta}[1+\underline{l}(t)^{p/q}]) \quad \text{if} \quad \beta=2.$$

If ν is symmetric then this holds for every $q\in[p,\beta)$.

(b) Assume $\beta<1$. Then as $t\to 0$, for every $r\in(\beta,1]$, $q\in[p,\beta)$

$$\mathbb{E}\sup_{s\leq t}|Y_s|^p=O(t^{p/\beta}[\underline{l}(t)^{p/r}+\underline{l}(t)^{p/q}])$$

where $Y_t=X_t-t\left(a-\int_{\{|x|\leq 1\}}x\,d\nu(x)\right)$. If ν is symmetric this holds for every $r\in(\beta,2]$.

(c) Assume $\beta=1$ and ν is symmetric. Then as $t\to 0$, for every $r\in(\beta,2], q\in[p,\beta)$

$$\mathbb{E}\sup_{s\leq t}|X_s-as|^p=O(t^{p/\beta}[\underline{l}(t)^{p/r}+\underline{l}(t)^{p/q}]).$$

It can be seen from strictly α-stable Lévy processes where $\beta=\alpha$ that the above estimates are best possible as concerns powers of t.
Observe that condition (1.5) is satisfied for a broad class of Lévy processes. For absolutely continuous Lévy measures one may consider the condition

$$\exists c\in(0,1], 1_{\{0<|x|\leq c\}}\nu(dx)\leq\psi(|x|)1_{\{0<|x|\leq c\}}dx \tag{1.6}$$

where $\psi:(0,c]\to(0,\infty)$ is a regularly varying function at zero of index $-(\beta+1)$ and $\psi(1/x)$ is locally bounded, $x\geq 1/c$. It implies that the tail function of the Lévy measure is dominated, for $x\leq c$, by $2\int_x^c\psi(s)ds+\underline{\nu}(c)$, a regularly varying function at zero with index $-\beta$, so that (1.5) holds with $\varphi(x)=C\,x\psi(x)$ (see [1], Theorem 1.5.11).

Important special cases are as follows.

Corollary 1.1. *Assume the situation of Theorem 2 (with ν symmetric if $\beta=1$) and let U denote any of the processes X, Y, $(X_t-at)_{t\geq 0}$.*

(a) Assume that the slowly varying part l of φ is decreasing and unbounded on $(0,c]$ (e.g. $(-\log x)^a, a>0$). Then as $t\to 0$, for every $\varepsilon\in(0,\beta)$,

$$\mathbb{E}\sup_{s\leq t}|U_s|^p=O(t^{p/\beta}\underline{l}(t)^{p/(\beta-\varepsilon)}).$$

(b) Assume that l is increasing on $(0, c]$ satisfying $l(0+) = 0$ (e.g. $(-\log x)^{-a}, a > 0, c < 1$) and $\beta \in (0, 2)$. Then as $t \to 0$, for every $\varepsilon > 0$,

$$\mathbb{E} \sup_{s \le t} |U_s|^p = O(t^{p/\beta} \underline{l}(t)^{p/(\beta+\varepsilon)}).$$

The remaining cases $p = \beta \in (0, 2)$ if $\beta \neq 1$ and $p \le 1$ if $\beta = 1$ are solved under the assumption that the slowly varying part of the function φ in (1.5) is constant.

Theorem 3. *Let $(X_t)_{t\ge 0}$ be a Lévy process with characteristics $(a, 0, \nu)$ and index β such that $\beta \in (0, 2)$ and $\mathbb{E} |X_1|^\beta < +\infty$ if $\beta \neq 1$ and $\mathbb{E} |X_1|^p < +\infty$ for some $p \le 1$ if $\beta = 1$. Assume that the tail function of the Lévy measure satisfies*

$$\exists\, c \in (0, 1],\ \exists\, C \in (0, \infty), \qquad \underline{\nu}(x) \le C x^{-\beta} \ \text{ on } \ (0, c]. \tag{1.7}$$

Then as $t \to 0$

$$\begin{aligned} \mathbb{E} \sup_{s \le t} |X_s|^\beta &= O(t(-\log t)) \ \text{ if } \ \beta > 1, \\ \mathbb{E} \sup_{s \le t} |Y_s|^\beta &= O(t(-\log t)) \ \text{ if } \ \beta < 1 \end{aligned}$$

and

$$\mathbb{E} \sup_{s \le t} |X_s|^p = O((t(-\log t))^p) \quad \text{if} \quad \beta = 1,\ p \le 1$$

where the process Y is defined as in Theorem 2.

