On Wind-Driven Lake Circulation

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ABSTRACT

We consider here the flow induced by applying a wind stress at the surface of an initially quiescent lake. It is assumed that the Ekman number, based on an eddy viscosity, is small, and that the Rossby number is at most of the order of (Ekman number)$^{\frac{1}{2}}$. Under these conditions, which are met in practice, a linear theory is applicable. The linear problem is solved using boundary layer methods. There are essentially five distinct regions: an outer region in which the horizontal velocity is independent of depth, Ekman layers at the upper and lower boundaries, a corner region at the edge of the lake at which the Ekman layers meet, and a shear layer adjacent to the corner region. Study of the Ekman layers provides the equations which hold in the outer and shear layer regions, and consideration of the corner region provides the boundary condition. The outer flow proves to be geostrophic and directed along curves of constant depth. The shear layer is needed to satisfy the boundary condition of zero net outward transport at the edge of the lake. If the wind stress is constant, or, more generally, has zero line integral around curves of constant depth, the transport is confined to the shear layer.

1. Introduction

In recent years considerable attention has been devoted to the problem of calculating wind-driven lake circulations. The standard method employs a one-layer model, with the fluid assumed to be homogeneous. The motion is assumed to be in hydrostatic balance and the free surface at which the wind stress is applied is usually taken to be level. After linearization, the horizontal momentum equations and the continuity equation are integrated over depth and the problem is cast in terms of either the depth-averaged velocity or the transport. A simple form is assumed for the bottom stress, and the equations are integrated numerically. Examples of such calculations can be found in the proceedings of the Conferences of Great Lakes Research (e.g., Murty and Rao, 1970).

Though undoubtedly useful, these studies rest on somewhat shaky foundations. There is a degree of uncertainty about the boundary condition to apply at the edge of the lake, where the depth vanishes and the equations have what for an ordinary differential equation would be called a regular singular point. Horizontal diffusion is almost always neglected, despite its importance near the coast, and the equations are linearized even though the neglected advective terms are usually as large as the bottom friction. Finally, the form of the the bottom friction term, often described as the result of an Ekman layer calculation with constant eddy viscosity, varies from paper to paper. In addition to the theoretical difficulties, the numerical analysis is non-trivial, particularly in the transient case.

The aim of this work is to resolve some of the above difficulties by providing an analytical solution for a constant eddy viscosity model. The problem treated is that of calculating the flow due to a stress suddenly applied at the surface of a quiescent body of fluid. It is assumed that the fluid is homogeneous, that the bottom slope is everywhere finite, and that curves of constant depth are closed. The procedure, which involves the use of matched asymptotic expansions (Van Dyke, 1964) and is similar to that of Greenspan (1968), consists of isolating an outer problem by consideration of the Ekman layers and of the region near the shore where the Ekman layers join up. The problem can be cast in terms of the depth-averaged velocity, and in this formulation the bottom stress terms appear as a given function of the depth-averaged velocity.

The resulting equations can be solved by perturbation methods, but the solution fails to satisfy the condition of zero net outward transport at the shore. Consequently, a vertical shear layer near the shore is necessary. For spatially constant wind stress, the transport is confined to this region. The equations for the outer problem are valid provided that the Ekman number based on depth is small and that the Rossby number is less than or equal to (Ekman number)$^{\frac{1}{2}}$. These conditions are met in practice except for very shallow lakes. The split-up of the outer region into a basically inviscid interior and a shear layer is valid if (Ekman number)$^{\frac{1}{4}}$

$\ll 1$. If this condition is not met, the outer problem must be solved numerically. In this connection, it should be pointed out that a numerical solution of the transient, linear, shallow-water equations carried out for Grand Traverse Bay, Lake Michigan, does confirm the principal conclusion of this work, i.e., that the transport is directed along contours of constant depth, and this result is in accord with observations (Smith, 1972).

During the preparation of this manuscript the author learned of similar work by Birchfield (1972). Birchfield's calculation differs from that presented here in that 1) his results are limited to steady-state conditions, 2) he neglects horizontal mixing, and 3) he resolves the problem of the boundary condition at the coast by taking the boundary to be a vertical wall. In view of these differences, separate publication of the present work seems justifiable. The results of our solution as time $\rightarrow\infty$ and Birchfield's steady-state solution are in agreement outside the coastal boundary layer provided the horizontal eddy coefficient in our solution is set equal to zero.

2. Formulation of outer problem

Let $\mathbf{x}$ and $\mathbf{q}$ denote the horizontal position vector and particle velocity and z and w the vertical coordinate and particle velocity, and let ∇ denote the horizontal gradient. We assume constant horizontal and vertical eddy viscosities ν_H and ν_V and constant Coriolis parameter f_0, and we neglect nonlinear and nonhydrostatic effects. Given these assumptions, the equations governing the flow of a homogeneous fluid are

$$\frac{\partial\mathbf{q}}{\partial t}+f_0\hat{\mathbf{k}}\times\mathbf{q}+\frac{1}{\rho}\nabla p=\nu_H\nabla^2\mathbf{q}+\nu_V\frac{\partial^2\mathbf{q}}{\partial z^2}, \tag{1}$$

$$\frac{\partial p}{\partial z}=-g\rho, \tag{2}$$

$$\nabla\cdot\mathbf{q}+\frac{\partial w}{\partial z}=0. \tag{3}$$

For the flow considered here the fluid is bounded above by a free surface whose undisturbed position is $z=0$ and below by a rigid surface $z=-H(\mathbf{x})$. For $t<0$, $\mathbf{q}=w=0$, and at $t=0$ a stress $\boldsymbol{\tau}(x)$ is applied at the upper surface. We filter out gravity waves by taking the upper surface to be undistorted. Then the boundary conditions for $t>0$ are

$$w=0,\quad \rho\nu_V\frac{\partial\mathbf{q}}{\partial z}=\boldsymbol{\tau}(x), \tag{4}$$

for $z=0$, and $\mathbf{q}=w=0$ for $z=-H(\mathbf{x})$.

