Elect. Comm. in Probab. **13** (2008), 606–613

ELECTRONIC COMMUNICATIONS in PROBABILITY

A MARTINGALE ON THE ZERO-SET OF A HOLOMORPHIC FUNCTION

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Submitted August 28, 2008, *accepted in final form* November 4, 2008

AMS 2000 Subject classification: Primary 60H30; secondary 60G46,60H10;
Keywords: complex martingales, stochastic differential equations, holomorphic functions

Abstract

We give a simple probabilistic proof of the classical fact from complex analysis that the zeros of a holomorphic function of several variables are never isolated and that they are not contained in any compact set. No facts from complex analysis are assumed other than the Cauchy-Riemann definition. From stochastic analysis only the Itô formula and the standard existence theorem for stochastic differential equations are required.

1 Introduction

It is well known that numerous facts from complex analysis can be given probabilistic proofs or interpretation, many of which stem from Paul Lévy's striking observation that the image of a complex Brownian motion is simply a time changed complex Brownian motion. Itô calculus in particular seems to be well-suited for giving interesting insights into classical facts as well as being a tool for deriving new or refined results. We refer the reader to [Bas95] and the bibliography therein for some examples. In this article we give a probabilistic approach to analyzing the zero-set of a holomorphic function on $\mathbb{C}^n$ (where $n \geq 2$).

Here we use a stochastic differential equation (SDE) which has a strong solution that remains on the zero-set of a given holomorphic function of several variables. We then prove that the solution reaches any given radius with a strictly positive probability by computing the radial process and the gradient norm process. The novelty in this application of stochastic calculus seems to be the use of the gradient norm process and the fact that this process happens to be a submartingale along the solution to the equation (1).

In section 2 we introduce the notation and basic facts to be used. The formalism developed here is tailored to the specific situation, but the notions should be recognizable to anyone familiar with conformal martingales, see for instance [RY91]. In section 3 we state and prove the theorem. The main claim is a well known fact from complex analysis of several variables (see for instance [Kra92] or [Ran86]) and can be proved in several ways, but the probabilistic argument used here has apparently not been observed. The proof here also shows how to construct semimartingales with desirable properties on analytic sets or complex manifolds without the use of local coordinates which are common in most literature on stochastic analysis on manifolds. Conceivably these kinds of SDE's could be useful or interesting in other contexts. In the last section we conclude with a numerical illustration and remarks.

2 Preliminaries and notation

Let $\mathscr{A}$ and $\mathscr{M}_c^{loc}$ denote the space of continuous real valued processes with bounded total variation and the space of continuous real valued local martingales respectively. We use a complexification of the space of $\mathbb{R}^2$ valued semimartingales.

Definition 1. *A* complex semimartingale *S is a $\mathbb{C}$-valued process of the form*

$$S = S_0 + X + iY = S_0 + A + M + i(B + N)$$

where $A, B \in \mathscr{A}$ and $M, N \in \mathscr{M}_c^{loc}$ are orthogonal local martingales with $\langle M, N\rangle = 0$ and $\langle M\rangle = \langle N\rangle$. Let the space of complex semimartingales be denoted by $\mathscr{Q}$.

Definition 2. *The* complex covariation *of a pair $S^1, S^2 \in \mathscr{Q}$ is defined to be the $\mathbb{C}$-bilinear form*

$$\langle S^1, S^2\rangle := \langle X^1 + iY^1, X^2 + iY^2\rangle = \langle X^1, X^2\rangle - \langle Y^1, Y^2\rangle + i(\langle X^1, Y^2\rangle + \langle X^2, Y^1\rangle)$$

A pair $S^1, S^2 \in \mathscr{Q}$ of complex semimartingales are orthogonal *iff their complex covariation is $\langle S^1, S^2\rangle = 0$.*

For the process $Z = B^1 + iB^2$ where B^1 and B^2 are orthogonal Brownian motions we therefore have $\langle Z, \overline{Z}\rangle = 2t$.