The above estimates are optimal (see Section 3). Condition (1.7) is satisfied if

$$\exists c \in (0, 1],\ \exists C \in (0, \infty),\ \mathbf{1}_{\{0<|x|\le c\}} \nu(dx) \le \frac{C}{|x|^{\beta+1}} \mathbf{1}_{\{0<|x|\le c\}} dx. \tag{1.8}$$

The paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1, 2 and 3. Section 3 contains a collection of examples.

2 Proofs

We will extensively use the following compensation formula (see e.g. [4])

$$\mathbb{E} \int_0^t \int f(s, x) \mu(ds, dx) = \mathbb{E} \sum_{s \le t} f(s, \Delta X_s) \mathbf{1}_{\{\Delta X_s \neq 0\}} = \int_0^t \int f(s, x) d\nu(x) ds$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is a Borel function.

Proof of Theorem 1. Since $\mathbb{E} |X_1|^p < +\infty$ and $p > \beta$ (or $p = \beta$ provided $\int_{\{|x|\le 1\}} |x|^\beta d\nu(x) < +\infty$ and $\beta > 0$), it follows from (1.2) that

$$\int |x|^p d\nu(x) < +\infty.$$

Case 1 ($0 < p < 1$). In this case we have $\beta < 1$ and hence $\int_{\{x| \le 1\}} |x| d\nu(x) < +\infty$. Consequently, X $a.s.$ has finite variation on finite intervals. By (1.4),

$$Y_t = X_t - t\left(a - \int_{\{|x|\le 1\}} x d\nu(x)\right) = \int_0^t \int x\mu(ds, dx) = \sum_{s\le t} \triangle X_s$$

so that, using the elementary inequality $(u+v)^p \le u^p + v^p$,

$$\sup_{s\le t} |Y_s|^p \le \left(\sum_{s\le t} |\triangle X_s|\right)^p \le \sum_{s\le t} |\triangle X_s|^p = \int_0^t \int |x|^p \mu(ds, dx).$$

Consequently,

$$\mathbb{E} \sup_{s\le t} |Y_s|^p \le t \int |x|^p d\nu(x) \text{ for every } t \ge 0.$$

Case 2 ($1 \le p \le 2$). Introduce the martingale

$$M_t := X_t - t\,\mathbb{E}\, X_1 = X_t - t\left(a + \int_{\{|x|>1\}} x d\nu(x)\right) = \int_0^t \int x(\mu - \lambda \otimes \nu)(ds, dx).$$

It follows from the Burkholder-Davis-Gundy inequality (see [5], p. 524) that

$$\mathbb{E} \sup_{s\le t} |M_s|^p \le C\mathbb{E}\,[M]_t^{p/2}$$

for some finite constant C. Since $p/2 \le 1$, the quadratic variation $[M]$ of M satisfies

$$[M]_t^{p/2} = \left(\sum_{s\le t} |\triangle X_s|^2\right)^{p/2} \le \sum_{s\le t} |\triangle X_s|^p$$

so that

$$\mathbb{E} \sup_{s\le t} |M_s|^p \le Ct \int |x|^p d\nu(x) \text{ for every } t \ge 0.$$

CASE 3: $p > 2$. One considers again the martingale Lévy process $M_t = X_t - t\,\mathbb{E}\, X_1$. For $k \ge 1$ such that $2^k \le p$, introduce the martingales

$$N_t^{(k)} := \int_0^t \int |x|^{2^k} (\mu - \lambda \otimes \nu)(ds, dx) = \sum_{s\le t} |\triangle X_s|^{2^k} - t \int |x|^{2^k} d\nu(x).$$

Set $m := \max\{k \ge 1 : 2^k < p\}$. Again by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}
\mathbb{E} \sup_{s\le t} |M_s|^p &\le C\,\mathbb{E}\,[M]_t^{p/2} \\
&= C\,\mathbb{E}\left(t \int x^2 d\nu(x) + N_t^{(1)}\right)^{p/2} \\
&\le C\left(t^{p/2}\left(\int x^2 d\nu(x)\right)^{p/2} + \mathbb{E}\,|N_t^{(1)}|^{p/2}\right) \\
&\le C\,(t + \mathbb{E}\,|N_t^{(1)}|^{p/2})
\end{aligned}$$

for every $t \in [0,1]$ where C is a finite constant that may vary from line to line. Applying successively the Burkholder-Davis-Gundy inequality to the martingales $N^{(k)}$ and exponents $p/2^k > 1, 1 \le k \le m$, finally yields

$$\mathbb{E} \sup_{s\le t} |M_s|^p \le C(t + \mathbb{E}\,[N^{(m)}]_t^{p/2^{m+1}}) \text{ for every } t\in [0,1].$$

