Elect. Comm. in Probab. **13** (2008), 614–627

ELECTRONIC COMMUNICATIONS in PROBABILITY

MARKOV PROCESSES WITH PRODUCT-FORM STATIONARY DISTRIBUTION

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Submitted November 3, 2007, *accepted in final form* November 18, 2008

AMS 2000 Subject classification: 60J25
Keywords: Markov process, stationary distribution, inert drift

Abstract

We consider a continuous time Markov process (X, L), where X jumps between a finite number of states and L is a piecewise linear process with state space $\mathbb{R}^d$. The process L represents an "inert drift" or "reinforcement." We find sufficient and necessary conditions for the process (X, L) to have a stationary distribution of the product form, such that the marginal distribution of L is Gaussian. We present a number of conjectures for processes with a similar structure but with continuous state spaces.

1 Introduction

This research has been inspired by several papers on processes with inert drift [5, 4, 3, 1]. The model involves a "particle" X and an "inert drift" L, neither of which is a Markov process by itself, but the vector process (X, L) is Markov. It turns out that for some processes (X, L), the stationary measure has the product form; see [1]. The first goal of this note is to give an explicit characterization of all processes (X, L) with a finite state space for X and a product form stationary distribution—see Theorem 2.1.

The second, more philosophical, goal of this paper is to develop a simple tool that could help generate conjectures about stationary distributions for processes with *continuous* state space and inert drift. So far, the only paper containing a rigorous result about the stationary distribution for a process with continuous state space and inert drift, [1], was inspired by computer simulations. Examples presented in Section 3 lead to a variety of conjectures that would be hard to arrive at using pure intuition or computer simulations.

[1]PARTIALLY SUPPORTED BY NSF GRANT DMS-0600206

2 The model

Let $\mathscr{S} = \{1, 2, \dots, N\}$ for some integer $N > 1$ and let $d \geq 1$ be an integer. We define a continuous time Markov process $(X(t), L(t))$ on $\mathscr{S} \times \mathbb{R}^d$ as follows. We associate with each state $j \in \mathscr{S}$ a vector $v_j \in \mathbb{R}^d, 1 \leq j \leq N$. Define $L_j(t) = \mu(\{s \in [0,t] : X(s) = j\})$, where μ is Lebesgue measure, and let $L(t) = \sum_{j\in\mathscr{S}} v_j L_j(t)$. To make the "reinforcement" non-trivial, we assume that at least one of v_j's is not 0. Since L will always belong to the hyperplane spanned by v_j's, we also assume that $d = \dim(\text{span}\{v_1, \dots, v_N\})$.
We also select non-negative functions $a_{ij}(l)$ which define the Poisson rates of jumps from state i to j. The rates depend on $l = L(t)$. We assume that a_{ij}'s are right-continuous with left limits. Formally speaking, the process (X, L) is defined by its generator A as follows,

$$Af(j,l) = v_j \cdot \nabla_l f(j,l) + \sum_{i\neq j} a_{ji}(l)[f(i,l) - f(j,l)], \quad j = 1, \dots, N, \ l \in \mathbb{R}^d,$$

for $f : \{1, \dots, N\} \times \mathbb{R}^d \to \mathbb{R}$.
We assume that (X, L) is irreducible in the sense of Harris, i.e., for some open set $U \subset \mathbb{R}^d$ and some $j_0 \in \mathscr{S}$, for all $(x, l) \in \mathscr{S} \times \mathbb{R}^d$, we have for some $t > 0$,

$$P((X(t), L(t)) \in \{j_0\} \times U) > 0.$$

We are interested only in processes satisfying (14) below. Using that condition, it is easy to check Harris irreducibility for each of our models by a direct argument. A standard coupling argument shows that Harris irreducibility implies uniqueness of the stationary probability distribution (assuming existence of such).
The (formal) adjoint of A is given by

$$A^* g(j,l) = -v_j \cdot \nabla_l g(j,l) + \sum_{i\neq j} [a_{ij}(l) g(i,l) - a_{ji}(l) g(j,l)], \quad j = 1, \dots, N, \ l \in \mathbb{R}^d. \tag{1}$$

We are interested in invariant measures of product form so suppose that $g(j,l) = p_j g(l)$, where $\sum_{j\in\mathscr{S}} p_j = 1$ and $\int_{\mathbb{R}^d} g(l) dl = 1$. We may assume that $p_j > 0$ for all j; otherwise some points in $\mathscr{S}$ are never visited. Under these assumptions, (1) becomes

$$A^* g(j,l) = -p_j v_j \cdot \nabla g(l) + \sum_{i\neq j} [p_i a_{ij}(l) g(l) - p_j a_{ji}(l) g(l)], \quad j = 1, \dots, N, \ l \in \mathbb{R}^d.$$

Theorem 2.1. *Assume that for every i and j, the function $l \to a_{ij}(l)$ is continuous. A probability measure $p_j g(l) dj dl$ is invariant for the process (X, L) if and only if*

$$-p_j v_j \cdot \nabla g(l) + \sum_{i\neq j} [p_i a_{ij}(l) g(l) - p_j a_{ji}(l) g(l)] = 0, \quad j = 1, \dots, N, \ l \in \mathbb{R}^d. \tag{2}$$