Definition 3. *The Itô integral of a complex valued process along a complex $(\mathscr{F}_t)$-semimartingale is*

$$\int_0^t (u_s + iv_s)d(X_s + iY_s) := \int_0^t (u_s dX_s - v_s dY_s) + i\int_0^t (v_s dX_s + u_s dY_s).$$

where u and v are $(\mathscr{F}_t)$-adapted.

We note that complex orthogonality of two complex martingales does not require the orthogonality of the corresponding real valued coordinate martingales, indeed, $\langle S, S\rangle = \langle \overline{S}, \overline{S}\rangle = 0$ for any complex semimartingale.
The bilinearity of the complex quadratic variation extends to complex integrals in the same way as it does in the real case. In particular, we make use of the following statements which are easily verified using the definitions above and basic properties for real semimartingales.

Proposition 1. *Let f be an integrable $(\mathscr{F}_t)$-adapted $\mathbb{C}$-valued process and $S \in \mathscr{Q}$ a complex semimartingale (respectively martingale). Then*

$$\int f\, dS$$

is a complex semimartingale (respectively martingale).

Let $S^1, S^2 \in \mathcal{Q}$ and let f and g be integrable $(\mathscr{F}_t)$-adapted $\mathbb{C}$-valued processes. Then

$$\langle \int f dS^1, \int g dS^2 \rangle = \int f g d\langle S^1, S^2 \rangle.$$

In particular, $\int f dS$ and $\int g dS$ are orthogonal semimartingales,

$$\langle \int f dS, \int g dS \rangle = \int f g d\langle S, S \rangle = 0.$$

Proof. Compute the quadratic variations and covariations of the real and imaginary parts of the according functions and integrals. □

Using the customary $z = x + iy$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, $\partial_{xx} + \partial_{yy} = 4\partial_{z\overline{z}}$ and $i^2 = -1$, the Itô formula takes the familiar form.

Theorem 1 (Itô formula)**.** *For a twice differentiable complex function*

$$f : \mathbb{C} \to \mathbb{C}$$

and a complex semimartingale $S \in \mathcal{Q}$ we have

$$f(S, \overline{S}) - f(S_0, \overline{S}_0) = \int \partial_z f(S, \overline{S}) dS + \int \partial_{\overline{z}} f(S, \overline{S}) d\overline{S} + \int \partial_{z\overline{z}} f(S, \overline{S}) d\langle S, \overline{S} \rangle.$$

This generalizes to complex functions of severable variables. We need this generalization only for the case of two complex variables and orthogonal martingales and we write it out explicitly for the sake of clarity.

Theorem 2 (Itô formula)**.** *Let f be a $\mathscr{C}^2$ complex function of two complex variables*

$$f : \mathbb{C}^2 \to \mathbb{C}$$

and let $Z, W \in \mathcal{Q}$ be orthogonal complex semimartingales. Then

$$\begin{aligned} f(Z, W) - f(Z_0, W_0) &= \int \partial_z f\, dZ + \int \partial_w f\, dW + \int \partial_{\overline{z}} f\, d\overline{Z} + \int \partial_{\overline{w}} f\, d\overline{W} + \\ &+ \int \partial_{z\overline{z}} f\, d\langle Z, \overline{Z} \rangle + \int \partial_{z\overline{w}} f\, d\langle Z, \overline{W} \rangle + \\ &+ \int \partial_{w\overline{z}} f\, d\langle W, \overline{Z} \rangle + \int \partial_{w\overline{w}} f\, d\langle W, \overline{W} \rangle. \end{aligned}$$

Proof. Complex orthogonality implies

$$\langle Z, Z \rangle = \langle Z, W \rangle = \langle W, W \rangle = \langle \overline{Z}, \overline{Z} \rangle = \langle \overline{Z}, \overline{W} \rangle = \langle \overline{W}, \overline{W} \rangle = 0.$$

□

In the case of a holomorphic function this simplifies still further by the Cauchy-Riemann equations

$$\partial_{\overline{z}} f = \partial_{\overline{w}} f = 0.$$

Theorem 3 (Itô formula)**.** *Let f be a holomorphic function of two variables and let $Z, W \in \mathcal{Q}$ be orthogonal complex semimartingales. Then*

$$f(Z,W) - f(Z_0,W_0) = \int \partial_z f\, dZ + \int \partial_w f\, dW.$$

We shall also use the existence theorem for SDE for the case of bounded and Lipschitz coefficients. As a basic reference for all these facts we cite [RY91].