Let D and L be characteristic vertical and horizontal length scales and c a characteristic magnitude of $\boldsymbol{\tau}$. Introducing scaled variables through

$$\left.\begin{array}{l}\mathbf{x}=L\mathbf{x}',\quad z=Dz',\quad t=(2/f_0)t',\quad \boldsymbol{\tau}=c\boldsymbol{\tau}',\quad H=DH'\\ \mathbf{q}=U\mathbf{q}',\quad w=U(D/L)w',\quad (p/\rho)+gz=\tfrac{1}{2}f_0ULp'\end{array}\right\}, \tag{5}$$

where

$$U=\frac{c}{\rho}(2/f_0\nu_V)^{\frac{1}{2}}, \tag{6}$$

and omitting the primes, we obtain the dimensionless equations

$$\frac{\partial\mathbf{q}}{\partial t}+2\hat{\mathbf{k}}\times\mathbf{q}+\nabla p=\mathrm{E}\left(\gamma\nabla^2\mathbf{q}+\frac{\partial^2\mathbf{q}}{\partial z^2}\right), \tag{7}$$

$$\frac{\partial p}{\partial z}=0, \tag{8}$$

$$\nabla\cdot\mathbf{q}+\frac{\partial w}{\partial z}=0, \tag{9}$$

to be solved subject to

$$w=0,\quad \mathrm{E}^{\frac{1}{2}}\frac{\partial\mathbf{q}}{\partial z}=\boldsymbol{\tau}(\mathbf{x}), \tag{10}$$

for $z=0$, $\mathbf{q}=w=0$ for $z=-H(\mathbf{x})$, and $\mathbf{q}=w=0$ for $t=0$. The parameters are

$$\mathrm{E}=2\nu_V/(f_0D^2),\quad \gamma=(D^2\nu_H)/(L^2\nu_V), \tag{11}$$

and are, respectively, the Ekman number and a scaled ratio of the eddy viscosities.

We consider here the case $\mathrm{E}\ll 1$, $\gamma\leqslant O(1)$, and we assume also that the bottom slope $|\nabla H|$ is finite everywhere. Under these conditions we can consider the flow domain to consist of an outer region away from the boundaries in which the viscous force can be neglected. Ekman layers at the upper and lower boundaries, and a corner region near the coast at which the Ekman layers meet (see Fig. 1). As it develops, there exists a shear layer adjacent to the corner region, but this can be considered at present as part of the outer region. Our aim in this section is to derive the equations and boundary condition governing the flow in the outer region.

In treating the outer problem we neglect the viscous force on the right side of (7) and note that in the outer region the horizontal velocity is independent of z and the vertical velocity is linear in z. Accordingly, we seek

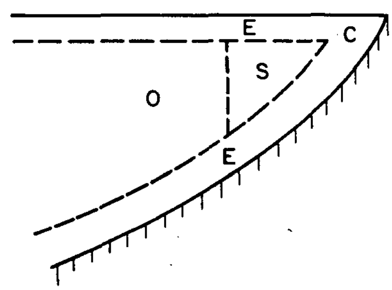

FIG. 1. Schematic drawing showing Ekman layers (E), corner region (C), shear layer (S), and outer region (O).

an outer solution of the form $\mathbf{q}=\mathbf{v}(\mathbf{x},t)$, $p=\theta(\mathbf{x},t)$, and

$$w(\mathbf{x},z,t)=a(\mathbf{x},t)+b(\mathbf{x},t)z. \tag{12}$$

To determine a and b we write the transport outside the corner region in the form

$$\int_{-H}^{0} \mathbf{q}dz=H\mathbf{v}+\mathrm{E}^{\frac{1}{2}}(\mathbf{f}-\mathbf{g}), \tag{13}$$

where $H\mathbf{v}$ is the transport in the outer region, $\mathrm{E}^{\frac{1}{2}}\mathbf{f}$ is the transport associated with the high-vorticity flow in the upper Ekman layer, and $-\mathrm{E}^{\frac{1}{2}}\mathbf{g}$ is the corresponding transport in the lower Ekman layer. Conservation of mass implies that at the edges of the Ekman layers the outer velocity obeys the conditions

$$w(\mathbf{x},0,t)=\mathrm{E}^{\frac{1}{2}}\nabla\cdot\mathbf{f},\quad w(\mathbf{x},-H,t)=-\mathbf{v}\cdot\nabla H+\mathrm{E}^{\frac{1}{2}}\nabla\cdot\mathbf{g}, \tag{14}$$

from which we obtain

$$a=\mathrm{E}^{\frac{1}{2}}\nabla\cdot\mathbf{f},\quad Hb=\mathbf{v}\cdot\nabla H+\mathrm{E}^{\frac{1}{2}}\nabla\cdot(\mathbf{f}-\mathbf{g}). \tag{15}$$

Hence the equations governing the outer flow are

$$\frac{\partial\mathbf{v}}{\partial t}+2\hat{\mathbf{k}}\times\mathbf{v}+\nabla\theta=0, \tag{16}$$

$$\nabla\cdot[H\mathbf{v}+\mathrm{E}^{\frac{1}{2}}(\mathbf{f}-\mathbf{g})]=0. \tag{17}$$

The last equation states that

$$\nabla\cdot\left(\int_{H}^{0} \mathbf{q}dz\right)=0, \tag{18}$$

a relation which can be derived independently by integrating (9) over depth.