Proof. Recall that the state space $\mathscr{S}$ for X is finite. Hence $v_* := \sup_{j\in\mathscr{S}} |v_j| < \infty$. Fix arbitrary $r, t_* \in (0, \infty)$. It follows that,

$$\sup_{i,j\in\mathscr{S}, l\in B(0, r+2t_* v_*)} a_{ij}(l) = a_* < \infty.$$

Note that we always have $|L(t) - L(u)| \leq v_* |t - u|$. Hence, if $|L(0)| \leq r + t_* v_*$ and $s, t > 0$, $s + t \leq t_*$, then $|L(s + t)| \leq r + 2t_* v_*$ and, therefore,

$$\sup_{j\in\mathscr{S}, u\leq s+t} a_{X(u),j}(L(u)) \leq a_* < \infty.$$

This implies that the probability of two or more jumps on the interval $[s, s+t]$ is $o(t)$. Assume that $|l| \le r + t_* v_*$ and $t \le t_*$. Then we have the following three estimates. First,

$$P(X(t) = j \mid X(0) = i, L(0) = l) = a_{ij}(l)t + R^1_{i,j,l}(t), \tag{3}$$

where the remainder $R^1_{i,j,l}(t)$ satisfies $\sup_{i,j\in\mathscr{S}, l\in B(0,t_* v_*)} |R^1_{i,j,l}(t)| \le R^1(t)$ for some $R^1(t)$ such that $\lim_{t\to 0} R^1(t)/t = 0$.
Let $a_{ii}(l) = -\sum_{j\neq i} a_{ij}(l)$. We have

$$P(X(t) = i, L(t) = l + tv_i \mid X(0) = i, L(0) = l) = 1 + a_{ii}(l)t + R^2_{i,l}(t), \tag{4}$$

where the remainder $R^2_{i,l}(t)$ satisfies $\sup_{i\in\mathscr{S}, l\in B(0,t_* v_*)} |R^2_{i,l}(t)| \le R^2(t)$ for some $R^2(t)$ such that $\lim_{t\to 0} R^2(t)/t = 0$.
Finally,

$$P(X(t) = i, L(t) \neq l + tv_i \mid X(0) = i, L(0) = l) = R^3_{i,l}(t), \tag{5}$$

where the remainder $R^3_{i,l}(t)$ satisfies $\sup_{i\in\mathscr{S}, l\in B(0,t_* v_*)} |R^3_{i,l}(t)| \le R^3(t)$ for some $R^3(t)$ such that $\lim_{t\to 0} R^3(t)/t = 0$.
Now consider any C^1 function $f(j,l)$ with support in $\mathscr{S} \times B(0,r)$. Recall that $|L(t) - L(u)| \le v_*|t-u|$. Hence, $E_{i,l} f(X_t, L_t) = 0$ for $t \le t_*$ and $|l| \ge r + v_* t_*$.
Suppose that $|l_0| \le r + v_* t_*$, $t_1 \in (0, t_*)$ and $s \in (0, t_* - t_1)$. Then

$$\begin{aligned}
&E_{i,l_0} f(X_{t_1+s}, L_{t_1+s}) - E_{i,l} f(X_{t_1}, L_{t_1}) \\
&= \sum_{j\in\mathscr{S}} \int_{\mathbb{R}^d} \sum_{k\in\mathscr{S}} \int_{\mathbb{R}^d} f(k,r) P(X(t_1+s) = k, L(t_1+s) \in dr \mid X(t_1) = j, L(t_1) = l) \\
&\quad \times P(X(t_1) = j, L(t_1) \in dl \mid X(0) = i, L(0) = l_0) \\
&- \sum_{j\in\mathscr{S}} \int_{\mathbb{R}^d} f(j,l) P(X(t_1) = j, L(t_1) \in dr \mid X(0) = i, L(0) = l_0).
\end{aligned}$$

We combine this formula with (3)-(5) to see that,

$$\begin{aligned}
&E_{i,l_0} f(X_{t_1+s}, L_{t_1+s}) - E_{i,l} f(X_{t_1}, L_{t_1}) \\
&= \sum_{j\in\mathscr{S}} \int_{\mathbb{R}^d} \sum_{k\in\mathscr{S}, k\neq j} (f(k,l) + O(s))(a_{jk}(l)s + R^1_{j,k,l}(s)) \\
&\quad \times P(X(t_1) = j, L(t_1) \in dl \mid X(0) = i, L(0) = l_0) \\
&+ \sum_{j\in\mathscr{S}} \int_{\mathbb{R}^d} f(j, l + sv_j)(1 + a_{jj}(l)s + R^2_{j,l}(s)) \\
&\quad \times P(X(t_1) = j, L(t_1) \in dl \mid X(0) = i, L(0) = l_0) \\
&+ \sum_{j\in\mathscr{S}} \int_{\mathbb{R}^d} (f(j,l) + O(s)) R^3_{j,l}(s) \\
&\quad \times P(X(t_1) = j, L(t_1) \in dl \mid X(0) = i, L(0) = l_0) \\
&- \sum_{j\in\mathscr{S}} \int_{\mathbb{R}^d} f(j,l) P(X(t_1) = j, L(t_1) \in dl \mid X(0) = i, L(0) = l_0),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}(f(j,l+sv_j)-f(j,l))P(X(t_1)=j,L(t_1)\in dl\mid X(0)=i,L(0)=l_0)\\
&+\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\left(f(j,l+sv_j)a_{jj}(l)+\sum_{k\in\mathscr{S},k\neq j}f(k,l)a_{jk}(l)\right)s\\
&\quad\times P(X(t_1)=j,L(t_1)\in dl\mid X(0)=i,L(0)=l_0)\\
&+\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\left(\left(\sum_{k\in\mathscr{S},k\neq j}f(k,l)R^1_{j,k,l}(s)+O(s)(a_{jk}(l)s+R^1_{j,k,l}(s))\right)\right.\\
&\quad\left.+f(j,l+sv_j)R^2_{j,l}(s)+(f(j,l)+O(s))R^3_{j,l}(s)\right)\\
&\quad\times P(X(t_1)=j,L(t_1)\in dl\mid X(0)=i,L(0)=l_0).
\end{aligned}$$