Using $p \le 2^{m+1}$, one gets

$$[N^{(m)}]_t^{p/2m+1} = \left(\sum_{s\le t} |\triangle X_s|^{2^{m+1}}\right)^{p/2^{m+1}} \le \sum_{s\le t} |\triangle X_s|^p$$

so that

$$\mathbb{E} \sup_{s\le t} |M_s|^p \le C\left(t + t\int |x|^p d\nu(x)\right) \qquad \text{for every } t\in [0,1].$$

This implies $\mathbb{E} \sup_{s\le t} |X_s|^p = O(t)$ as $t \to 0$. □

Proof of Theorems 2 and 3. Let $p \le \beta$ and fix $c \in (0,1]$. Let $\nu_1 = \mathbf{1}_{\{|x|\le c\}} \cdot \nu$ and $\nu_2 = \mathbf{1}_{\{|x|>c\}} \cdot \nu$. Construct Lévy processes $X^{(1)}$ and $X^{(2)}$ such that $X \overset{d}{=} X^{(1)} + X^{(2)}$ and $X^{(2)}$ is a compound Poisson process with Lévy measure ν_2. Then $\beta = \beta(X) = \beta(X^{(1)}), \beta(X^{(2)}) = 0$, $\mathbb{E}\,|X^{(1)}|^q < +\infty$ for every $q > 0$ and $\mathbb{E}\,|X_1^{(2)}|^p < +\infty$. It follows e.g. from Theorem 1 that for every $t \ge 0$,

$$\mathbb{E} \sup_{s\le t} |X_s^{(2)}|^p \le C_p\, t \qquad \text{if } p < 1, \tag{2.1}$$

$$\mathbb{E} \sup_{s\le t} |X^{(2)} - s\,\mathbb{E}\,X_1^{(2)}|^p \le C_p\, t \qquad \text{if } 1 \le p \le 2$$

where $\mathbb{E}\,X_1^{(2)} = \int x d\nu_2(x) = \int_{\{|x|>c\}} x d\nu(x)$.
As concerns $X^{(1)}$, consider the martingale

$$Z_t^{(1)} := X_t^{(1)} - tEX_1^{(1)} = X_t^{(1)} - t\left(a - \int x\mathbf{1}_{\{c<|x|\le 1\}} d\nu(x)\right) = \int_0^t \int x(\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

where μ_1 denotes the Poisson random measure associated with the jumps of $X^{(1)}$. The starting idea is to separate the "small" and the "big" jumps of $X^{(1)}$ in a non homogeneous way with respect to the function $s \mapsto s^{1/\beta}$. Indeed one may decompose $Z^{(1)}$ as follows

$$Z^{(1)} = M + N$$

where

$$M_t := \int_0^t \int x\mathbf{1}_{\{|x|\le s^{1/\beta}\}}(\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

and

$$N_t := \int_0^t \int x\mathbf{1}_{\{|x|>s^{1/\beta}\}}(\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

are martingales. Observe that for every $q > 0$ and $t \geq 0$,

$$\begin{aligned}\int_0^t \int |x|^q \mathbf{1}_{\{|x|>s^{1/\beta}\}} d\nu_1(x)ds &= \int |x|^q (|x|^\beta \wedge t) d\nu_1(x) \\ &\leq \int_{\{|x|\leq c\}} |x|^{\beta+q} d\nu(x) < +\infty.\end{aligned}$$

Consequently,

$$N_t = \int_0^t \int x\mathbf{1}_{\{|x|>s^{1/\beta}\}} d\mu_1(s,x) - g(t)$$

where $g(t) := \int_0^t \int x\mathbf{1}_{\{|x|>s^{1/\beta}\}} d\nu_1(x)ds$. Furthermore, for every $r > \beta$ or $r = 2$ and $t \geq 0$

$$\int_0^t \int |x|^r \mathbf{1}_{\{|x|\leq s^{1/\beta}\}} d\nu_1(x)ds \leq t \int_{\{|x|\leq c\}} |x|^r d\nu(x) < +\infty. \tag{2.2}$$