3 Statement and proof

In what follows we denote the complex derivative of a holomorphic function f by $f_z := \partial_z f$. Let

$$N := \{z \in \mathbb{C}^n : f = 0\}$$

be the zero-set of f and let

$$D := \{z \in \mathbb{C}^n : |f_{z^1}|^2 + \ldots + |f_{z^2}|^2 = 0\}$$

be the set of critical points of f.

Theorem 4. *Let f be a holomorphic function of two complex variables*

$$f : \mathbb{C}^2 \to \mathbb{C}.$$

Assuming the zero-set N is not empty, then the zeros are not isolated and N is not contained in any compact subset of $\mathbb{C}^2$. Moreover, the same is true for any component of $N \setminus D$.

The proof is organized into three parts.
First we prove the statement for the non-degenerate case $N \cap D = \emptyset$, in other words, when the zero-set is a manifold by the implicit function theorem.
Secondly, we prove the second assertion, concerning the non-degenerate components of the zero-set N. The argument is essentially equivalent to the maximum principle for a subharmonic function.
Strictly speaking, the distinction between the first part and second part is redundant, but it should make the proof clearer.
Thirdly, we consider the completely degenerate case.
Proof. **Part 1.** Assuming $N \neq \emptyset$ and $N \cap D = \emptyset$. We choose such a point $(z_0, w_0) \in \mathbb{C}^2$ that $f(z_0, w_0) = 0$. Denote the norm of the gradient of f as a function on $\mathbb{C}^2$

$$\begin{aligned} \|\nabla f\| &: \quad \mathbb{C}^2 \to \mathbb{R}^+ \\ \|\nabla f\| &: \quad (u,v) \mapsto \sqrt{|f_z(u,v)|^2 + |f_w(u,v)|^2} = (f_z \overline{f_z} + f_w \overline{f_w})^{\frac{1}{2}}, \end{aligned}$$

with slight abuse of notation. This function is used here as a measure of the distance from the set of degenerate points D. Let

$$B_R = \{(z,w) : \|(z,w) - (z_0,w_0)\| \leq R\}$$

be the ball of radius R around (z_0, w_0). Since N and D are closed sets the norm $\|\nabla f\|$ has a strictly positive lower bound $N \cap B_R$, say $\|\nabla f\| > 2\varepsilon$. Denote

$$D_\varepsilon = \{(u,v) : \|\nabla f\|(u,v) < \varepsilon\}$$

which we name the ε-neighborhood of D.
Let $Q = B^1 + iB^2$ where (B^1, B^2) is a standard Brownian motion so that $\langle Q, \overline{Q}\rangle = 2t$ and $\langle Q, Q\rangle = \langle \overline{Q}, \overline{Q}\rangle = 0$ and consider the following sde

$$\begin{aligned} dZ &= -\frac{f_w}{\| \nabla f \|} dQ \qquad (1)\\ dW &= \frac{f_z}{\| \nabla f \|} dQ \\ Z_0 &= z_0 \\ W_0 &= w_0. \end{aligned}$$

The map

$$g := \frac{1}{\| \nabla f \|} \begin{pmatrix} -f_w \\ f_z \end{pmatrix} \qquad (2)$$

and the corresponding degenerate dispersion matrix $\sigma = (g\ 0)$ for (1) are globally bounded and Lipschitz continuous on an open (Euclidean) neighborhood of $B_R \cap D^C_\varepsilon$, and likewise for the 4-dimensional real counterpart of the equation (1). Then the standard existence theorem for SDE guarantees a unique local strong solution, local meaning for all $0 \le t \le \tau$, where $\tau := \tau_{{B_R}^C} \wedge \tau_{D_\varepsilon}$ and τ_A is the hitting time of an open set A.
The solutions Z and W are clearly complex martingales by Proposition 1. Their covariations are