We will analyze the limit

$$\lim_{s\downarrow 0}\frac{1}{s}(E_{i,l_0}f(X_{t_1+s},L_{t_1+s})-E_{i,l_0}f(X_{t_1},L_{t_1})).$$

Note that

$$\begin{aligned}
&\lim_{s\downarrow 0}\frac{1}{s}\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\left(\left(\sum_{k\in\mathscr{S},k\neq j}f(k,l)R^1_{j,k,l}(s)+O(s)(a_{jk}(l)s+R^1_{j,k,l}(s))\right)\right.\\
&\quad\left.+f(j,l+sv_j)R^2_{j,l}(s)+(f(j,l)+O(s))R^3_{j,l}(s)\right)\\
&\quad\times P(X(t_1)=j,L(t_1)\in dl\mid X(0)=i,L(0)=l_0)\\
&\quad=0.
\end{aligned}$$

We also have

$$\begin{aligned}
&\lim_{s\downarrow 0}\frac{1}{s}\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}(f(j,l+sv_j)-f(j,l))P(X(t_1)=j,L(t_1)\in dl\mid X(0)=i,L(0)=l_0)\\
&=\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\nabla_l f(j,l)\cdot v_j\,P(X(t_1)=j,L(t_1)\in dl\mid X(0)=i,L(0)=l_0),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{s\downarrow 0}\frac{1}{s}\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\left(f(j,l+sv_j)a_{jj}(l)+\sum_{k\in\mathscr{S},k\neq j}f(k,l)a_{jk}(l)\right)s\\
&\quad\times P(X(t_1)=j,L(t_1)\in dl\mid X(0)=i,L(0)=l_0)\\
&=\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\sum_{k\in\mathscr{S}}f(k,l)a_{jk}(l)P(X(t_1)=j,L(t_1)\in dl\mid X(0)=i,L(0)=l_0).
\end{aligned}$$

This implies that

$$\frac{d}{dt}E_{i,l_0}f(X_t,L_t)\Big|_{t=t_1} = \lim_{s\downarrow 0}\frac{1}{s}(E_{i,l_0}f(X_{t_1+s},L_{t_1+s}) - E_{i,l_0}f(X_{t_1},L_{t_1}))$$

$$= \sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\left(\nabla_l f(j,l)\cdot v_j + \sum_{k\in\mathscr{S}} f(k,l)a_{jk}(l)\right)P(X(t_1)=j, L(t_1)\in dl \mid X(0)=i, L(0)=l_0) \quad (6)$$

$$= \sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d} Af(j,l)P(X(t_1)=j, L(t_1)\in dl \mid X(0)=i, L(0)=l_0)$$

$$= E_{i,l_0}Af(X_{t_1},L_{t_1}), \quad (7)$$

for all i and $|l_0|\le r+v_*t_*$.
We will argue that

$$P((X(t_1),L(t_1))\in\cdot\mid X(0)=i,L(0)=l)\to P((X(t_1),L(t_1))\in\cdot\mid X(0)=i,L(0)=l_0), \quad (8)$$

weakly when $l\to l_0$. Let T_k be the time of the k-th jump of X. We have

$$P(T_1>t\mid X(0)=i,L(0)=l)=\exp\left(\int_0^t a_{ii}(l+sv_i)ds\right).$$

Since $l\to a_{ii}(l)$ is continuous, we conclude that

$$P(T_1>t\mid X(0)=i,L(0)=l)\to P(T_1>t\mid X(0)=i,L(0)=l_0),$$

weakly as $l\to l_0$. This and continuity of $l\to a_{ij}(l)$ for every j implies that

$$P((X(T_1),L(T_1))\in\cdot\mid X(0)=i,L(0)=l)\to P((X(T_1),L(T_1))\in\cdot\mid X(0)=i,L(0)=l_0),$$

weakly when $l\to l_0$. By the strong Markov property applied at T_k's, we obtain inductively that

$$P((X(T_k),L(T_k))\in\cdot\mid X(0)=i,L(0)=l)\to P((X(T_k),L(T_k))\in\cdot\mid X(0)=i,L(0)=l_0),$$

when $l\to l_0$, for every $k\ge 1$. This easily implies (8), because the number of jumps is stochastically bounded on any finite interval.
Since $(j,l)\hookrightarrow \nabla_l f(j,l)\cdot v_j+\sum_{k\in\mathscr{S}} f(k,l)a_{jk}(l)$ is a continuous function, it follows from (6) and (8) that $l_0\hookrightarrow \frac{d}{dt}E_{i,l_0}f(X_t,L_t)\Big|_{t=t_1}$ is continuous on the set $|l_0|\le r+v_*t_*$.
Recall that $E_{i,l_0}f(X_t,L_t)=0$ for $t\le t_*$ and $|l_0|\ge r+v_*t_*$. Hence, $l_0\to\frac{d}{dt}E_{i,l_0}f(X_t,L_t)\Big|_{t=t_1}$ is continuous for all $t_1\le t_*$ and all values of i.
Fix some $t\le t_*$ and let $u_t(j,l)=E_{j,l}f(X_t,L_t)$. We have just shown that for a fixed $t\le t_*$ and any j, the function $l\to u_t(j,l)$ is C^1. Hence we can apply (7) with $f(j,l)=u_t(j,l)$ to obtain,

