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AN EXTENSION OF THE INDUCTIVE APPROACH TO THE LACE EXPANSION

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Abstract
We extend the inductive approach to the lace expansion, previously developed to study models with critical dimension 4, to be applicable more generally. In particular, the result of this note has recently been used to prove Gaussian asymptotic behaviour for the Fourier transform of the two-point function for sufficiently spread-out lattice trees in dimensions $d > 8$, and it is potentially also applicable to percolation in dimensions $d > 6$.

1 Motivation

The lace expansion has been used since the mid-1980s to study a wide variety of problems in high-dimensional probability, statistical mechanics, and combinatorics [12]. One of the most flexible approaches to the lace expansion is the inductive method, first developed in [2] in the context of weakly self-avoiding walks in dimensions $d > 4$, and subsequently extended to a much more general setting in [6]. The inductive approach of [6] was successfully used to prove Gaussian asymptotic behavior for the Fourier transform of the critical two-point function $c_n(x; z_c)$ for a sufficiently spread-out model of self-avoiding walk in dimensions $d > 4$ [8]. Up to a constant, $c_n(x; z_c)$ is the probability that a randomly chosen n-step self-avoiding walk ends at x. Other models to which [6] applies include sufficiently spread-out models of oriented

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percolation in dimensions $d > 4$ [7], where the corresponding quantity is the critical two-point function $\tau_n(x; z_c) = \mathbb{P}((0,0) \to (x,n))$, and self-avoiding walks with nearest-neighbour attraction in dimensions $d > 4$ [13]. More generally, an inductive analysis of lace expansion recursions has been useful in studying the contact process [5] (extension to continuous time), self-interacting random walks (such as excited random walk) [3] and the ballistic behavior of 1-dimensional weakly self-avoiding walk [1].

As it is stated in [6], the general inductive method is limited to models with critical dimension 4. Thus it does not apply directly to percolation, which has critical dimension 6, or to lattice trees, which have critical dimension 8. In this paper, we show that the method and results of [6] are robust to appropriate changes in various parameters and exponents, so that one can indeed extend the results to more general critical dimensions.

Our extension has been applied already to prove Gaussian asymptotic behavior for the two-point function $t_n(x; z_c)$ for sufficiently spread-out lattice trees in dimensions $d > d_c = 8$ in [9, 10]. Up to a constant, $t_n(x; z_c)$ is the probability (under a particular critical weighting scheme) that a randomly chosen finite lattice tree contains the point x, with the unique path in the tree from 0 to x consisting of exactly n bonds. The asymptotic behavior of the Fourier transform of the two-point function provides a first but significant step towards proving convergence of the finite-dimensional distributions of the associated sequence of measure-valued processes to those of the canonical measure of super-Brownian motion [10, 11].

A possible future application of our results is to study the critical two-point function $\tau_n(x; z_c)$ for sufficiently spread-out percolation in dimensions $d > d_c = 6$. Here, $\tau_n(x; z_c)$ is the probability that x is in the open cluster of the origin, with the open path of minimum length connecting the origin and x consisting of exactly n bonds, or, alternatively, with the open path connecting the origin and x containing exactly n bonds that are *pivotal* for the connection.

2 The recursion relation

The lace expansion typically gives rise to a recursion relation for a sequence f_n depending on parameters $k \in [-\pi, \pi]^d$ and positive z. We may assume that $f_0 = 1$. The recursion relation takes the form

$$f_{n+1}(k; z) = \sum_{m=1}^{n+1} g_m(k; z) f_{n+1-m}(k; z) + e_{n+1}(k; z), \quad (n \geq 0), \tag{1}$$

with given sequences $g_m(k; z)$ and $e_{n+1}(k; z)$. The goal is to understand the behaviour of the solution $f_n(k; z)$ of (1).