$$\begin{aligned} \langle Z, \overline{Z}\rangle &= 2\int \frac{f_w \cdot \overline{f_w}}{\| \nabla f \|^2} dt \qquad (3)\\ \langle W, \overline{W}\rangle &= 2\int \frac{f_z \cdot \overline{f_z}}{\| \nabla f \|^2} dt \\ \langle Z, \overline{W}\rangle &= -2\int \frac{f_w \cdot \overline{f_z}}{\| \nabla f \|^2} dt \\ \langle W, \overline{Z}\rangle &= -2\int \frac{f_z \cdot \overline{f_w}}{\| \nabla f \|^2} dt \\ \langle Z, W\rangle &= 0. \end{aligned}$$

Z and W are orthogonal in the complex sense, so Theorem 3 above applies:

$$\begin{aligned} f(Z, W) &= f(z_0, w_0) + \int f_z dZ + \int f_w dW \\ &= 0. \qquad (4) \end{aligned}$$

The annihilation occurs at the path level for the pathwise local solution. The local solution of (1) therefore stays on the zero-set. It cannot be constant since a real local martingale is constant iff its quadratic variation is constant (see for instance [RY91]), which is never the case for all the coordinate martingales of the local solution at the same time by the assumption $N \cap D = \emptyset$.
Clearly, $\tau_{D_\varepsilon} = \infty$ since $(Z, W)_{t\wedge\tau}$ remains on $N \cap B_R \subset D^C_{2\varepsilon}$, so here we have $\tau = \tau_{{B_R}^C}$.

For the claim that $\tau_{B_R{}^C} < \infty$, let us observe the square of the radial process of (Z,W):

$$\begin{aligned} Z_t\overline{Z_t} + W_t\overline{W}_t - z_0\overline{z}_0 - w_0\overline{w_0} &= \int_0^t \overline{Z}_s dZ_s + \int_0^t Z_s d\overline{Z}_s + \langle Z,\overline{Z}\rangle_t \\ &\quad + \int_0^t \overline{W}_s dW_s + \int_0^t W_s d\overline{W}_s + \langle W,\overline{W}\rangle_t \\ &= \text{a real valued local martingale} + 2t, \end{aligned} \tag{5}$$

where we write t instead of $t \wedge \tau$ for simplicity. Using optional stopping for this bounded local submartingale at $t \wedge \tau$ for a fixed $t < \infty$ one gets

$$\mathbb{E}\|(Z_{t\wedge\tau}, W_{t\wedge\tau})\|^2 - \|(z_0,w_0)\|^2 = 2\mathbb{E}(t\wedge\tau).$$

Since $\|(Z_{t\wedge\tau}, W_{t\wedge\tau})\|^2 \le R^2$ for all t, we have the following inequality

$$R^2 \ge \mathbb{E}\|(Z_{t\wedge\tau}, W_{t\wedge\tau})\|^2 \ge \mathbb{E}(t\wedge\tau) \ge 2t\mathbb{P}(\tau \ge t), \tag{6}$$

from which

$$\mathbb{P}(\tau = \infty) = \lim_{t\to\infty} \mathbb{P}(\tau \ge t) = 0$$

follows, proving the existence of continuous paths on the null-set that reach any distance R from (z_0, w_0). In fact, the solution to equation (1) actually reaches R with probability 1 and $2\mathbb{E}(\tau) = R^2 - \|(z_0,w_0)\|^2$.

Part 2. Assuming $N \setminus D \ne \varnothing$. We can still start equation (1) at a zero-point (z_0, w_0) outside a 2ε neighborhood $D_{2\varepsilon}$ of the set D by continuity of the norm $\|\nabla f\|$ at (z_0, w_0). A unique local strong solution for the equation (1) up to τ exists as before. The equations (4) and (5) still apply up to τ and hence so does $\mathbb{P}(\tau < \infty) = 1$ and (6), only we can no longer guarantee $\tau_{D_\varepsilon} = \infty$. But we need only to prove that $\mathbb{P}(\tau_{D_\varepsilon} < \infty) < 1$.