$$\frac{d}{dt}E_{j,l}f(X_t,L_t)=\lim_{s\downarrow 0}\frac{1}{s}(E_{j,l}f(X_{t+s},L_{t+s})-E_{j,l}f(X_t,L_t)) \quad (9)$$

$$=\lim_{s\downarrow 0}\frac{1}{s}(E_{j,l}u_t(X_s,L_s)-u_t(j,l))$$

$$=(Au_t)(j,l).$$

Since $\sup_{j,l}\left(\nabla_l f(j,l)\cdot v_j+\sum_{k\in\mathscr{S}} f(k,l)a_{jk}(l)\right)<\infty$, formula (6) shows that

$$\sup_{j,l,s\le t_*}\left(\frac{d}{dt}E_{j,l}f(X_s,L_s)\right)<\infty. \tag{10}$$

Now assume that (2) is true and let $\pi(dj,dl)=p_j g(l)djdl$. In view of (10), we can change the order of integration in the following calculation. For $0\le t_1<t_2\le t_*$, using (9),

$$\begin{aligned}
&E_\pi f(X(t_2),L(t_2))-E_\pi f(X(t_1),L(t_1)) \\
&=\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}E_{j,l}f(X_{t_2},L_{t_2})p_j g(l)dl-\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}E_{j,l}f(X_{t_1},L_{t_1})p_j g(l)dl \\
&=\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\int_{t_1}^{t_2}\frac{d}{ds}E_{j,l}f(X_s,L_s)ds\,p_j g(l)dl \\
&=\int_{t_1}^{t_2}\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}\frac{d}{ds}E_{j,l}f(X_s,L_s)p_j g(l)dl\,ds \\
&=\int_{t_1}^{t_2}\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}(Au_s)(j,l)p_j g(l)dl\,ds.
\end{aligned} \tag{11}$$

Let $h(j,l)=p_j g(l)$. For a fixed j and $s\le t_*$, the function $u_s(j,l)=0$ outside a compact set, so we can use integration by parts to show that

$$\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}(Au_s)(j,l)p_j g(l)dl=\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}u_s(j,l)(A^*h)(j,l)dl. \tag{12}$$

We combine this with the previous formula and the assumption that $A^*h\equiv 0$ to see that

$$E_\pi f(X(t_2),L(t_2))-E_\pi f(X(t_1),L(t_1))=\int_{t_1}^{t_2}\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}u_s(j,l)(A^*h)(j,l)dl\,ds=0.$$

It follows that $t\to E_\pi f(X(t),L(t))$ is constant for every C^1 function $f(j,l)$ with compact support. This proves that the distributions of $(X(t_1),L(t_1))$ and $(X(t_2),L(t_2))$ are identical under π, for all $0\le t_1<t_2\le t_*$.

Conversely, assume that $\pi(dj,dl)=p_j g(l)djdl$ is invariant. Then the left hand side of (11) is zero for all $0\le t_1<t_2\le t_*$. This implies that

$$\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}(Au_s)(j,l)p_j g(l)dl=0$$

for a set of s that is dense on $[0,\infty)$. By (12),

$$\sum_{j\in\mathscr{S}}\int_{\mathbb{R}^d}u_s(j,l)(A^*h)(j,l)dl=0 \tag{13}$$

for a set of s that is dense on $[0,\infty)$. Note that $\lim_{s\downarrow 0}u_s(j,l)=f(j,l)$. Hence, the collection of C^1 functions $u_s(j,l)$, obtained by taking arbitrary C^1 functions $f(j,l)$ with compact support and positive reals s dense in $[0,\infty)$, is dense in the family of C^1 functions with compact support. This and (13) imply that $A^*h\equiv 0$, that is, (2) holds. □

Corollary 2.2. *If a probability measure $p_j g(l) dj dl$ is invariant for the process (X, L) then*

$$\sum_{j\in\mathcal{S}} p_j v_j = 0. \tag{14}$$

Proof. Summing (2) over j, we obtain

$$\sum_{j\in\mathcal{S}} -p_j v_j \cdot \nabla g(l) = 0, \tag{15}$$

for all l. Since g is integrable over $\mathbb{R}^d$, it is standard to show that there exist $l_1, l_2, \dots, l_d$ which span $\mathbb{R}^d$. Applying (15) to all $l_1, l_2, \dots$, we obtain (14). □

It will be convenient to use the following notation,

$$b_{ij}(l) = p_i a_{ij}(l) - p_j a_{ji}(l). \tag{16}$$

Note that $b_{ij} = -b_{ji}$.

Corollary 2.3. *A probability measure $p_j g(l) dj dl$ is invariant for the process (X, L) and $g(l)$ is the Gaussian density*

$$g(l) = (2\pi)^{-d/2} \exp(-|l|^2/2), \tag{17}$$

if and only if the following equivalent conditions hold,

$$p_j v_j \cdot l + \sum_{i\neq j} [p_i a_{ij}(l) - p_j a_{ji}(l)] = 0, \quad j = 1, \dots, N,\ l \in \mathbb{R}^d, \tag{18}$$

$$p_j v_j \cdot l + \sum_{i\neq j} b_{ij}(l) = 0, \quad j = 1, \dots, N,\ l \in \mathbb{R}^d. \tag{19}$$

Proof. If $g(l)$ is the Gaussian density then $\nabla g(l) = -l g(l)$ and (2) is equivalent to (18). Conversely, if (2) and (18) are satisfied then $\nabla g(l) = -l g(l)$, so $g(l)$ must have the form (17). □

In the rest of the paper we will consider only processes satisfying (18)-(19).