A rough idea of the behaviour we seek to prove can be obtained from the following (nonrigorous) argument. Suppose for simplicity that $D(x)$ is uniformly distributed on a finite box centred at the origin (so that $\sum_x D(x) = 1$), that $g_1(k; 1) = \widehat{D}(k) \approx 1 - |k|^2\sigma^2/(2d)$, and that $e_m, g_{m+1} \approx 0$ for $m \geq 1$. Then we have $f_{n+1} \approx g_1 f_n$, so $f_n(k) \approx g_1(k)^n \approx \left(1 - \frac{|k|^2\sigma^2}{2d}\right)^n$, and thus

$$f_n\left(\frac{k}{\sqrt{\sigma^2 n}}; 1\right) \approx \left(1 - \frac{|k|^2}{2dn}\right)^n \to e^{-\frac{|k|^2}{2d}}, \quad \text{as } n \to \infty.$$

The above argument is, however, overly simplistic, and misses important effects on the asymptotic behaviour of the solution to (1) due to the presence of $e_m(k; z)$ and $g_m(k; z)$. The inductive method of [6] details specific bounds on g_m and e_{n+1} that ensure that there exists a critical value z_c and positive constants A, v such that the true asymptotic behaviour is

$f_n\left(\frac{k}{\sqrt{v\sigma^2 n}}; z_c\right) \to Ae^{-\frac{|k|^2}{2d}}$. Verification of these bounds has been carried out for sufficiently spread-out models of self-avoiding walk [8], oriented percolation [7] and the contact process [5], by estimating certain Feynman diagrams in dimensions $d > 4$. The required bounds are typically of the form $|h_m(k,z)| \le Cm^{b-\frac{d}{2}}$, for some functions h_m and exponent $b \ge 0$ that varies from bound to bound. What turns out to be important in the analysis is that $\frac{d}{2} = 2 + \frac{d-4}{2}$ is greater than 2 when $d > 4$.

In our analysis we introduce two new parameters $\theta(d)$, p^* and a set $B \subset [1, p^*]$. We will discuss the significance of p^* and B following Assumption D in the next section. The most important parameter, $\theta(d)$, takes the place of $\frac{d}{2}$ in exponents appearing in various bounds. As in [6] we require that $\theta > 2$. In [10], the result of this note is applied to lattice trees with the choice $\theta = 2 + \frac{d-8}{2}$, with $d > 8$. In general, when the critical dimension is d_c, we expect that the correct parameter value is $\theta = 2 + \frac{d-d_c}{2}$, e.g., we expect that $\theta = 2 + \frac{d-6}{2}$ is the appropriate choice for percolation. A detailed proof of the results in this note is available in [4], however, most of the changes to the proof in [6] simply involve replacing $\frac{d}{2}$ in [6] with θ in [4]. In this note we state the new assumptions and results explicitly, but for the sake of brevity, we present only significant changes in the proof and refer the reader to [6] when the changes are merely cosmetic.

The remainder of this note is organised as follows. In Section 3 we state the Assumptions S, D, E_θ, and G_θ on the quantities appearing in the recursion relation, and the main theorem to be proved. In Section 4, we introduce the induction hypotheses on f_n that will be used to prove the main theorem. We then discuss the necessary changes to the advancement of the induction hypotheses of [6]. Once the induction hypotheses have been advanced, the main theorem follows without difficulty.

3 Assumptions and main result

Suppose that for $z > 0$ and $k \in [-\pi, \pi]^d$, we have $f_0(k; z) = 1$ and that (1) holds for all $n \ge 0$, where the functions g_m and e_m are to be regarded as given. Fix $\theta > 2$.

The first assumption, Assumption S, remains unchanged from [6]. It requires that the functions appearing in the recursion relation (1) respect the lattice symmetries of reflection and rotation, and that f_n remains bounded in a weak sense.

Assumption S. For every $n \in \mathbb{N}$ and $z > 0$, the mapping $k \mapsto f_n(k; z)$ is symmetric under replacement of any component k_i of k by $-k_i$, and under permutations of the components of k. The same holds for $e_n(\cdot; z)$ and $g_n(\cdot; z)$. In addition, for each n, $|f_n(k; z)|$ is bounded uniformly in $k \in [-\pi, \pi]^d$ and z in a neighbourhood of 1 (both the bound and the neighbourhood may depend on n).

The next assumption, Assumption D, is only cosmetically changed from [6]. It introduces a probability mass function $D = D_L$ on $\mathbb{Z}^d$ which defines an underlying random walk model and involves a non-negative parameter L which will typically be large. This serves to spread out the steps of the random walk over a large set. An example of a family of D's obeying the assumption is taking D uniform on a box of side $2L + 1$ centred at the origin. In particular, Assumption D implies that D has a finite second moment, and we define

$$\sigma^2 \equiv -\nabla^2 \hat{D}(0) = \sum_x |x|^2 D(x), \tag{2}$$

where $\hat{D}(k) = \sum_{x\in\mathbb{Z}^d} D(x)e^{ik\cdot x}$ is the Fourier transform of D, and $\nabla^2 = \sum_{j=1}^d \frac{\partial^2}{\partial k_j^2}$ with $k = (k_1, \ldots, k_d)$.