For this purpose we use the Itô formula for the squared gradient norm process

$$\|\nabla f\|^2(Z,\overline{Z},W,\overline{W})$$

along the local solution of (1). We get

$$\begin{aligned} f_z\overline{f_z} + f_w\overline{f_w} &= \|\nabla f\|_0^2 + \int f_{zz}\overline{f_z}dZ + \int f_z\overline{f_{zz}}d\overline{Z} + \int f_{zw}\overline{f_z}dW + \int f_z\overline{f_{zw}}d\overline{W} \\ &+ \int f_{ww}\overline{f_w}dW + \int f_w\overline{f_{ww}}d\overline{W} + \int f_{wz}\overline{f_w}dZ + \int f_w\overline{f_{wz}}d\overline{Z} \\ &+ \int f_{zz}\overline{f_{zz}}d\langle Z,\overline{Z}\rangle + \int f_{zw}\overline{f_{zw}}d\langle W,\overline{W}\rangle \\ &+ \int f_{zw}\overline{f_{zz}}d\langle W,\overline{Z}\rangle + \int f_{zz}\overline{f_{zw}}d\langle Z,\overline{W}\rangle \\ &+ \int f_{wz}\overline{f_{wz}}d\langle Z,\overline{Z}\rangle + \int f_{ww}\overline{f_{ww}}d\langle W,\overline{W}\rangle \\ &+ \int f_{ww}\overline{f_{wz}}d\langle W,\overline{Z}\rangle + \int f_{wz}\overline{f_{ww}}d\langle Z,\overline{W}\rangle \\ &= \text{a real local martingale} + \int \text{a real non-negative process } dt. \end{aligned} \tag{7}$$

In other words, the squared gradient norm process is a submartingale. That the integrand above is indeed non-negative can be seen using the covariations (3) computed above and the elementary inequality

$$\alpha\overline{\alpha}+\beta\overline{\beta}-2\mathrm{Re}(\alpha\cdot\beta)\geq 0 \text{ for } \alpha,\beta\in\mathbb{C}$$

for the pairs $\alpha=\overline{f_{zz}}\cdot f_w, \beta=f_{wz}\cdot\overline{f_z}$ and $\alpha=f_{ww}\cdot\overline{f_z}, \beta=\overline{f_{wz}}\cdot f_w$.
Since the local submartingale $\|\nabla f\|^2$ is bounded as a function of a bounded martingale and the stopping time τ is also bounded, this justifies the use of optional stopping for the squared gradient norm process (7) at τ.
Now assume $\mathbb{P}(\tau_{D_\varepsilon}<\infty)=1$, so the solution is almost always stopped at a point where $\|\nabla f\|=\varepsilon$. At $t=0$ we have

$$\|\nabla f\|_0>2\varepsilon$$

so we get a contradiction

$$\varepsilon^2>4\varepsilon^2$$

when taking expectation at the stopping time τ of (7). Hence the solution to equation (1) hits B_R^C with a strictly positive probability.
Part 3. Assume a non-degenerate point of N cannot be found. Choose a point from $(z_0,w_0)\in N\cap D$. We cannot start a sensible SDE at this point because the function and derivative values of f at this point are null. Rather, choose a non-degenerate point $(z_1,w_1)\in\mathbb{C}^2$ of the gradient of f in a δ neighborhood of (z_0,w_0), even though this means leaving the zero-set. Such a point must exist for any $\delta>0$. Here we can use the elementary fact from complex analysis that a holomorphic function of one variable is constant on an open set iff it is constant. Alternatively, one can use a general topological argument: assuming all points on the zero-set have an open (Euclidean) neighborhood where the gradient is null, the zero-set is both open and closed, implying $N=\varnothing$ or $N=\mathbb{C}^2$. Choosing a $\delta>0$ sufficiently small, one get can $f(z_1,w_1)=\varepsilon$ for arbitrary $\varepsilon>0$, since f is continuous.
Now (z_1,w_1) is a non-degenerate point on the zero set of $f-\varepsilon$. Using the equation (1) for $(Z_0,W_0)=(z_1,w_1)$, one can get zeros of $f-\varepsilon$ at any distance from (z_0,w_0). These are not zeros of f, but they are close, measured by f itself.
The proof can now be concluded in several ways. To keep in line with the reasoning above, choose such a sequence of non-degenerate points $(z_i,w_i)\in\mathbb{C}^2$ of f that converge to (z_0,w_0). Then the sequence $\varepsilon_i:=f(z_i,w_i)$ converges to 0. Applying part 2 to $f-\varepsilon_i$ one obtains a sequence of points (z_i,w_i) on the two-sphere $S(R)$ such that $f(z_i,w_i)=\varepsilon_i$ and by compactness this sequence has a Cauchy subsequence. For the limit (z,w) of any such subsequence one has $f(z,w)=0$ by continuity of f. □