Example 2.4. We now present some choices for a_{ij}'s. Recall the notation $x^+ = \max(x, 0)$, $x^- = -\min(x, 0)$, and the fact that $x^+ - x^- = x$. Given v_j's, p_j's and b_{ij}'s which satisfy (19) and the condition $b_{ij} = -b_{ji}$, we may take

$$a_{ij}(l) = \left(b_{ij}(l)\right)^+ / p_i. \tag{20}$$

Then

$$a_{ji}(l) = \left(b_{ji}(l)\right)^+ / p_j = \left(-b_{ij}(l)\right)^+ / p_j = \left(b_{ij}(l)\right)^- / p_j,$$

so

$$p_i a_{ij}(l) - p_j a_{ji}(l) = \left(b_{ij}(l)\right)^+ - \left(b_{ij}(l)\right)^- = b_{ij}(l),$$

as desired.

The above is a special case, in a sense, of the following. Suppose that $p_j = p_i$ for all i and j. Assume that v_j's and b_{ij}'s satisfy (19) and the condition $b_{ij} = -b_{ji}$. Fix some $c > 0$ and let

$$a_{ij}(l) = \frac{b_{ij}(l) \exp(c b_{ij}(l))}{\exp(c b_{ij}(l)) - \exp(-c b_{ij}(l))}. \tag{21}$$

It is elementary to check that with this definition, (16) is satisfied for all i and j, because $b_{ij}(l) = -b_{ji}(l)$. The formula (21) arose naturally in [5]. Note that (20) (with all p_i's equal) is the limit of (21) as $c \to \infty$.

3 Approximation of processes with continuous state space

This section is contains examples of processes (X, L) with finite state space for X, and conjectures concerned with processes with continuous state space. There are no proofs in this section
First we will consider processes that resemble diffusions with reflection. In these models, the "inert drift" is accumulated only at the "boundary" of the domain.
We will now assume that elements of $\mathscr{S}$ are points in a Euclidean space $\mathbb{R}^n$ with $n \le N$. We denote them $\mathscr{S} = \{x_1, x_2, \dots, x_N\}$. In other words, by abuse of notation, we switch from j to x_j. We also take $v_j \in \mathbb{R}^n$, i.e. $d = n$. Moreover, we limit ourselves to functions $b_{ij}(l)$ of the form $b_{ij} \cdot l$ for some vector $b_{ij} \in \mathbb{R}^n$. Then (19) becomes

$$\begin{aligned} 0 + b_{12} \ + b_{13} \ + \cdots + b_{1N} &= -p_1 v_1 \\ -b_{12} - 0 \quad + b_{23} \ + \cdots + b_{2n} &= -p_2 v_2 \\ \cdots \\ -b_{1N} - b_{2N} - b_{3N} - \cdots - 0 \quad &= -p_N v_N. \end{aligned} \tag{22}$$

Consider any orthogonal transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$. If $\{b_{ij}, v_j, p_j\}$ satisfy (22) then so do $\{\Lambda b_{ij}, \Lambda v_j, p_j\}$.
Suppose that $a_{ij}(l)$ have the form $a_{ij} \cdot l$ for some $a_{ij} \in \mathbb{R}^n$. If $\{a_{ij}, v_j, p_j\}$ satisfy (18) then so do $\{\Lambda a_{ij}, \Lambda v_j, p_j\}$. Moreover, the process with parameters $\{a_{ij}, v_j\}$ has the same transition probabilities as the one with parameters $\{\Lambda a_{ij}, \Lambda v_j\}$.

Example 3.1. Our first example is a reflected random walk on the interval $[0,1]$. Let $x_j = (j-1)/(N-1)$ for $j = 1, \dots, N$. We will construct a process with all p_j's equal to each other, i.e., $p_j = 1/N$. We will take $l \in \mathbb{R}^1$, $v_1 = \alpha$ and $v_N = -\alpha$, for some $\alpha = \alpha(N) > 0$, and all other $v_j = 0$, so that the "inert drift" L changes only at the endpoints of the interval. We also allow jumps only between adjacent points, so $b_{ij} = 0$ for $|i - j| > 1$. Then (22) yields

$$\begin{aligned} b_{12} &= -\alpha/N \\ -b_{12} + b_{23} &= 0 \\ &\cdots \\ -b_{(N-1)N} &= \alpha/N. \end{aligned}$$

Solving this, we obtain $b_{i(i+1)} = -\alpha/N$ for all i.
We would like to find a family of semi-discrete models indexed by N that would converge to a continuous process with product-form stationary distribution as $N \to \infty$. For $1 < i < N$, we set $a_{i(i+1)}(l) = A_R(l, N)$ and $a_{(i+1)i}(l) = A_L(l, N)$. We would like the random walk to have variance of order 1 at time 1, for large N, so we need

$$A_R + A_L = N^2. \tag{23}$$

Since $b_{i(i+1)} = -\alpha/N$ for all i, A_R and A_L have to satisfy

$$A_L - A_R = \alpha l. \tag{24}$$

When l is of order 1, we would like to have drift of order 1 at time 1, so we take $\alpha = N$. Then (24) becomes

$$A_L - A_R = N l. \tag{25}$$

Solving (23) and (25) gives

$$A_L = \frac{N^2 + Nl}{2}, \qquad A_R = \frac{N^2 - Nl}{2}.$$

Unfortunately, A_L and A_R given by the above formula can take negative values—this is not allowed because a_{ij}'s have to be positive. However, for every N, the stationary distribution of L is standard normal, so l typically takes values of order 1. We are interested in large N so, intuitively speaking, A_R and A_L are not likely to take negative values. To make this heuristics rigorous, we modify the formulas for A_R and A_L as follows,

$$A_L = \frac{N^2 + Nl}{2} \vee 0 \vee Nl, \qquad A_R = \frac{N^2 - Nl}{2} \vee 0 \vee (-Nl). \tag{26}$$