In the corner region the flow variables must vary on the length scale $\mathrm{E}^{\frac{1}{2}}$, the Ekman layer thickness, both in the vertical direction and in the horizontal direction perpendicular to the coast. The equations governing the flow in the corner region are easily obtained and the necessary boundary condition on $\mathbf{v}$ derived by integrating the corner region form of the continuity equation over depth. The result implies that in the corner region

$$\int_{-H}^{0} \hat{\mathbf{n}}\cdot\mathbf{q}dz=\mathrm{O}(\mathrm{E}^{\frac{1}{2}}), \tag{19}$$

where

$$\hat{\mathbf{n}}=-\nabla H/|\nabla H| \tag{20}$$

is the normal in the horizontal plane to the curves $H=$constant. The physical content of (19) is that the transport normal to the coast vanishes throughout the corner region, subject to the error estimate. Hence, from (13) and (19), the boundary condition at the coast is

$$\lim_{H\to 0} \hat{\mathbf{n}}\cdot[H\mathbf{v}+\mathrm{E}^{\frac{1}{2}}(\mathbf{f}-\mathbf{g})]=0, \tag{21}$$

with an $\mathrm{O}(\mathrm{E}^{\frac{1}{2}})$ error.

We now turn to study of the Ekman layers with the aim of determining explicit expressions for the Ekman layer transports. In treating the Ekman layers we take the coordinates to be $(\mathbf{x},\zeta)$, where $\zeta=-\mathrm{E}^{-\frac{1}{2}}z$ for the upper layer, and $(\mathbf{x},\eta)$, where $\eta=\mathrm{E}^{-\frac{1}{2}}(z+H)$ for the lower layer, and expand the dependent variables in powers of $\mathrm{E}^{\frac{1}{2}}$ as in

$$\mathbf{q}=\mathbf{q}_0+\mathrm{E}^{\frac{1}{2}}\mathbf{q}_1+\cdots. \tag{22}$$

The Ekman layer equations are solved through use of the Laplace transform, denoted here by an uppercase letter as in

$$\mathbf{Q}=\mathcal{L}\,\mathbf{q}=\int_0^\infty e^{-st}\mathbf{q}dt. \tag{23}$$

Omitting the analysis, which is straightforward but tedious, we obtain the solutions

$$\mathbf{q}_0^U=\mathbf{v}_0+\mathrm{Re}\,\mathcal{L}^{-1}\left\{\left[\frac{\boldsymbol{\tau}+i\hat{\mathbf{k}}\times\boldsymbol{\tau}}{s(s-2i)^{\frac{1}{2}}}\right]\exp[-(s-2i)^{\frac{1}{2}}\zeta]\right\}, \tag{24}$$

$$w_0^U=0,\quad w_1^U=\nabla\cdot\left(\int_0^\zeta \mathbf{q}_0^U d\zeta'\right),$$

$$\mathbf{q}_0^L=\mathbf{v}_0-\mathrm{Re}\,\mathcal{L}^{-1}\{(\mathbf{V}_0+i\hat{\mathbf{k}}\times\mathbf{V}_0)\times\exp[-(s-2i)^{\frac{1}{2}}\eta/A]\}, \tag{25}$$

$$w_0^L=-\mathbf{q}_0^L\cdot\nabla H,\quad w_1^L=-\mathbf{q}_1^L\cdot\nabla H-\nabla\cdot\left(\int_0^\eta \mathbf{q}_0^L d\eta'\right),$$

where the superscripts U and L refer to the upper and lower Ekman layers, Re means "real part of," and

$$A=(1+\gamma|\nabla H|^2)^{\frac{1}{2}}. \tag{26}$$

In accord with the previous definitions of $\mathbf{f}$ and $\mathbf{g}$, we have

$$\left.\begin{aligned}\mathbf{f}&=\int_0^\infty(\mathbf{q}_0^U-\mathbf{v}_0)d\zeta\\&=\tfrac{1}{2}[\boldsymbol{\tau}\sin 2t+(\hat{\mathbf{k}}\times\boldsymbol{\tau})(\cos 2t-1)]\\ \mathbf{g}&=\mathbf{g}[\mathbf{v}_0]=-\int_0^\infty(\mathbf{q}_0^L-\mathbf{v}_0)d\eta\\&=\mathrm{Re}\,A\,\mathcal{L}^{-1}[(\mathbf{V}_0+i\hat{\mathbf{k}}\times\mathbf{V}_0)/(s-2i)^{\frac{1}{2}}]\end{aligned}\right\}. \tag{27}$$

In writing the governing equations for the outer problem we can approximate $\mathbf{g}[\mathbf{v}_0]$ by $\mathbf{g}[\mathbf{v}]$, with a resulting error in the equations of $\mathrm{O}(E^{\frac{1}{2}})$. Consequently, the equations for the outer problem, correct to order $\mathrm{E}^{\frac{1}{2}}$, are

$$\frac{\partial\mathbf{v}}{\partial t}+2\hat{\mathbf{k}}\times\mathbf{v}+\nabla\theta=0\ [\text{as given in (16)}],$$

$$\nabla\cdot\{H\mathbf{v}+\mathrm{E}^{\frac{1}{2}}(\mathbf{f}-\mathbf{g}[\mathbf{v}])\}=0, \tag{28}$$

and the boundary condition is

$$\lim_{H\to 0} \hat{\mathbf{n}}\cdot\{H\mathbf{v}+\mathrm{E}^{\frac{1}{2}}(\mathbf{f}-\mathbf{g}[\mathbf{v}])\}=0, \qquad (29)$$

with $\mathbf{f}$ and $\mathbf{g}$ given by (27). In this formulation the role of the Ekman layer analysis is to provide explicit expressions for the transports $\mathrm{E}^{\frac{1}{2}}\mathbf{f}$ in the upper Ekman layer and $-\mathrm{E}^{\frac{1}{2}}\mathbf{g}$ in the lower layer.