Assumption D. We assume that

$$f_1(k;z) = z\hat{D}(k) \quad \text{and} \quad e_1(k;z) = 0.$$

In particular, this implies that $g_1(k;z) = z\hat{D}(k)$. In addition, we also assume:
(i) D is normalised so that $\hat{D}(0) = 1$, and has $2+2\epsilon$ moments for some $0 < \epsilon < \theta - 2$, i.e.,

$$\sum_{x\in\mathbb{Z}^d} |x|^{2+2\epsilon} D(x) < \infty. \tag{3}$$

(ii) There is a constant C such that, for all $L \geq 1$,

$$\sup_{x\in\mathbb{Z}^d} D(x) \leq CL^{-d} \quad \text{and} \quad \sigma^2 \leq CL^2. \tag{4}$$

(iii) Let $a(k) = 1 - \hat{D}(k)$. There exist constants $\eta, c_1, c_2 > 0$ such that

$$c_1 L^2|k|^2 \leq a(k) \leq c_2 L^2 |k|^2 \quad (\|k\|_\infty \leq L^{-1}), \tag{5}$$

$$a(k) > \eta \quad (\|k\|_\infty \geq L^{-1}), \tag{6}$$

$$a(k) < 2 - \eta \quad (k \in [-\pi,\pi]^d). \tag{7}$$

Assumptions E and G of [6] are adapted to general $\theta > 2$ as follows. The relevant bounds on f_m, which *a priori* may or may not be satisfied, are that for some $p^* \geq 1$ and some nonempty $B \subset [1, p^*]$, we have for every $p \in B$,

$$\|\hat{D}^2 f_m(\cdot;z)\|_p \leq \frac{K}{L^{\frac{d}{p}} m^{\frac{d}{2p}\wedge\theta}}, \quad |f_m(0;z)| \leq K, \quad |\nabla^2 f_m(0;z)| \leq K\sigma^2 m, \tag{8}$$

for some positive constant K, where the norm is defined by $\|f\|_p^p = (2\pi)^{-d}\int_{[-\pi,\pi]^d} |f(k)|^p d^dk$. The bounds in (8) are identical to the ones in [6, (1.27)], except the first bound, which only appears in [6] with $p = 1$ and $\theta = \frac{d}{2}$. It may be that $B = \{p^*\}$ (i.e. B is a singleton), and then $p = p^*$. This is the case in [10], where the choices $p^* = 2$ and $B = \{2\}$ are sufficient, as only the $p = 2$ case in (8) is required to estimate the diagrams arising from the lace expansion and verify the assumptions E_θ, G_θ which follow below. The set B allows for the possibility that in other applications a larger collection of $\|\cdot\|_p$ norms may be required to verify the assumptions. Let

$$\beta = \beta(p^*) = L^{-\frac{d}{p^*}}.$$

Since $p^* < \infty$, $\beta(p^*)$ is small for large L.

Assumption $\mathbf{E}_\theta$. There is an L_0, an interval $I \subset [1-\alpha, 1+\alpha]$ with $\alpha \in (0,1)$, and a function $K \mapsto C_e(K)$, such that if (8) holds for some $K > 1$, $L \geq L_0$, $z \in I$ and for all $1 \leq m \leq n$, then for that L and z, and for all $k \in [-\pi,\pi]^d$ and $2 \leq m \leq n+1$, the following bounds hold:

$$|e_m(k;z)| \leq C_e(K)\beta m^{-\theta}, \quad |e_m(k;z) - e_m(0;z)| \leq C_e(K)a(k)\beta m^{-\theta+1}.$$

Assumption $\mathbf{G}_\theta$. There is an L_0, an interval $I \subset [1-\alpha, 1+\alpha]$ with $\alpha \in (0,1)$, and a function $K \mapsto C_g(K)$, such that if (8) holds for some $K > 1$, $L \geq L_0$, $z \in I$ and for all $1 \leq m \leq n$, then for that L and z, and for all $k \in [-\pi,\pi]^d$ and $2 \leq m \leq n+1$, the following bounds hold:

$$|g_m(k;z)| \leq C_g(K)\beta m^{-\theta}, \quad |\nabla^2 g_m(0;z)| \leq C_g(K)\sigma^2\beta m^{-\theta+1},$$

$$|\partial_z g_m(0;z)| \le C_g(K)\beta m^{-\theta+1},$$
$$|g_m(k;z) - g_m(0;z) - a(k)\sigma^{-2}\nabla^2 g_m(0;z)| \le C_g(K)\beta a(k)^{1+\epsilon'} m^{-\theta+1+\epsilon'},$$

with the last bound valid for any $\epsilon' \in [0,\epsilon]$, with $0 < \epsilon < \theta - 2$ given by (3).