In the sequel let C denote a finite constant that may vary from line to line.
We first claim that for every $t \geq 0, r \in (\beta, 2] \cap [1, 2]$ and for $r = 2$,

$$\mathbb{E} \sup_{s\leq t} |M_s|^p \leq C \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x|\leq s^{1/\beta}\}} d\nu_1(x)ds \right)^{p/r}. \tag{2.3}$$

In fact, it follows from the Burkholder-Davis-Gundy inequality and from $p/r \leq 1, r/2 \leq 1$ that

$$\begin{aligned}\mathbb{E} \sup_{s\leq t} |M_s|^p &\leq \left(\mathbb{E} \sup_{s\leq t} |M_s|^r \right)^{p/r} \\ &\leq C \left(\mathbb{E} \, [M]_t^{r/2} \right)^{p/r} \\ &= C \left(\mathbb{E} \left(\sum_{s\leq t} |\triangle X_s^{(1)}|^2 \mathbf{1}_{\{|\triangle X_s^{(1)}|\leq s^{1/\beta}\}} \right)^{r/2} \right)^{p/r} \\ &\leq C \left(\mathbb{E} \sum_{s\leq t} |\triangle X_s^{(1)}|^r \mathbf{1}_{\{|\triangle X_s^{(1)}|\leq s^{1/\beta}\}} \right)^{p/r} \\ &= C \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x|\leq s^{1/\beta}\}} d\nu_1(x)ds \right)^{p/r}.\end{aligned}$$

Exactly as for M, one gets for every $t \geq 0$ and every $q \in [p, 2] \cap [1, 2]$ that

$$\mathbb{E} \sup_{s\leq t} |N_s|^p \leq C \left(\int_0^t \int |x|^q \mathbf{1}_{\{|x|>s^{1/\beta}\}} d\nu_1(x)ds \right)^{p/q}. \tag{2.4}$$

If ν is symmetric then (2.4) holds for every $q \in [p, 2]$ (which of course provides additional information in case $p < 1$ only). Indeed, $g = 0$ by the symmetry of ν so that

$$N_t = \int_0^t \int x\mathbf{1}_{\{|x|>s^{1/\beta}\}} d\mu_1(s,x)$$

and for $q \in [p, 1]$

$$\begin{aligned}
\mathbb{E}\sup_{s\le t}\left|\int_0^s\int x\mathbf{1}_{\{|x|>u^{1/\beta}\}}\mu_1(du,dx)\right|^p &\le \left(\mathbb{E}\sup_{s\le t}\left|\int_0^t\int x\mathbf{1}_{\{|x|>u^{1/\beta}\}}\mu_1(du,dx)\right|^q\right)^{p/q} \quad (2.5)\\
&\le \left(\mathbb{E}\sum_{s\le t}\left|\triangle X_s^{(1)}\right|^q\mathbf{1}_{\{|\triangle X_s^{(1)}|>s^{1/\beta}\}}\right)^{p/q}\\
&= \left(\int_0^t\int |x|^q\mathbf{1}_{\{|x|>s^{1/\beta}\}}d\nu_1(x)ds\right)^{p/q}.
\end{aligned}$$

In the case $\beta < 1$ we consider the process

$$\begin{aligned}
Y_t^{(1)} &:= Z_t^{(1)} + t\int x d\nu_1(x) = X_t^{(1)} - t\left(a - \int_{\{|x|\le 1\}} x d\nu(x)\right)\\
&= M_t + N_t + t\int x d\nu_1(x)\\
&= \int_0^t\int x\mathbf{1}_{\{|x|\le s^{1/\beta}\}}\mu_1(ds,dx) + \int_0^t\int x\mathbf{1}_{\{|x|>s^{1/\beta}\}}\mu_1(ds,dx).
\end{aligned}$$

Exactly as in (2.5) one shows that for $t \ge 0$ and $r \in (\beta, 1]$

$$\mathbb{E}\sup_{s\le t}\left|\int_0^s\int x\mathbf{1}_{\{|x|\le u^{1/\beta}\}}\mu_1(du,dx)\right|^p \le \left(\int_0^t\int |x|^r\mathbf{1}_{\{|x|\le s^{1/\beta}\}}d\nu_1(x)ds\right)^{p/r}. \quad (2.6)$$