Though the outer problem can be worked out in terms of $\mathbf{v}$, there are advantages to use of the depth-averaged velocity

$$\mathbf{u}=\frac{1}{H}\int_{-H}^{0}\mathbf{q}dz=\mathbf{v}+\frac{\mathrm{E}^{\frac{1}{2}}}{H}(\mathbf{f}-\mathbf{g}[\mathbf{v}]) \qquad (30)$$

as a dependent variable. Solving for $\mathbf{v}$ in terms of $\mathbf{u}$, again neglecting $\mathrm{O}(\mathrm{E}^{\frac{1}{2}})$ terms, yields the result

$$\mathbf{v}=\mathbf{u}+\frac{\mathrm{E}^{\frac{1}{2}}}{H}(\mathbf{g}[\mathbf{u}]-\mathbf{f}), \qquad (31)$$

and substitution into the momentum and continuity equations yields

$$\frac{\partial\mathbf{u}}{\partial t}+2\hat{\mathbf{k}}\times\mathbf{u}+\nabla\theta =\frac{\mathrm{E}^{\frac{1}{2}}}{H}\{\boldsymbol{\tau}-\tfrac{1}{2}A\ \mathcal{L}^{-1}[(s-2i)^{\frac{1}{2}}(\mathbf{U}+i\hat{\mathbf{k}}\times\mathbf{U}) +(s+2i)^{\frac{1}{2}}(\mathbf{U}-i\hat{\mathbf{k}}\times\mathbf{U})]\}, \qquad (32)$$

$$\nabla\cdot(H\mathbf{u})=0, \qquad (33)$$

with boundary condition

$$\lim_{H\to 0}\hat{\mathbf{n}}\cdot(H\mathbf{u})=0, \qquad (34)$$

and initial condition $\mathbf{u}=0$ at $t=0$. For this formulation the Ekman layer analysis serves to provide the explicit expression for the bottom stress on the right side of (32).

The sinusoidal oscillation at the inertial frequency in the expression for $\mathbf{f}$ does not appear in the solution for the total transport because of cancellation, but nevertheless deserves comment. A detailed analysis of the limit process used in the derivation shows that these oscillations decay on the diffusion time scale, $t\sim E^{-1}$, so that as $t\to\infty$ the transport settles down to its steady value, $-\frac{1}{2}\mathrm{E}^{\frac{1}{2}}(\hat{\mathbf{k}}\times\boldsymbol{\tau})$. It is therefore incorrect to state, as some authors have done, that the transport in the upper layer is oscillatory in time.

3. Solution

From previous work on rotating fluids we expect that the solution consists of a geostrophic mode, a function of $\mathbf{x}$ and a slow time $T=\mathrm{E}^{\frac{1}{2}}t$, and inertial modes with time dependence of the form

$$h(t)=f(T)\exp(i\omega t),\quad |\omega|<2, \qquad (35)$$

where h stands for any of the flow variables. Thus, if $F(s)$ is the Laplace transform of $f(t)$, $\mathrm{E}^{-\frac{1}{2}}F[\mathrm{E}^{-\frac{1}{2}}(s-i\omega)]$ is the Laplace transform of $h(t)$, and the inverse transforms which occur in the expression for the bottom stress are of the form

$$I_{\pm}=\mathcal{L}^{-1}(s\pm 2i)^{\frac{1}{2}}\mathcal{L}\,h =\frac{\mathrm{E}^{-\frac{1}{2}}}{2\pi i}\int(s\pm 2i)^{\frac{1}{2}}F[\mathrm{E}^{-\frac{1}{2}}(s-i\omega)]e^{st}ds, \qquad (36)$$

where the integral is taken along a Bromwich contour in the complex s plane. We assume, subject to later verification, that $F(s)$ is meromorphic, with singularities inside a circle of $\mathrm{O}(1)$ radius about $s=0$, and that $F(s)=\mathrm{O}(|s|^{-2})$ as $|s|\to\infty$. Hence $F[\mathrm{E}^{-\frac{1}{2}}(s-i\omega)]$ has its singularities in a circle of $\mathrm{O}(\mathrm{E}^{\frac{1}{2}})$ radius about $s=i\omega$ and behaves like $\mathrm{E}|s-i\omega|^{-2}$ for $|s-i\omega|\geqslant\mathrm{O}(1)$. It follows that there are two contributions to $I_{\pm}$, a loop integral about the branch cut for $(s\pm 2i)^{\frac{1}{2}}$, and a sum of residues at the poles of $F[\mathrm{E}^{-\frac{1}{2}}(s-i\omega)]$. The loop integral is $\mathrm{O}(\mathrm{E}^{\frac{1}{2}})$ in magnitude and can be ignored, and in the residue calculation $(s\pm 2i)^{\frac{1}{2}}$ can be approximated by $(i\omega\pm 2i)^{\frac{1}{2}}$, again with an $\mathrm{O}(\mathrm{E}^{\frac{1}{2}})$ error. Hence

$$I_{\pm}=(2\pm\omega)^{\frac{1}{2}}\exp[i(\omega t\pm\pi/4)]f(T)+\mathrm{O}(\mathrm{E}^{\frac{1}{2}}). \qquad (37)$$