Our main result is the following theorem. (There is a misprint in [6, Theorem 1.1(a)] whose restrictions should require $\gamma, \delta < \frac{d-4}{2}$ rather than $\gamma, \delta < \frac{d-4}{4}$; our assumption $\epsilon < \theta - 2$ makes the restriction redundant here.)

Theorem 3.1. *Let $d > d_c$ and $\theta(d) > 2$, and assume that Assumptions S, D, E_θ and G_θ all hold. There exist positive $L_0 = L_0(d,\epsilon)$, $z_c = z_c(d,L)$, $A = A(d,L)$, and $v = v(d,L)$, such that for $L \ge L_0$, the following statements hold.*
(a) Fix $\gamma \in (0, 1\wedge\epsilon)$ and $\delta \in (0, (1\wedge\epsilon) - \gamma)$. Then

$$f_n\Big(\frac{k}{\sqrt{v\sigma^2 n}}; z_c\Big) = Ae^{-\frac{|k|^2}{2d}}[1 + \mathcal{O}(|k|^2 n^{-\delta}) + \mathcal{O}(n^{-\theta+2})],$$

with the error estimate uniform in $\{k \in \mathbb{R}^d : a(k/\sqrt{v\sigma^2 n}) \le \gamma n^{-1}\log n\}$.
(b)

$$-\frac{\nabla^2 f_n(0;z_c)}{f_n(0;z_c)} = v\sigma^2 n[1 + \mathcal{O}(\beta n^{-\delta})].$$

(c) For all $p \ge 1$,

$$\|\hat{D}^2 f_n(\cdot;z_c)\|_p \le \frac{C}{L^{\frac{d}{p}} n^{\frac{d}{2p}\wedge\theta}}.$$

(d) The constants z_c, A and v obey

$$1 = \sum_{m=1}^{\infty} g_m(0;z_c), \qquad A = \frac{1 + \sum_{m=1}^\infty e_m(0;z_c)}{\sum_{m=1}^\infty m g_m(0;z_c)}, \qquad v = -\frac{\sum_{m=1}^\infty \nabla^2 g_m(0;z_c)}{\sigma^2 \sum_{m=1}^\infty m g_m(0;z_c)}.$$

As in the proof of [6, Theorem 1.1], the proof of Theorem 3.1 establishes the bounds (8) for all non-negative integers m, with z in an m-dependent interval containing z_c. Consequently, all bounds appearing in Assumptions E_θ and G_θ follow as a corollary, for $z = z_c$ and all m. Also, it follows immediately from Theorem 3.1(d) and the bounds of Assumptions E_θ and G_θ that

$$z_c = 1 + \mathcal{O}(\beta), \quad A = 1 + \mathcal{O}(\beta), \quad v = 1 + \mathcal{O}(\beta).$$

Finally, we remark that it is straightforward to extend [6, Theorem 1.2] for the susceptibility to our present setting, with the assumption $\theta > 2$ replacing $d > 4$. On the other hand, the proof of the local central limit theorem [6, Theorem 1.3] does require $\theta = \frac{d}{2}$, and does not extend to the more general setting considered in this paper.

4 Induction hypotheses and their consequences

4.1 Induction hypotheses

Theorem 3.1 is proved via induction on n, as in [6]. The induction hypotheses involve a sequence v_n, which is defined exactly as in [6] as follows. We set $v_0 = b_0 = 1$, and for $n \ge 1$ we define

$$b_n = -\frac{1}{\sigma^2}\sum_{m=1}^{n} \nabla^2 g_m(0;z), \quad c_n = \sum_{m=1}^{n}(m-1)g_m(0;z), \quad v_n = \frac{b_n}{1+c_n}.$$