4 Conclusions and remarks

We include a numerical illustration of the pair of processes Z and W of equation (1) on the zero-set of the function

$$f(z,w)=4z^4+3z^2+2wz+w^3+\sin(z+w)-10-\sin(2).$$

Obviously, the theorem can be extended to holomorphic functions $f:D\subset\mathbb{C}^n\to\mathbb{C}$ for $n>2$ (simply by freezing $n-2$ coordinates) and to arbitrary domains $D\subset\mathbb{C}^n$. Namely, replacing

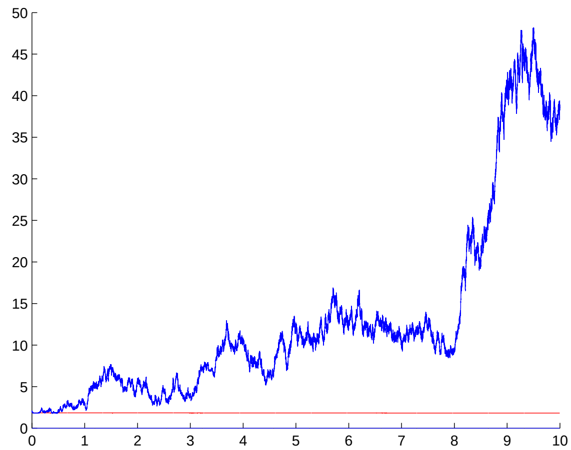

Figure 1: The process $f(Z,W)$ (red) and the square radial process $Z\overline{Z}+W\overline{W}$ (blue) for the example above

the balls B_R with any compact exhaustion of a domain D that contains an initial point where $f(z_0,w_0)=0$ for the SDE is inconsequential to the proof.

In effect, equation(1) defines a Brownian-like martingale on a Euclidean submersion of a complex manifold. The radial process $\|(Z_t,W_t)\|$ behaves like the radial process of 2-dimensional Brownian motion in the ambient space stopped at R in the sense that

$$2\mathbb{E}(\tau)=R^2$$

when $t\to\infty$ and

$$\mathbb{E}\|(Z_t,W_t)\|^2=2t,$$

when $R\to\infty$.

In the manifold case the zero-set N (viewed as an embedded manifold in $\mathbb{C}^2$) has a natural Riemannian structure and hence there is a canonical Brownian motion induced by the Kähler metric, see for instance [IW81]. In general, Brownian motion on an embedded manifold has non-zero drift coefficients and is not a martingale. However, in the special case when the embedded manifold has a holomorphic defining function, so there exist holomorphic local coordinates by the implicit function theorem, the intrinsic Brownian motion also turns out to be a martingale and the process defined by equation (1) is its time change. One can see this by using the implicit holomorphic local coordinates to map both processes from the ambient space $\mathbb{C}^2$ onto a region of $\mathbb{C}$ where they are both complex martingales and hence time changed Brownian motions by Lévy's theorem.

In the manifold case equation (1) can be generalized to dimensions $n>2$ simply by choosing any vector field with values in the kernel of the derivative of the defining function. It can also be generalized to real manifolds (with non-holomorphic defining functions) by replacing Itô integrals with Stratonovich integrals.

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