The inertial modes must be included if $\mathbf{u}$ satisfies an arbitrary initial condition. In the present case, for which $\mathbf{u}=0$ at $t=0$, the inertial modes are not needed, since we can find a geostrophic solution which vanishes to lowest order at $t=0$. Hence we assume $\mathbf{u}=\mathbf{u}(\mathbf{x},T)$, $\theta=\theta(\mathbf{x},T)$, and accordingly set $\omega=0$ in (37). The resulting equations for $\mathbf{u}$ and θ are

$$\mathrm{E}^{\frac{1}{2}}\frac{\partial\mathbf{u}}{\partial T}+2\hat{\mathbf{k}}\times\mathbf{u}+\nabla\theta=\frac{\mathrm{E}^{\frac{1}{2}}}{H}[\boldsymbol{\tau}-A(\mathbf{u}+\hat{\mathbf{k}}\times\mathbf{u})], \qquad (38)$$

$$\nabla\cdot(H\mathbf{u})=0\ [\text{as given in (33)}],$$

and $\mathbf{v}$ is given in terms of $\mathbf{u}$ by

$$\mathbf{v}=\mathbf{u}+\frac{\mathrm{E}^{\frac{1}{2}}}{2H}[A(\mathbf{u}-\hat{\mathbf{k}}\times\mathbf{u})-2\mathbf{f}]+\mathrm{O}(\mathrm{E}^{\frac{1}{2}}). \qquad (39)$$

The solution is obtained by expanding $\mathbf{u}$ and θ in powers of $\mathrm{E}^{\frac{1}{2}}$, as in (22). To lowest order,

$$2\hat{\mathbf{k}}\times\mathbf{u}_0+\nabla\theta_0=0, \qquad (40)$$

$$\nabla\cdot(H\mathbf{u}_0)=0, \qquad (41)$$

and it follows that

$$\theta_0=\theta_0(H,T),\quad \mathbf{u}_0=\tfrac{1}{2}(\hat{\mathbf{k}}\times\nabla H)\frac{\partial\theta_0}{\partial H}. \qquad (42)$$

We can now eliminate $\mathbf{u}_1$ and θ_1 from the $\mathrm{O}(\mathrm{E}^{\frac{1}{2}})$ equations,

$$2\hat{\mathbf{k}}\times\mathbf{u}_1+\nabla\theta_1=-\frac{\partial\mathbf{u}_0}{\partial T}+\frac{1}{H}[\boldsymbol{\tau}-A(\mathbf{u}_0+\hat{\mathbf{k}}\times\mathbf{u}_0)],\quad(43)$$

$$\nabla\cdot(H\mathbf{u}_1)=0,\quad(44)$$

by taking the dot product of $Hd\mathbf{s}$ and the momentum equation, where $d\mathbf{s}$ is a directed segment of arc along a curve $H=$constant, and integrating around the curve. The result is

$$\frac{\partial\theta_0}{\partial H}=-2G(H)\{1-\exp[-C(H)T/H]\},\quad(45)$$

where

$$\left.\begin{aligned}G(H)&=\oint\boldsymbol{\tau}\cdot d\mathbf{s}\Big/\oint A|\nabla H|ds\\ C(H)&=\oint A|\nabla H|ds\Big/\oint|\nabla H|ds\end{aligned}\right\},\quad(46)$$

and the integrals are taken around a curve of depth H. Hence, if $\hat{\mathbf{s}}$ is the unit vector tangent to the curve,

$$\mathbf{u}_0=\hat{\mathbf{s}}|\nabla H|G(H)\{1-\exp[-C(H)T/H]\}.\quad(47)$$

Except for a shear layer term to be described shortly, this is the O(1) solution for $\mathbf{u}$. This type of result, stating that the O(1) solution for the transport is directed along the depth contours and varies on the spin-up time scale, is familiar from previous work in rotating fluids. The new feature here is the explicit form of the expression for the model under consideration, that of shallow water theory with wind stress as the driving force. It should be noted that for a constant wind stress the transport vanishes. In this case the transport is confined to a shear layer near the coast, as will be seen shortly.

The cross-contour component of $\mathbf{u}_1$ can be calculated and behaves as

$$\mathbf{n}\cdot\mathbf{u}_1\sim\frac{1}{2H}\{\hat{\mathbf{s}}\cdot\boldsymbol{\tau}-G(H)|\nabla H|\times[A+(C-A)\exp(-CT/H)]\},\quad(48)$$

as $H\rightarrow 0$. Hence the $O(E^{1/2})$ term in the solution for $\mathbf{u}$ does not satisfy the boundary condition of vanishing outward transport at the coast; consequently, the ordinary perturbation solution given above is locally invalid. This implies the existence of a shear layer near the coast in which the outward transport changes rapidly from its value just outside the shear layer to zero in the corner region. A treatment of the shear layer is necessary to complete the solution, and will be given below.

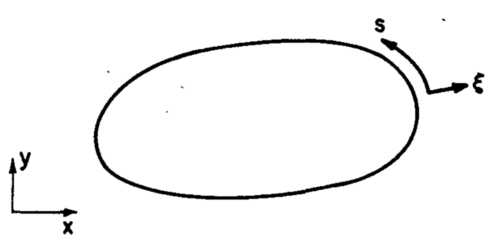

FIG. 2. Coordinate system used in describing shear layer.