The induction hypotheses also involve several constants. Let $\theta > 2$, and recall from (3) that $\epsilon < \theta - 2$. We fix $\gamma, \delta > 0$ and $\lambda > 2$ according to

$$0 < \gamma < 1 \wedge \epsilon, \qquad 0 < \delta < (1 \wedge \epsilon) - \gamma, \qquad \theta - \gamma < \lambda < \theta. \tag{9}$$

Here λ replaces $\rho + 2$ from [6], which is merely a change of notation.
We also introduce constants $K_1, \ldots, K_5$, which are independent of β. We define

$$K_4' = \max\{C_e(cK_4), C_g(cK_4), K_4\}, \tag{10}$$

where c is a constant determined in the proof of Lemma 4.6 below. To advance the induction, we need to assume that

$$K_3 \gg K_1 > K_4' \geq K_4 \gg 1, \quad K_2 \geq K_1, 3K_4', \quad K_5 \gg K_4. \tag{11}$$

Here $a \gg b$ denotes the statement that a/b is sufficiently large. The amount by which, for instance, K_3 must exceed K_1 is independent of β, but may depend on p^*, and is determined during the course of the advancement of the induction.
Let $z_0 = z_1 = 1$, and define z_n recursively by

$$z_{n+1} = 1 - \sum_{m=2}^{n+1} g_m(0; z_n), \qquad n \geq 1.$$

For $n \geq 1$, we define intervals

$$I_n = [z_n - K_1\beta n^{-\theta+1}, z_n + K_1\beta n^{-\theta+1}]. \tag{12}$$

In particular this gives $I_1 = [1 - K_1\beta, 1 + K_1\beta]$.
Recall the definition $a(k) = 1 - \hat{D}(k)$. Our induction hypotheses are that the following four statements hold for all $z \in I_n$ and all $1 \leq j \leq n$.

(H1) $|z_j - z_{j-1}| \leq K_1\beta j^{-\theta}$.

(H2) $|v_j - v_{j-1}| \leq K_2\beta j^{-\theta+1}$.

(H3) For k such that $a(k) \leq \gamma j^{-1}\log j$, $f_j(k; z)$ can be written in the form

$$f_j(k; z) = \prod_{i=1}^{j} \left[1 - v_i a(k) + r_i(k)\right],$$

with $r_i(k) = r_i(k; z)$ obeying

$$|r_i(0)| \leq K_3\beta i^{-\theta+1}, \quad |r_i(k) - r_i(0)| \leq K_3\beta a(k) i^{-\delta}.$$

(H4) For k such that $a(k) > \gamma j^{-1}\log j$, $f_j(k; z)$ obeys the bounds

$$|f_j(k; z)| \leq K_4 a(k)^{-\lambda} j^{-\theta}, \quad |f_j(k; z) - f_{j-1}(k; z)| \leq K_5 a(k)^{-\lambda+1} j^{-\theta}.$$

Note that these four statements are those of [6] with the replacement

$$\rho + 2 \mapsto \lambda \tag{13}$$

in (H4) and the global replacement

$$\frac{d}{2} \mapsto \theta. \tag{14}$$

By global replacement we also mean that $\frac{d-2}{2} \mapsto \theta - 1$, $\frac{d-4}{2} \mapsto \theta - 2$, etc. whenever such quantities appear in exponents.

4.2 Initialisation of the induction

The verification that the induction hypotheses hold for $n = 0$ remains unchanged from the $p = 1$ case, up to the replacements (13-14).

4.3 Consequences of induction hypotheses

The key result of this section is that the induction hypotheses imply (8) for all $1 \leq m \leq n$, from which the bounds of Assumptions E_θ and G_θ then follow, for $2 \leq m \leq n+1$.
Throughout this note:

- C denotes a strictly positive constant that may depend on $d, \gamma, \delta, \lambda$, but *not* on the K_i, k, n, and not on β (which must, however, be chosen sufficiently small, possibly depending on the K_i). The value of C may change from one occurrence to the next.
- We frequently assume $\beta \ll 1$ (i.e., $L \gg 1$) without explicit comment.

Lemmas 4.1 and 4.3 are proved in [6] and the proof in our context requires only the global change (14).