Combining (2.1) and (2.3) - (2.6) we obtain the following estimates. Let

$$Z_t = X_t - t\left(a - \int x\mathbf{1}_{\{c<|x|\le 1\}}d\nu(x)\right).$$

CASE 1: $\beta \ge 1$ and $p < 1$. Then for every $t \ge 0, r \in (\beta, 2] \cup \{2\}, q \in [1, 2]$,

$$\begin{aligned}
\mathbb{E}\sup_{s\le t}|Z_s|^p &\le C\Bigg(t + \left(\int_0^t\int |x|^r\mathbf{1}_{\{|x|\le s^{1/\beta}\}}d\nu_1(x)ds\right)^{p/r}\\
&\quad + \left(\int_0^t\int |x|^q\mathbf{1}_{\{|x|>s^{1/\beta}\}}d\nu_1(x)ds\right)^{p/q}\Bigg). \quad (2.7)
\end{aligned}$$

If ν is symmetric (2.7) is even valid for every $q \in [p, 2]$.

CASE 2: $\beta \ge 1$ and $p \ge 1$. Then for every $t \ge 0, r \in (\beta, 2] \cup \{2\}, q \in [p, 2]$,

$$\begin{aligned}
\mathbb{E}\sup_{s\le t}|X_s - s\,\mathbb{E}X_1|^p &\le C\Bigg(t + \left(\int_0^t\int |x|^r\mathbf{1}_{\{|x|\le s^{1/\beta}\}}d\nu_1(x)ds\right)^{p/r}\\
&\quad + \left(\int_0^t |x|^q\mathbf{1}_{\{|x|>s^{1/\beta}\}}d\nu_1(x)ds\right)^{p/q}\Bigg). \quad (2.8)
\end{aligned}$$

CASE 3: $\beta < 1$. Then for every $t \geq 0, r \in (\beta, 1], q \in [p, 1]$

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |Y_s|^p &\leq C \left(t + \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r} \right. \\ &\quad \left. + \left(\int_0^t |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q} \right). \end{aligned} \tag{2.9}$$

If ν is symmetric then $Y = Z = (X_t - at)_{t \geq 0}$ and (2.9) is valid for every $r \in (\beta, 2], q \in [p, 2]$.

Now we deduce Theorem 2. Assume $p \in (0, \beta)$ and (1.5). The constant c in the above decomposition of X is specified by the constant from (1.5). Then one just needs to investigate the integrals appearing in the right hand side of the inequalities (2.7) - (2.10). One checks that for $a > 0, s \leq c^\beta$

$$\int |x|^a \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) \leq a \int_0^{s^{1/\beta}} x^{a-1} \underline{\nu}(x) dx \leq a \int_0^{s^{1/\beta}} x^{a-1} \varphi(x) dx$$

and

$$\begin{aligned} \int |x|^a \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) &\leq a \int_{s^{1/\beta}}^c x^{a-1} \underline{\nu}(x) dx + s^{a/\beta} \underline{\nu}(s^{1/\beta}) \\ &\leq a \int_{s^{1/\beta}}^c x^{a-1} \varphi(x) dx + s^{\frac{q}{\beta}-1} l(s^{1/\beta}). \end{aligned}$$

Now, Theorem 1.5.11 in [1] yields for $r > \beta$,

$$\int_0^{s^{1/\beta}} x^{r-1} \varphi(x) dx \sim \frac{1}{r - \beta} s^{\frac{r}{\beta}-1} l(s^{1/\beta}) \quad \text{as} \quad s \to 0$$

which in turn implies that for small t,

$$\begin{aligned} \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds &\leq r \int_0^t \int_0^{s^{1/\beta}} x^{r-1} \varphi(x) dx ds \\ &\sim \frac{\beta}{(r - \beta)} t^{r/\beta} l(t^{1/\beta}) \quad \text{as} \quad t \to 0. \end{aligned} \tag{2.10}$$

Similarly, for $0 < q < \beta$,

$$\int_{s^{1/\beta}}^c x^{q-1} \varphi(x) dx \sim \frac{1}{\beta - q} s^{\frac{q}{\beta}-1} l(s^{1/\beta}) \text{ as } s \to 0$$

and thus

$$\begin{aligned} \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds &\leq q \int_0^t \int_{s^{1/\beta}}^c x^{q-1} \varphi(x) dx ds + \int_0^t s^{\frac{q}{\beta}-1} l(s^{1/\beta})\, ds \\ &\sim \frac{\beta^2}{(\beta - q) q} t^{q/\beta} l(t^{1/\beta}) \quad \text{as} \quad t \to 0. \end{aligned} \tag{2.11}$$

Using (2.2) for the case $\beta = 2$ and $t + t^p = o(t^{p/\beta} \underline{l}(t)^\alpha)$ as $t \to 0, \alpha > 0$, for the case $\beta > 1$ one derives Theorem 2.