In studying the shear layer it is convenient to describe the curve $H=0$ parametrically by $\mathbf{x}=\mathbf{r}(s)$, where s is arc length along the curve, and to let ξ denote distance out from the coast along a normal (see Fig. 2). The horizontal position vector to any point in the shear layer can be written as

$$\mathbf{x}=\mathbf{r}(s)-\xi\frac{\partial}{\partial s}[\hat{\mathbf{k}}\times\mathbf{r}(s)],\quad(49)$$

and (49) defines s and ξ as orthogonal coordinates in the $\mathbf{x}$ plane. Letting u and v be velocity components in the directions of increase of ξ and s and writing out the equations of motion in component form yields, by trial and error, the result that the shear layer thickness is $E^{1/4}$. It follows that the depth in the shear layer region is $O(E^{1/4})$ and that the local spin-up time is $O(E^{-1/4})$. In accord with these results we introduce new scaled variables

$$r=E^{-1/4}\xi,\quad \bar{t}=E^{1/4}t,\quad(50)$$

and obtain the governing equations for the shear layer in the form

$$E^{1/4}\frac{\partial u}{\partial\bar{t}}-2v+E^{-1/4}\frac{\partial\theta}{\partial r}=\frac{E^{1/2}}{H}[\tau_\xi-A(u-v)],\quad(51)$$

$$E^{1/4}\frac{\partial v}{\partial\bar{t}}+2u+\frac{1}{(1+E^{1/4}\kappa r)}\frac{\partial\theta}{\partial s}=\frac{E^{1/2}}{H}[\tau_s-A(u+v)],\quad(52)$$

$$E^{-1/4}\frac{\partial}{\partial r}[H(1+E^{1/4}\kappa r)u]+\frac{\partial}{\partial s}[Hv]=0,\quad(53)$$

where τ_ξ and τ_s are the components of $\boldsymbol{\tau}$, and κ is the curvature of the curve $H=0$. In the shear layer

$$H=-E^{1/4}\Delta(s)r+O(E^{1/2}),\quad(54)$$

where $\Delta(s)=|\nabla H|$ evaluated at the coast.

To solve the above system we expand the dependent variables in powers of $E^{1/4}$, as in

$$\mathbf{u}=\mathbf{u}_0+E^{1/4}\mathbf{u}_{1/2}+E^{1/2}\mathbf{u}_1+\cdots,\quad(55)$$

where the subscript denotes the power of $E^{1/4}$, substitute into the governing equations, and equate powers of $E^{1/4}$. From (51) and (53) there follows

$$\frac{\partial\theta_0}{\partial r}=\frac{\partial(ru_0)}{\partial r}=0,\quad(56)$$

and we now need to apply matching and boundary conditions. At the outer edge of the shear layer θ_0 is independent of s, from (42), and at $r=0$, $ru_0=0$ from (34). Hence $u_0=0$ and θ_0 is independent of s. It follows that the governing equations for determining the lowest order non-zero velocity components are

$$-2v_0+\frac{\partial\theta_1}{\partial r}=0, \tag{57}$$

$$\frac{\partial v_0}{\partial \bar{t}}+2u_{\frac{1}{2}}+\frac{\partial\theta_1}{\partial s}=-\frac{1}{\Delta(s)r}[\tau_s-Av_0], \tag{58}$$

$$\frac{\partial}{\partial r}[\Delta(s)ru_{\frac{1}{2}}]+\frac{\partial}{\partial s}[\Delta(s)rv_0]=0, \tag{59}$$

with τ_s and A evaluated at the coast. The matching condition, from (47), is

$$v_0\to G(0)\Delta(s)\{1-\exp[C(0)\bar{t}/\Delta(s)r]\}, \tag{60}$$

as $r\to-\infty$, and the boundary condition requires that $ru_{\frac{1}{2}}=0$ at $r=0$.

If we introduce a transport streamfunction Ψ through

$$H\mathbf{u}=\nabla\times\hat{\mathbf{k}}\Psi, \tag{61}$$

and note that $\Psi=O(E^{\frac{1}{2}})$ in the shear layer, we find that

$$\Delta rv_0=\frac{\partial\psi}{\partial r},\quad \Delta ru_{\frac{1}{2}}=-\frac{\partial\psi}{\partial s}, \tag{62}$$

where

$$\psi=E^{-\frac{1}{2}}\Psi. \tag{63}$$

It is convenient to employ the new independent variables

$$h=-\Delta(s)r,\quad \varphi=\sigma\int_0^s A\Delta ds, \tag{64}$$

where σ is a constant defined by

$$\sigma\oint A\Delta ds=2\pi, \tag{65}$$

with the integral taken around the curve $H=0$. In terms of the new independent variables ψ solves

$$\frac{1}{A}\frac{\partial^2}{\partial h\partial\bar{t}}\left(\frac{1}{h}\frac{\partial\psi}{\partial h}\right)+\frac{\partial}{\partial h}\left(\frac{1}{h^2}\frac{\partial\psi}{\partial h}\right)-\frac{2\sigma}{h^2}\frac{\partial\psi}{\partial\varphi}=-\frac{\tau_s}{A\Delta h^2}, \tag{66}$$

with boundary condition $\psi=0$ at $h=0$ and matching condition

$$\frac{\partial\psi}{\partial h}\to hG(0)[1-\exp(-C(0)\bar{t}/h)], \tag{67}$$

as $h\to\infty$. The new independent variables satisfy $h\geqslant 0$, $0\leqslant\varphi\leqslant 2\pi$, and it may be noted that

$$G(0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\tau_s}{A\Delta}d\varphi. \tag{68}$$