Let P_N denote the distribution of (X, L) with the above parameters. We conjecture that as $N \to \infty$, P_N converge to the distribution of reflected Brownian motion in $[0, 1]$ with inert drift, as defined in [5, 1]. The stationary distribution for this continuous time process is the product of the uniform measure in $[0, 1]$ and the standard normal; see [1].

Example 3.2. This example is a semi-discrete approximation to reflected Brownian motion in a bounded Euclidean subdomain of $\mathbb{R}^n$, with inert drift. In this example we proceed in the reversed order, starting with b_{ij}'s and a_{ij}'s.
Consider an open bounded connected set $D \subset \mathbb{R}^n$. Let K be a (large) integer and let $D_K = \mathbb{Z}^n/K \cap D$, i.e., D_K is the subset of the square lattice with mesh $1/K$ that is inside D. We assume that D_K is connected, i.e., any vertices in D_K are connected by a path in D_K consisting of edges of length $1/K$. We take $\mathscr{S} = D_K$ and $l \in \mathbb{R}^n$.
We will consider nearest neighbor random walk, i.e., we will take $a_{ij}(l) = 0$ for $|x_i - x_j| > 1/K$. In analogy to (26), we define

$$a_{ij}(l) = \frac{K^2}{2}(1 + (x_i - x_j) \cdot l) \vee 0 \vee K^2 (x_i - x_j) \cdot l. \tag{27}$$

Then $b_{ij}(l) = (K^2/N)(x_i - x_j) \cdot l$. Let us call a point in $\mathscr{S} = D_K$ an interior point if it has $2n$ neighbors in D_K. We now define v_j's using (22) with $p_j = 1/|D_K|$. For all interior points x_j, the vector v_j is 0, by symmetry. For all boundary (that is, non-interior) points x_j, the vector v_j is not 0.
Fix $D \subset \mathbb{R}^n$ and consider large K. Let P_K denote the distribution of (X, L) constructed in this example. We conjecture that as $K \to \infty$, P_K converge to the distribution of normally reflected Brownian motion in D with inert drift, as defined in [5, 1]. If D is C^2 then it is known that the stationary distribution for this continuous time process is the product of the uniform measure in D and the standard Gaussian distribution; see [1].

The next two examples are discrete counterparts of processes with continuous state space and smooth inert drift. The setting is similar to that in Example 3.2. We consider an open bounded connected set $D \subset \mathbb{R}^n$. Let K be a (large) integer and let $D_K = \mathbb{Z}^n/K \cap D$, i.e., D_K is the subset of the square lattice with mesh $1/K$ that is inside D. We assume that D_K is connected, i.e., any vertices in D_K are connected by a path in D_K consisting of edges of length $1/K$. We take $\mathscr{S} = D_K$ and $l \in \mathbb{R}^n$.

Example 3.3. This example is concerned with a situation when the stationary distribution has the form $p_j g(l)$ where p_j's are not necessarily equal. We start with a C^2 "potential" $V : D \to \mathbb{R}$. We will write V_j instead of $V(x_j)$. Let $p_j = c\exp(-V_j)$. We need an auxiliary function

$$d_{ij} = \frac{2(p_i - p_j)}{p_i(V_j - V_i) - p_j(V_i - V_j)}.$$

Note that $d_{ij} = d_{ji}$ and for a fixed i, we have $d_{ij_K} \to 1$ when $K \to \infty$ and $|i - j_K| = 1/K$.
Let $a_{ij}(l) = 0$ for $|x_i - x_j| > 1/K$, and for $|x_i - x_j| = 1/K$,

$$\widetilde{a}_{ij}(l) = \frac{K^2}{2}(2 + d_{ij}(V_i - V_j) + (x_j - x_i) \cdot l).$$

We set

$$a_{ij}(l) = \begin{cases} \widetilde{a}_{ij}(l) \vee 0 & \text{if } \widetilde{a}_{ji}(l) > 0, \\ (K^2/2p_i)(p_i + p_j)(x_j - x_i) \cdot l & \text{otherwise.} \end{cases} \tag{28}$$

If $\widetilde{a}_{ji}(l) > 0$ and $\widetilde{a}_{ij}(l) > 0$ then

$$\begin{aligned} b_{ij}(l) &= p_i a_{ij}(l) - p_j a_{ji}(l) \\ &= \frac{K^2}{2}(2(p_i - p_j) + (p_i(V_i - V_j) - p_j(V_j - V_i))d_{ij} + (p_i(x_j - x_i) - p_j(x_i - x_j)) \cdot l) \\ &= \frac{K^2}{2}(2(p_i - p_j) - 2(p_i - p_j) + (p_i + p_j)(x_j - x_i) \cdot l) \\ &= \frac{K^2}{2}(p_i + p_j)(x_j - x_i) \cdot l. \end{aligned}$$

It follows from (28) that the above formula holds also if $\widetilde{a}_{ji}(l) \le 0$ or $\widetilde{a}_{ij}(l) \le 0$. Consider an interior point x_j. For (19) to be satisfied, we have to take

$$v_j = -\frac{1}{p_j} \sum_{|x_i - x_j| = 1/K} \frac{K^2}{2}(p_i + p_j)(x_j - x_i).$$