As for Theorem 3, one just needs a suitable choice of q in (2.7) - (2.9). Note that by (1.7) for every $\beta\in(0,2)$ and $t \le c^{\beta}$,

$$\int_0^t \int |x|^{\beta} \mathbf{1}_{\{|x|>s^{1/\beta}\}} d\nu_1(x) ds \le \int_0^t \left(C\beta \int_{s^{1/\beta}}^{c} x^{-1} dx + 1 \right) ds \le C_1 t(-\log t)$$

so that $q = \beta$ is the right choice. (This choice of q is optimal.) Since by (2.10), for $r \in (\beta, 2]$ $(\neq \emptyset)$,

$$\int_0^t \int |x|^r \mathbf{1}_{\{|x|\le s^{1/\beta}\}} d\nu_1(x) ds = O(t^{r/\beta})$$

the assertions follow from (2.7) - (2.9). □

3 Examples

Let K_ν denote the modified Bessel function of the third kind and index $\nu > 0$ given by

$$K_\nu(z) = \frac{1}{2} \int_0^\infty u^{\nu-1} \exp\left(-\frac{z}{2}(u + \frac{1}{u}) \right) du, \quad z > 0.$$

• *The* Γ*-process* is a subordinator (increasing Lévy process) whose distribution $\mathbb{P}_{X_t}$ at time $t > 0$ is a $\Gamma(1,t)$-distribution

$$\mathbb{P}_{X_t}(dx) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} \mathbf{1}_{(0,\infty)}(x) dx.$$

The characteristics are given by

$$\nu(dx) = \frac{1}{x} e^{-x} \mathbf{1}_{(0,\infty)}(x) ds$$

and $a = \int_0^1 x d\nu(x) = 1 - e^{-1}$ so that $\beta = 0$ and $Y = X$. It follows from Theorem 1 that

$$\mathbb{E} \sup_{s\le t} X_s^p = \mathbb{E} X_t^p = O(t)$$

for every $p > 0$. This is clearly the true rate since

$$\mathbb{E} X_t^p = \frac{\Gamma(p+t)}{\Gamma(t+1)} t \sim \Gamma(p)\, t \quad \text{as} \quad t \to 0.$$

• The α*-stable Lévy Processes* indexed by $\alpha\in(0,2)$ have Lévy measure

$$\nu(dx) = \left(\frac{C_1}{x^{\alpha+1}} \mathbf{1}_{(0,\infty)}(x) + \frac{C_2}{|x|^{\alpha+1}} \mathbf{1}_{(-\infty,0)}(x) \right) dx$$

with $C_i \ge 0, C_1 + C_2 > 0$ so that $\mathbb{E}\,|X_1|^p < +\infty$ for $p\in(0,\alpha), \mathbb{E}\,|X_1|^\alpha = \infty$ and $\beta = \alpha$. It follows from Theorems 2 and 3 that for $p\in(0,\alpha)$,

$$\begin{aligned} \mathbb{E} \sup_{s\le t} |X_s|^p &= O(t^{p/\alpha}) \quad \text{if} \quad \alpha > 1, \\ \mathbb{E} \sup_{s\le t} |Y_s|^p &= O(t^{p/\alpha}) \quad \text{if} \quad \alpha < 1, \\ \mathbb{E} \sup_{s\le t} |X_s|^p &= O((t(-\log t))^p) \quad \text{if} \quad \alpha = 1. \end{aligned}$$

Here Theorem 3 gives the true rate provided X is not strictly stable. In fact, if $\alpha = 1$ the scaling property in this case says that $X_t \stackrel{d}{=} t\,X_1 + Ct\log t$ for some real constant $C \neq 0$ (see [7], p.87) so that for $p < 1$

$$\mathbb{E}\,|X_t|^p = t^p \mathbb{E}\,|X_1 + C\log t|^p \sim |C|^p t^p |\log t|^p \qquad \text{as} \qquad t \to 0.$$