The equation for ψ is non-separable and hence intractible for the transient case. However, for $\bar{t}\gg 1$, a steady-state solution is readily obtained. Expanding $(\tau_s/A\Delta)$ in Fourier series,

$$\tau_s/A\Delta=\sum c_n e^{in\varphi}, \tag{69}$$

and noting that $G(0)=c_0$, we obtain the solution

$$\psi=\tfrac{1}{2}c_0h^2+\sum{}'\,(c_n/\nu_n^2)[1-(1+\nu_nh)e^{-\nu_nh}]e^{in\varphi}, \tag{70}$$

where $\nu_n=(2in\sigma)^{\frac{1}{3}}$, with positive real part, and the prime on the summation sign denotes omission of the term $n=0$. To lowest order, the velocity component parallel to the depth contours is given by

$$v_0=\Delta[c_0+\sum{}'\,c_ne^{(in\varphi-\nu_nh)}], \tag{71}$$

and has the value

$$v_0=\tau_s/A \tag{72}$$

at $H=0$. We note again that if $\boldsymbol{\tau}$ is constant, or, more generally, has zero curl, the transport is confined to the shear layer region.

Combining previous results, we find that to lowest order the composite solution for the depth-averaged velocity in the limit $t\to\infty$ is

$$\mathbf{u}(\mathbf{x},\infty)=\hat{\mathbf{s}}[|\nabla H|G(H)+\Delta\sum{}'\,c_ne^{(in\varphi-\nu_nh)}], \tag{73}$$

while the ultimate steady-state surface current, from (24) and (39), is

$$\mathbf{q}^U(\mathbf{x},0,\infty)=\tfrac{1}{2}(\boldsymbol{\tau}-\hat{\mathbf{k}}\times\boldsymbol{\tau})+\mathbf{u}(\mathbf{x},\infty), \tag{74}$$

with $\mathbf{u}(\mathbf{x},\infty)$ given by (73). Also, again using (24), the transient surface current in the region outside the shear layer is

$$\mathbf{q}^U(\mathbf{x},0,t)=C[(4t/\pi)^{\frac{1}{2}}]\boldsymbol{\tau}-S[(4t/\pi)^{\frac{1}{2}}](\hat{\mathbf{k}}\times\boldsymbol{\tau})+\mathbf{u}_0, \tag{75}$$

where $\mathbf{u}_0$ is given by (47) and C and S are the Fresnel integrals defined by

$$C(t)+iS(t)=\int_0^t\exp(i\pi s^2/2)ds. \tag{76}$$

These results can be generalized to include winds of limited duration. If the stress is taken to be $\boldsymbol{\tau}(\mathbf{x})P(t)$, where $P=0$ for $t<0$, and if $\mathbf{u}^{(1)}$ is the depth-averaged velocity computed previously, then, by Duhamel's theorem,

$$\mathbf{u}(\mathbf{x},t)=\frac{\partial}{\partial t}\int_0^t\mathbf{u}^{(1)}(\mathbf{x},t-t')P(t')dt'. \tag{77}$$

If, for example,

$$\boldsymbol{\tau}=\begin{cases}0, & t<0\\ \boldsymbol{\tau}(\mathbf{x}), & 0<t<t_1\\ 0, & t_1<t\end{cases} \tag{78}$$

for some $t_1>0$, then

$$\mathbf{u}=\mathbf{u}^{(1)}(\mathbf{x},t)H(t)-\mathbf{u}^{(1)}(\mathbf{x},t-t_1)H(t-t_1), \tag{79}$$

where $H(t)$ is the Heaviside step function. The surface velocities can be computed in a similar way.

The preceding analysis has been carried out under the conditions $E^{\frac{1}{4}} \ll 1$, $\gamma \leqslant 1$, and the assumption that advection, the β-effect, and nonhydrostatic effects can be neglected. These neglects are permissible provided that the aspect ratio δ, the Rossby number Ro, and a scaled gradient of the Coriolis parameter b, defined here by

$$\delta = D/L, \quad \mathrm{Ro} = 2U/(f_0 L), \quad b = \beta L/f_0, \tag{80}$$

are all of order $E^{\frac{1}{2}}$ or smaller. Given this condition, it can be shown that the only difference in the analysis due to terms hitherto neglected arises in the $O(E^{\frac{1}{2}})$ momentum equation (43) which contains the additional terms

$$\epsilon_1 \nabla(\tfrac{1}{2}\mathbf{u}_0 \cdot \mathbf{u}_0) + [\epsilon_1 \hat{\mathbf{k}} \cdot \nabla \times \mathbf{u}_0 + 2\epsilon_2 y](\hat{\mathbf{k}} \times \mathbf{u}_0),$$

where

$$\epsilon_1 = \mathrm{Ro}E^{-\frac{1}{2}} \leqslant 1, \quad \epsilon_2 = bE^{-\frac{1}{2}} \leqslant 1, \tag{81}$$

and y is the north-south coordinate. In the elimination of $\mathbf{u}_1$ and θ_1 the momentum equation is multiplied scalarly by $Hd\mathbf{s}$ and integrated around a curve of constant depth. Since $(\hat{\mathbf{k}} \times \mathbf{u}_0)$ is perpendicular to $d\mathbf{s}$, the only extra term which survives is

$$\epsilon_1 \oint \nabla(\tfrac{1}{2}\mathbf{u}_0 \cdot \mathbf{u}_0) \cdot Hd\mathbf{s},$$

and this term integrates to zero when integrated around a curve of constant depth. Consequently, advection and the β-effect play no role provided (81) is satisfied.