Lemma 4.1. *Assume (H1) for $1 \leq j \leq n$. Then $I_1 \supset I_2 \supset \cdots \supset I_n$.*

Remark 4.2. *The bound [6, (2.19)] is missing a constant. Instead of [6, (2.19)] we use*

$$|s_i(k)| \leq K_3(2 + C(K_2 + K_3)\beta)\beta a(k) i^{-\delta}, \tag{15}$$

the only difference being that the constant 2 appears here instead of a constant 1 in [6, (2.19)]. This does not affect the proof in [6]. To verify (15), we use the fact that $\frac{1}{1-x} \leq 1 + 2x$ for $0 \leq x \leq \frac{1}{2}$ and note that for small enough β it follows from [6, (2.20)] that

$$\begin{aligned} |s_i(k)| &\leq [1 + 2K_3\beta]\,[(1 + |v_i - 1|)a(k)r_i(0) + |r_i(k) - r_i(0)|] \\ &\leq [1 + 2K_3\beta]\left[(1 + CK_2\beta)a(k)\frac{K_3\beta}{i^{\theta-1}} + \frac{K_3\beta a(k)}{i^\delta}\right] \\ &\leq \frac{K_3\beta a(k)}{i^\delta}[1 + 2K_3\beta][2 + CK_2\beta] \leq \frac{K_3\beta a(k)}{i^\delta}[2 + C(K_2 + K_3)\beta]. \end{aligned}$$

Here we have used the bounds of (H2-H3) as well as the fact that $\theta - 1 > \delta$.

Lemma 4.3. *Let $z \in I_n$ and assume (H2-H3) for $1 \leq j \leq n$. Then for k with $a(k) \leq \gamma j^{-1} \log j$,*

$$|f_j(k; z)| \leq e^{CK_3\beta} e^{-(1 - C(K_2+K_3)\beta) j a(k)}.$$

The middle bound of (8) follows, for $1 \leq m \leq n$ and $z \in I_m$, directly from Lemma 4.3. We next state two lemmas which provide the other two bounds of (8). The first concerns the $\|\cdot\|_p$ norms and contains the most significant changes to [6]. As such we present the full proof of this lemma.

Lemma 4.4. *Let $z \in I_n$ and assume (H2), (H3) and (H4). Then for all $1 \leq j \leq n$, and $p \geq 1$,*

$$\|\hat{D}^2 f_j(\cdot; z)\|_p \leq \frac{C(1 + K_4)}{L^{\frac{d}{p}} j^{\frac{d}{2p} \wedge \theta}},$$

where the constant C may depend on p, d.

Proof. We show that

$$\|\hat{D}^2 f_j(\cdot;z)\|_p^p \le \frac{C(1+K_4)^p}{L^d j^{\frac{d}{2}\wedge\theta p}}.$$

For $j=1$ the result holds since $|f_1(k)| = |z\widehat{D}(k)| \le z \le 2$, and, since $p \ge 1$, it therefore follows from (4) and the Parseval relation that $\|\hat{D}^2 f_1(\cdot;z)\|_p^p \le 2^p \|\hat{D}^{2p}\|_1 \le 2^p\|\hat{D}^2\|_1 = 2^p\|D\|_2^2 \le 2^p C L^{-d}$. We may therefore assume that $j \ge 2$ where needed in what follows, so that in particular $\log j \ge \log 2$.

Fix $z \in I_n$ and $1 \le j \le n$, and define

$$\begin{aligned}
R_1 &= \{k \in [-\pi,\pi]^d : a(k) \le \gamma j^{-1}\log j,\ \|k\|_\infty \le L^{-1}\},\\
R_2 &= \{k \in [-\pi,\pi]^d : a(k) \le \gamma j^{-1}\log j,\ \|k\|_\infty > L^{-1}\},\\
R_3 &= \{k \in [-\pi,\pi]^d : a(k) > \gamma j^{-1}\log j,\ \|k\|_\infty \le L^{-1}\},\\
R_4 &= \{k \in [-\pi,\pi]^d : a(k) > \gamma j^{-1}\log j,\ \|k\|_\infty > L^{-1}\}.
\end{aligned}$$

The set R_2 is empty if j is sufficiently large. Then

$$\|\hat{D}^2 f_j\|_p^p = \sum_{i=1}^4 \int_{R_i} \Big(\hat{D}(k)^2|f_j(k)|\Big)^p \frac{d^dk}{(2\pi)^d}.$$

We will treat each of the four terms on the right side separately.