Now assume that X is *strictly* α*-stable.* If $\alpha < 1$, then $a = \int_{|x|\le 1} x d\nu(x)$ and thus $Y = X$ and if $\alpha = 1$, then ν is symmetric (see [7]). Consequently, by Theorem 2, for every $\alpha \in (0,2)$, $p \in (0,\alpha)$,

$$\mathbb{E}\sup_{s\le t}|X_s|^p = O(t^{\,p/\alpha}).$$

In this case Theorem 2 provides the true rate since the self-similarity property of strictly stable Lévy processes implies

$$\mathbb{E}\sup_{s\le t}|X_s|^p = t^{\,p/\alpha}\mathbb{E}\sup_{s\le 1}|X_s|^p.$$

- *Tempered stable processes* are subordinators with Lévy measure

$$\nu(dx) = \frac{2^\alpha\cdot\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} \exp(-\frac{1}{2}\gamma^{1/\alpha}x)\mathbf{1}_{\,(0,\infty)}(x)dx$$

and first characteristic $a = \int_0^1 x d\nu(x), \alpha \in (0,1), \gamma > 0$ (see [8]) so that $\beta = \alpha$, $Y = X$ and $\mathbb{E}\,X_1^p < +\infty$ for every $p > 0$. The distribution of X_t is not generally known. It follows from Theorems 1,2 and 3 that

$$\begin{aligned} \mathbb{E}\,X_t^p &= O(t) \quad \text{if} \quad p > \alpha, \\ \mathbb{E}\,X_t^p &= O(t^{\,p/\alpha}) \quad \text{if} \quad p < \alpha \\ \mathbb{E}\,X_t^\alpha &= O(t(-\log t)) \quad \text{if} \quad p = \alpha. \end{aligned}$$

For $\alpha = 1/2$, the process reduces to the *inverse Gaussian process* whose distribution $\mathbb{P}_{X_t}$ at time $t > 0$ is given by

$$\mathbb{P}_{X_t}(dx) = \frac{t}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{1}{2}(\frac{t}{\sqrt{x}} - \gamma\sqrt{x})^2\right)\mathbf{1}_{(0,\infty)}(x)dx.$$

In this case all rates are the true rates. In fact, for $p > 0$,

$$\begin{aligned} \mathbb{E}X_t^p &= \frac{t}{\sqrt{2\pi}} e^{t\gamma} \int_0^\infty x^{p-3/2} \exp\left(-\frac{1}{2}(\frac{t^2}{x} + \gamma^2 x)\right) dx \\ &= \frac{t}{\sqrt{2\pi}} e^{t\gamma} \left(\frac{t}{\gamma}\right)^{p-1/2} \int_0^\infty y^{p-3/2} \exp\left(-\frac{t\gamma}{2}(\frac{1}{y} + y)\right) dy \\ &= \frac{2}{\sqrt{2\pi}} \left(\frac{1}{\gamma}\right)^{p-1/2} t^{p+1/2} e^{t\gamma} K_{p-1/2}(t\gamma) \end{aligned}$$

and, as $z \to 0$,

$$\begin{aligned} K_{p-1/2}(z) &\sim \frac{C_p}{z^{p-1/2}} \quad \text{if} \quad p > \frac{1}{2}, \\ K_{p-1/2}(z) &\sim \frac{C_p}{z^{1/2-p}} \quad \text{if} \quad p < \frac{1}{2}, \\ K_0(z) &\sim |\log z| \end{aligned}$$

where $C_p = 2^{p-3/2}\Gamma(p-1/2)$ if $p > 1/2$ and $C_p = 2^{-p-1/2}\Gamma(\frac{1}{2}-p)$ if $p < 1/2$.

• *The Normal Inverse Gaussian (NIG)* process was introduced by Barndorff-Nielsen and has been used in financial modeling (see [8]), in particular for energy derivatives (electricity). The NIG process is a Lévy process with characteristics $(a, 0, \nu)$ where

$$\begin{aligned} \nu(dx) &= \frac{\delta\alpha}{\pi} \frac{\exp(\gamma x) K_1(\alpha|x|)}{|x|} dx, \\ a &= \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\gamma x) K_1(\alpha x)\, dx, \end{aligned}$$