For large K, series expansion shows that

$$v_j \approx -\nabla V.$$

Fix $D \subset \mathbb{R}^n$ and consider large K. Let P_K denote the distribution of (X, L) constructed in this example. We recall the following SDE from [1],

$$\begin{aligned} dY_t &= -\nabla V(Y_t)\, dt + S_t\, dt + dB_t\,, \\ dS_t &= -\nabla V(Y_t)\, dt\,, \end{aligned}$$

where B is standard n-dimensional Brownian motion and V is as above. Let P_* denote the distribution of (Y, S). We conjecture that as $K \to \infty$, P_K converge to P_*. Under mild assumptions on V, it is known that the stationary distribution for (Y, S) is the product of the measure $\exp(-V(x))dx$ and the standard Gaussian distribution; see [1].

Example 3.4. We again consider the situation when all p_j's are equal, i.e., $p_j = 1/N$. Consider a C^2 function $V : D \to \mathbb{R}$. We let $a_{ij}(l) = 0$ for $|x_i - x_j| > 1/K$. If $|x_i - x_j| = 1/K$, we let

$$\widetilde{a}_{ij}(l) = \frac{K^2}{2}(1 + (V_j + V_i)(x_j - x_i) \cdot l).$$

We set

$$a_{ij}(l) = \begin{cases} \widetilde{a}_{ij}(l) \vee 0 & \text{if } \widetilde{a}_{ji}(l) > 0, \\ K^2(V_j + V_i)(x_j - x_i) \cdot l & \text{otherwise.} \end{cases} \tag{29}$$

Then $b_{ij}(l) = (1/N)K^2(V_j + V_i)(x_j - x_i) \cdot l$ and

$$v_j = K^2 \sum_{|x_i - x_j| = 1/K} (V_j + V_i)(x_j - x_i).$$

For large K, we have $v_j \approx -2\nabla V$.
Fix $D \subset \mathbb{R}^n$ and consider large K. Let P_K denote the distribution of (X, L) constructed in this example. Consider the following SDE,

$$dY_t = V(Y_t)S_t\, dt + dB_t\ ,$$
$$dS_t = -2\nabla V(Y_t)\, dt\ ,$$

where B is standard n-dimensional Brownian motion and V is as above. Let P_* denote the distribution of (Y, S). We conjecture that as $K \to \infty$, P_K converge to P_*, and that the stationary distribution for (Y, S) is the product of the uniform measure on D and the standard Gaussian distribution.

The next example and conjecture are devoted to examples where the inert drift is related to the curvature of the state space, in a suitable sense.

Example 3.5. In this example, we will identify $\mathbb{R}^2$ and $\mathbb{C}$. The imaginary unit will be denoted by i, as usual. Let $\mathscr{S}$ consist of N points on a circle with radius $r > 0$, $x_j = r\exp(j2\pi i/N)$, $j = 1, \dots, N$. We assume that the p_j's are all equal to each other.
For any pair of adjacent points x_j and x_k, we let

$$\widetilde{a}_{jk}(l) = \frac{N^2}{2}(1 + (x_k - x_j) \cdot l),$$

and

$$a_{jk}(l) = \begin{cases} \widetilde{a}_{jk}(l) \vee 0 & \text{if } \widetilde{a}_{kj}(l) > 0, \\ N^2(x_k - x_j) \cdot l & \text{otherwise,} \end{cases}$$

with the other $a_{kj}(l) = 0$. Then $b_{j(j+1)} = N(x_{j+1} - x_j) \cdot l$, and by (19) we have

$$v_j = N^2(x_{j-1} - x_j) + N^2(x_{j+1} - x_j) = 2N^2(\cos(2\pi/N) - 1)x_j.$$

Note that $v_j \to -4\pi^2 x_j$ when $N \to \infty$.
Let P_N be the distribution of (X, L) constructed above.
Let $\mathscr{C}$ be the circle with radius $r > 0$ and center 0, and let T_y be the projection of $\mathbb{R}^2$ onto the tangent line to $\mathscr{C}$ at $y \in \mathscr{C}$. Consider the following SDE,

$$dY_t = T_{Y_t}(S_t)\, dt + dB_t\ ,$$
$$dS_t = -4\pi^2 Y_t\, dt\ ,$$

where Y takes values in $\mathscr{C}$ and B is Brownian motion on this circle. Let P_* be the distribution of (Y,S). We conjecture that as $N \to \infty$, P_N converge to P_*, and that the stationary distribution for (Y,S) is the product of the uniform measure on the circle and the standard Gaussian distribution.

Conjecture 3.6. We propose a generalization of the conjecture stated in the previous example. We could start with an explicit discrete approximation, just like in other examples discussed so far. The notation would be complicated and the whole procedure would not be illuminating, so we skip the approximation and discuss only the continuous model.
Let $\mathscr{S} \subset \mathbb{R}^n$ be a smooth $(n-1)$-dimensional surface, let T_y be the projection of $\mathbb{R}^n$ onto the tangent space to $\mathscr{S}$ at $y \in \mathscr{S}$, let $\mathbf{n}(y)$ be the inward normal to $\mathscr{S}$ at $y \in \mathscr{S}$, and let ρ be the mean curvature at $y \in \mathscr{S}$. Consider the following SDE,

$$\begin{aligned} dY_t &= T_{Y_t}(S_t)\,dt + dB_t\,, \\ dS_t &= c_0\rho^{-1}\mathbf{n}(Y_t)\,dt\,, \end{aligned}$$

where Y takes values in $\mathscr{S}$ and B is Brownian motion on this surface. We conjecture that for some c_0 depending only on the dimension n, the stationary distribution for (Y,S) exists, is unique and is the product of the uniform measure on $\mathscr{S}$ and the standard Gaussian distribution.