In checking the usefulness of the theory as applied to flow in the Great Lakes, we have assumed eddy viscosities $\nu_V = 30$ cm^2 sec^{-1}, $\nu_H = 3 \times 10^4$ cm^2 sec^{-1} (Huang, 1972), and a wind stress expressed in dimensional terms by

$$\boldsymbol{\tau} = \rho_{\mathrm{air}} C_D (\bar{u}_{10})^2 \hat{\mathbf{u}}_{10}, \tag{82}$$

where C_D is a drag coefficient, $\bar{u}_{10}$ is the mean wind speed at 10 m above the water surface, and $\hat{\mathbf{u}}_{10}$ a unit vector in the direction of the mean wind. For the conditions $\rho_{\mathrm{air}} = 1.25 \times 10^{-3}$ gm cm^{-3}, $C_D = 10^{-3}$, $\bar{u}_{10} = 8$ m sec^{-1}, we obtain $U = 20$ cm sec^{-1} as the characteristic magnitude for the velocity. For each of the Great Lakes we have taken D as the maximum depth and L as the width in computing the values of the dimensionless parameters. For all cases

$$\delta E^{-\frac{1}{2}} < 0.2, \quad \gamma < 0.1.$$

TABLE 1. Parameters for the Great Lakes.

Lake	$E^{\frac{1}{2}}$	$E^{\frac{1}{4}}$	ϵ_1	ϵ_2
Superior	0.02	0.14	0.16	1.32
Michigan	0.03	0.17	0.15	0.69
Ontario	0.03	0.18	0.28	0.26
Huron	0.03	0.18	0.14	0.48
Erie	0.12	0.35	0.07	0.08

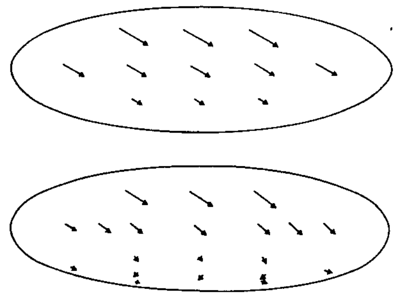

FIG. 3. Surface currents at one-fourth pendulum day (upper) and three pendulum days (lower), for stress $\boldsymbol{\tau} = \hat{\mathbf{i}}(1+y)$.

The other parameters of interest are given in Table 1. On the basis of these results, we feel that the theory is applicable for all of the Great Lakes except for Lake Erie provided that the constant eddy coefficient assumption is appropriate and that density variations are unimportant.

As an illustration of the theory, we have computed the velocity for the stress

$$\boldsymbol{\tau} = \hat{\mathbf{i}}(1+y)H(t) \tag{83}$$

for an idealized lake with depth

$$H(x,y) = 1 - (x^2/9) - y^2, \tag{84}$$

under the conditions $E^{\frac{1}{2}} = 0.03$ and $\gamma = 0$. The surface current distribution is shown in Fig. 3 and the transports in the Ekman layers, averaged over an inertial period, in Fig. 4.

4. Concluding remarks

For mathematical convenience we have filtered out gravity waves, and this is one of the defects of the model. However, numerical integrations for the wind driven circulation in Grand Traverse Bay (Smith, 1972), with gravity waves included, indicate that the current patterns are not strongly affected by the presence of the waves, and consequently we consider the filtering approximation to be acceptable.

An interesting feature of the solution is the behavior of the transport at the inner edge of the shear layer, $r = 0$. From (58), it can be seen that at $r = 0$ Eq. (72) is valid for all $t > 0$, and from (27) that the outward transports at $r = 0$, averaged over an inertial period, are $\frac{1}{2}E^{\frac{1}{2}}Av_0$ in the upper Ekman layer, zero in the outer

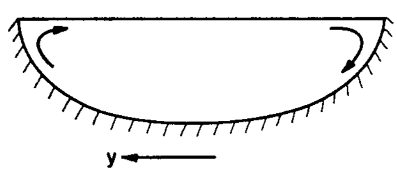

FIG. 4. Ekman layer transports at north and south shores for stress $\boldsymbol{\tau} = \hat{\mathbf{i}}(1+y)$.

region, and $-\frac{1}{2}E^{\frac{1}{2}}Av_0$ in the lower Ekman layer. Consequently, there is a secondary circulation near the corner which obeys a right-hand screw rule, with the point of the screw in the direction of the horizontal transport parallel to $H=0$ and the circulation in the sense of rotation of the screw (see Fig. 4). This is of interest with regard to theories of upwelling.

It should be noted that except for a slow viscous decay the vertical velocity at the edge of the upper Ekman layer oscillates with the inertial frequency. For a confined homogeneous fluid the inertial frequency is not a resonant frequency and forcing at the inertial frequency does not induce a large response. However, when the fluid is stratified there is a class of internal waves with frequencies close to the inertial frequency, and it is likely that the small forcing at this frequency does induce a resonant response.

The major defect of this work lies in the use of constant eddy coefficients. We believe that for a more realistic model, with variable eddy viscosity, the same type of boundary layer behavior would be found, i.e., Ekman layers, a corner region, and a shear layer near the corner region would be present. The quantitative behavior of the solution for a variable eddy coefficient model would, of course, be somewhat different, but the qualitative behavior should be similar to that calculated here.

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