$\alpha > 0,\ \gamma \in (-\alpha, \alpha), \delta > 0$. Since $K_1(|z|) \sim |z|^{-1}$ as $z \to 0$, the Lévy density behaves like $\delta\pi^{-1}|x|^{-2}$ as $x \to 0$ so that (1.8) is satisfied with $\beta = 1$. One also checks that $\mathbb{E}\,|X_1|^p < +\infty$ for every $p > 0$. It follows from Theorems 1 and 3 that, as $t \to 0$

$$\begin{aligned} \mathbb{E} \sup_{s \le t} |X_s|^p &= O(t) \quad \text{if} \quad p > 1, \\ \mathbb{E} \sup_{s \le t} |X_s|^p &= O((t(-\log t))^p) \quad \text{if} \quad p \le 1. \end{aligned}$$

If $\gamma = 0$, then ν is symmetric and by Theorem 2,

$$\mathbb{E} \sup_{s \le t} |X_s|^p = O(t^p) \quad \text{if} \quad p < 1.$$

The distribution $\mathbb{P}_{X_t}$ at time $t > 0$ is given by

$$\mathbb{P}_{X_t}(dx) = \frac{t\delta\alpha}{\pi} \exp(t\,\delta\sqrt{\alpha^2 - \gamma^2} + \gamma x) \frac{K_1(\alpha\sqrt{t^2\delta^2 + x^2})}{\sqrt{t^2\delta^2 + x^2}} dx$$

so that Theorem 3 gives the true rate for $p = \beta = 1$ in the symmetric case. In fact, assuming $\gamma = 0$, we get as $t \to 0$

$$\begin{aligned} \mathbb{E}\,|X_t| &= \frac{2t\delta\alpha}{\pi} e^{t\delta\alpha} \int_0^\infty \frac{x K_1(\alpha\sqrt{t^2\delta^2 + x^2})}{\sqrt{t^2\delta^2 + x^2}} dx \\ &= \frac{2t\delta\alpha}{\pi} e^{t\delta\alpha} \int_{t\delta}^\infty K_1(\alpha y) dy \\ &\sim \frac{2\delta}{\pi}\, t \int_{t\delta}^1 \frac{1}{y} dy \\ &\sim \frac{2\delta}{\pi}\, t\,(-\log(t)). \end{aligned}$$

• *Hyperbolic Lévy motions* have been applied to option pricing in finance (see [3]). These processes are Lévy processes whose distribution $\mathbb{P}_{X_1}$ at time $t = 1$ is a symmetric (centered) hyperbolic distribution

$$\mathbb{P}_{X_1}(dx) = C\, \exp(-\delta\sqrt{1 + (x/\gamma)^2})\, dx, \quad \gamma,\, \delta > 0.$$

Hyperbolic Lévy processes have characteristics $(0, 0, \nu)$ and satisfy $\mathbb{E}\,|X_1|^p < +\infty$ for every $p > 0$. In particular, they are martingales. Their (rather involved) symmetric Lévy measure has a Lebesgue density that behaves like Cx^{-2} as $x \to 0$ so that (1.8) is satisfied with $\beta = 1$. Consequently, by Theorems 1, 2 and 3, as $t \to 0$

$$\begin{aligned}
\mathbb{E} \sup_{s \le t} |X_s|^p &= O(t) \quad \text{if} \quad p > 1, \\
\mathbb{E} \sup_{s \le t} |X_s|^p &= O(t^p) \quad \text{if} \quad p < 1, \\
\mathbb{E} \sup_{s \le t} |X_s| &= O(t\,(-\log t)) \quad \text{if} \quad p = 1.
\end{aligned}$$

• *Meixner processes* are Lévy processes without Brownian component and with Lévy measure given by

$$\nu(dx) = \frac{\delta\, e^{\gamma x}}{x \sinh(\pi x)} dx, \ \delta > 0, \ \gamma \in (-\pi, \pi)$$

(see [8]). The density behaves like $\delta / \pi x^2$ as $x \to 0$ so that (1.8) is satisfied with $\beta = 1$. Using (1.2) one observes that $\mathbb{E}\,|X_1|^p < +\infty$ for every $p > 0$. It follows from Theorems 1 and 3 that

$$\begin{aligned}
\mathbb{E} \sup_{s \le t} |X_s|^p &= O(t) \quad \text{if} \quad p > 1, \\
\mathbb{E} \sup_{s \le t} |X_s|^p &= O((t\,(-\log t))^p) \quad \text{if} \quad p \le 1.
\end{aligned}$$

If $\gamma = 0$, then ν is symmetric and hence Theorem 2 yields

$$\mathbb{E} \sup_{s \le t} |X_s|^p = O(t^p) \quad \text{if} \quad p < 1.$$

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