On R_1, we use (5) in conjunction with Lemma 4.3 and the fact that $\hat{D}(k)^2 \le 1$, to obtain for all $p > 0$,

$$\begin{aligned}
\int_{R_1} \Big(\hat{D}(k)^2|f_j(k)|\Big)^p \frac{d^dk}{(2\pi)^d} &\le \int_{R_1} Ce^{-cpj(L|k|)^2}\frac{d^dk}{(2\pi)^d}\\
&\le \int_{\mathbb{R}^d} Ce^{-cpj(L|k|)^2}dk \le \frac{C}{L^d(pj)^{d/2}} \le \frac{C}{L^d j^{d/2}}.
\end{aligned}$$

Here we have used the substitution $k_i' = Lk_i\sqrt{pj}$. On R_2, we use Lemma 4.3 and (6) to conclude that for all $p > 0$, there is an $\alpha(p) > 1$ such that

$$\int_{R_2} \Big(\hat{D}(k)^2|f_j(k)|\Big)^p \frac{d^dk}{(2\pi)^d} \le C\int_{R_2} \alpha^{-j}\frac{d^dk}{(2\pi)^d} = C\alpha^{-j}|R_2|,$$

where $|R_2|$ denotes the volume of R_2. For $j \ge 2$, $j^{-1}\log j$ takes its largest value when $j=3$, so $|R_2|$ is maximal when $j=3$ and

$$|R_2| \le \left|\{k : a(k) \le \tfrac{\gamma\log 3}{3}\}\right| \le \left|\{k : \hat{D}(k) \ge 1 - \tfrac{\gamma\log 3}{3}\}\right| \le \left(\frac{1}{1-\frac{\gamma\log 3}{3}}\right)^2 \|\hat{D}^2\|_1 \le \left(\frac{1}{1-\frac{\gamma\log 3}{3}}\right)^2 CL^{-d},$$

using (4) in the last step. Therefore $\alpha^{-j}|R_2| \le CL^{-d}j^{-d/2}$ since $\alpha^{-j}j^{\frac{d}{2}} \le C(\alpha,d)$ for every j, and

$$\int_{R_2} \Big(\hat{D}(k)^2|f_j(k)|\Big)^p \frac{d^dk}{(2\pi)^d} \le CL^{-d}j^{-d/2}.$$

On R_3 and R_4, we use (H4). As a result, the contribution from these two regions is bounded above by

$$\left(\frac{K_4}{j^\theta}\right)^p \sum_{i=3}^4 \int_{R_i} \frac{\hat{D}(k)^{2p}}{a(k)^{\lambda p}}\frac{d^dk}{(2\pi)^d}.$$

We first consider R_3, where we apply $\hat{D}(k)^2 \leq 1$. Recall that we can restrict our attention to $j \geq 2$. From (5), $k \in R_3$ implies that $L^2|k|^2 > Cj^{-1}\log j$, and we have the upper bound

$$\frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\int_{R_3}\frac{1}{|k|^{2\lambda p}}d^dk \leq \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\int_{\sqrt{\frac{C\log j}{L^2 j}}}^{\frac{C}{L}} r^{d-1-2\lambda p}dr. \tag{16}$$

For $d > 2\lambda p$, we have an upper bound on (16) of

$$\frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\int_0^{\frac{C}{L}} r^{d-1-2\lambda p}dr \leq \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\left(\frac{C}{L}\right)^{d-2\lambda p} \leq \frac{CK_4^p}{j^{\theta p}L^d}. \tag{17}$$

For $d = 2\lambda p$, (16) is

$$\frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\int_{\sqrt{\frac{C\log j}{L^2 j}}}^{\frac{C}{L}} \frac{1}{r}dr \leq \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\log\left(\frac{C\sqrt{L^2 j}}{L\sqrt{\log j}}\right) = \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\log\left(\frac{Cj}{\log j}\right), \tag{18}$$

and $\theta p = \frac{\theta d}{2\lambda} > \frac{d}{2}$ since $\lambda < \theta$. This gives an upper bound in this case of $CK_4^p j^{-\frac{d}{2}}L^{-d}$. Lastly, for $d < 2\lambda p$, since $\lambda < \theta$, (16) is bounded, as required, by

$$\frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\int_{\sqrt{\frac{C\log j}{CL^2 j}}}^{\infty} r^{d-1-2\lambda p}dr \leq \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}}\left(\frac{CL^2 j}{\log j}\right)^{\frac{2\lambda p-d}{2}} \leq \frac{CK_4^p}{j^{\frac{d}{2}}L^d}. \tag{19}$$