We end with examples of processes that are discrete approximations of continuous-space processes with jumps. It is not hard to construct examples of discrete-space processes that converge in distribution to continuous-space processes with jumps. Stable processes are a popular family of processes with jumps. These and similar examples of processes with jumps allow for jumps of arbitrary size, and this does not mesh well with our model because we assume a finite state space for X. Jump processes confined to a bounded domain have been defined (see, e.g., [2]) but their structure is not very simple. For these technical reasons, we will present approximations to processes similar to the stable process wrapped around a circle.
In both examples, we will identify $\mathbb{R}^2$ and $\mathbb{C}$. Let $\mathscr{S}$ consist of N points on the unit circle D, $x_j = \exp(j2\pi i/N), j = 1,\dots,N$. We assume that the p_j's are all equal to each other, hence, $p_j = 1/N$. In these examples, L takes values in $\mathbb{R}$, not $\mathbb{R}^2$.

Example 3.7. Consider a C^3-function $V : D \to \mathbb{R}$. We write $V_j = V(x_j)$. We define

$$A(j,k) = \begin{cases} 1 & \text{if } x_j \text{ and } x_k \text{ are adjacent on the unit circle,} \\ 0 & \text{otherwise.} \end{cases}$$

For any pair of points x_j and x_k, not necessarily adjacent, we let

$$\widetilde{a}_{jk}(l) = \frac{N^2}{2}(V_k - V_j)A(j,k)l + \frac{1}{N}\sum_{n\in\mathbb{Z}} |(k-j)+nN|^{-1-\alpha},$$

where $\alpha \in (0,2)$. We define

$$a_{jk}(l) = \begin{cases} \widetilde{a}_{jk}(l) \vee 0 & \text{if } \widetilde{a}_{kj}(l) > 0, \\ N^2(V_k - V_j)A(j,k)l & \text{otherwise.} \end{cases}$$

Then

$$b_{jk}(l) = N(V_k - V_j)A(j,k)l$$

and by (19) we have

$$v_k = -N^2 \sum_{j:A(k,j)=1} V_k - V_j.$$

Note that $v_k \to \Delta V(x) = V''(x)$ when $N \to \infty$ and $x_k \to x$.

Let P_N be the distribution of (X,L) constructed above. Let $W(x) = V(e^{ix})$ and let (Z,S) be a Markov process with the state space $\mathbb{R} \times \mathbb{R}$ and the following transition probabilities. The component Z is a jump process with the drift $\nabla W(Z)S = W'(Z)S$. The jump density for the process Z is $\sum_{n\in\mathbb{Z}} |(x-y)+n2\pi|^{-1-\alpha}$. We let $S_t = \int_0^t \Delta W(Z_s)ds$. Let $Y_t = \exp(iZ_t)$ and P_* be the distribution of (Y,S). We conjecture that $P_N \to P_*$ as $N \to \infty$ and the process (Y,S) has the stationary distribution which is the product of the uniform measure on D and the standard normal distribution. The process (Y,S) is a "stable process with index α, with inert drift, wrapped on the unit circle."

Example 3.8. Consider a continuous function $V : D \to \mathbb{R}$ with $\int_D V(x)dx = 0$. Recall the notation $V_j = V(x_j)$. For any pair of points x_j and x_k, not necessarily adjacent, we let

$$\widetilde{a}_{jk}(l) = \frac{1}{N}\left(\frac{1}{2}(V_k - V_j)l + \sum_{n\in\mathbb{Z}} |(k-j)+nN|^{-1-\alpha}\right),$$

where $\alpha \in (0,2)$. We define

$$a_{jk}(l) = \begin{cases} \widetilde{a}_{jk}(l) \vee 0 & \text{if } \widetilde{a}_{kj}(l) > 0, \\ \frac{1}{N}(V_k - V_j)l & \text{otherwise.} \end{cases}$$

Then $b_{jk}(l) = (1/N^2)(V_k - V_j)l$ and by (19) we have

$$v_k = \frac{1}{N} \sum_{1\le j\le N, j\ne k} V_k - V_j.$$

Note that if $\arg x_k \to y$ when $N \to \infty$ then $v_k \to V(e^{iy}) - \int_D V(x)dx = V(e^{iy})$.

Let P_N be the distribution of (X,L) constructed above. Let $W(x) = V(e^{ix})$ and let (Z,S) be a Markov process with the state space $\mathbb{R} \times \mathbb{R}$ and the following transition probabilities. The component Z is a jump process with the jump density $f(x) = (W(x) - W(y))s - \int \sum_{n\in\mathbb{Z}}((x-y) + n2\pi)^{-1-\alpha}$ at time t, given $\{Z_t = y, S_t = s\}$. We let $S_t = \int_0^t W(Z_s)ds$. Let $Y_t = \exp(iZ_t)$ and P_* be the distribution of (Y,S). We conjecture that $P_N \to P_*$ as $N \to \infty$ and the process (Y,S) has the stationary distribution which is the product of the uniform measure on D and the standard normal distribution.

4 Acknowledgments

We are grateful to Zhenqing Chen and Tadeusz Kulczycki for very helpful suggestions.

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