On R_4, we use (4), $p \geq 1$, $\hat{D}(k)^2 \leq 1$, and (6) to obtain the bound

$$\frac{CK_4^p}{j^{\theta p}}\int_{[-\pi,\pi]^d}\hat{D}(k)^{2p}\frac{d^dk}{(2\pi)^d} \leq \frac{CK_4^p}{j^{\theta p}}\int_{[-\pi,\pi]^d}\hat{D}(k)^2\frac{d^dk}{(2\pi)^d} \leq \frac{CK_4^p}{j^{\theta p}L^d}.$$

This completes the proof. □

Lemma 4.5. *Let $z \in I_n$ and assume (H2) and (H3). Then, for $1 \leq j \leq n$,*

$$|\nabla^2 f_j(0;z)| \leq (1 + C(K_2+K_3)\beta)\sigma^2 j.$$

The proof is identical to [6]. We merely point out one inconsequential correction to the first line of [6, (2.35)]: a constant 2 is missing and it should read

$$\nabla^2 s_i(0) = 2\sum_{l=1}^{d}\lim_{t\to 0}\frac{s_i(te_l)-s_i(0)}{t^2}. \tag{20}$$

The next lemma, whose proof proceeds exactly as in [6] with $\frac{d}{2}$ replaced by θ, is the key to advancing the induction, as it provides bounds for e_{n+1} and g_{n+1}. Recall that K_4' was defined in (10).

Lemma 4.6. *Let $z \in I_n$, and assume (H2), (H3) and (H4). For $k \in [-\pi,\pi]^d$, $2 \leq j \leq n+1$, and $\epsilon' \in [0,\epsilon]$, the following hold:*
(i) $|g_j(k;z)| \leq K_4'\beta j^{-\theta}$,
(ii) $|\nabla^2 g_j(0;z)| \leq K_4'\sigma^2\beta j^{-\theta+1}$,
(iii) $|\partial_z g_j(0;z)| \leq K_4'\beta j^{-\theta+1}$,
(iv) $|g_j(k;z) - g_j(0;z) - a(k)\sigma^{-2}\nabla^2 g_j(0;z)| \leq K_4'\beta a(k)^{1+\epsilon'}j^{-\theta+1+\epsilon'}$,
(v) $|e_j(k;z)| \leq K_4'\beta j^{-\theta}$,
(vi) $|e_j(k;z) - e_j(0;z)| \leq K_4' a(k)\beta j^{-\theta+1}$.

5 The induction advanced

The advancement of the induction is carried out as in [6] with a few minor changes corresponding to the global replacement (14), and also (13) for (H4). Full details can be found in [4], and here we only point out the main places where changes are required.
In adapting [6, (3.2)], we use the fact that $\sum_{m=2}^{\infty} m^{-\theta+1} < \infty$, since $\theta > 2$, and in adapting [6, (3.26)], we use $\sum_{j=n+2-m}^{n} j^{-\theta+1} \leq C(n+2-m)^{-\theta+2}$. For [6, (3.40)], we apply $\epsilon' \leq \epsilon < \theta - 2$ to conclude that $\sum_{m=2}^{\infty} m^{-\theta+1+\epsilon'} < \infty$. To adapt [6, (3.43)], we use the fact that $\delta+\gamma < 1 \wedge (\theta-2)$, by (9), to conclude that there exists a $q > 1$ sufficiently close to 1 so that

$$(n+1)^{-\delta} \geq (n+1)^{\gamma q - 1} \log(n+1) \times \begin{cases} (n+1)^{0 \vee (3-\theta)}, & (\theta \neq 3) \\ \log(n+1), & (\theta = 3). \end{cases}$$

Other similar bounds required to verify (H3) (corresponding to [6, (3.50)–(3.51)] and [6, (3.58)] for example) also follow from $\delta + \gamma < 1 \wedge (\theta - 2)$. For (H4), using the fact that $\gamma + \lambda - \theta > 0$, there exists q' close to 1 so that for $a(k) \leq \gamma n^{-1} \log n$,

$$\frac{C}{n^{\theta}} \frac{n^{\lambda}}{n^{q'\gamma+\lambda-\theta}} \leq \frac{C}{n^{\theta} a(k)^{\lambda}}.$$

This corresponds to [6, (3.62)], and is used to advance the first and second bounds of (H4).

Once the induction has been advanced, the proof of Theorem 3.1 is then completed exactly as in [6], with the global replacement (14). Full details can be found in [4].

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