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Invited review

ON RADIATIVE TRANSFER PROBLEMS AND THEIR SOLUTION

O. Atanacković-Vukmanović

Department of Astronomy, Faculty of Mathematics, University of Belgrade
Studentski trg 16, 11000 Belgrade, Serbia and Montenegro

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SUMMARY: A survey of the main stages in the solution of radiative transfer (RT) problems is given with special emphasis on recent approaches and powerful numerical techniques developed to solve non-LTE RT problems.

Key words. Methods: numerical - Radiative transfer - Stars: atmospheres

1. INTRODUCTION

Radiative transfer (RT) is the underlying physical phenomenon in many astrophysical problems. Its good understanding is important as the radiation field not only affects the structure and dynamics of the medium it propagates through, but also, being practically the only source of information about distant celestial objects, serves as an important diagnostic tool in establishing their properties. We shall focus here on radiative transfer problems relevant to stellar atmospheres as the most studied and among the best understood astrophysical objects. However, these problems and their solutions can be easily applied or extended to the study of radiative transfer in many other astrophysical systems.

The radiation released in nuclear reactions in deep stellar interior propagates through the outermost layers of the star into free space. In stellar interior its energy distribution is almost the equilibrium one. Through a very large number of interactions with the gas on its way out, the radiation affects the state of the gas, being at the same time altered itself. Eventually, its emergent spectral distribution differs from the equilibrium one, with the spectral lines as the most prominent features indicating the departure. Since the radiation leaving the star was last absorbed and emitted somewhere in its atmosphere, its character is fixed by conditions in these outer layers. However, the establishment of atmospheric properties from the observed spectrum is not an easy task because of complex mutual interactions among different physical phenomena.

Radiative transfer is among the most difficult problems to deal with due to an important role of scattering which makes the properties of well-distant regions non-locally and, in general, non-linearly coupled through the radiation field. The specific intensity of unpolarized and time-independent radiation field at frequency ν, $I(\vec{r},\vec{l},\nu)$, is a function of position $\vec{r}$ and of the direction of propagation specified by the unit vector $\vec{l}$. It is governed by the equation of radiative transfer

$$\vec{l}\cdot\nabla\, I(\vec{r},\vec{l},\nu) = -\chi(\vec{r},\vec{l},\nu)[I(\vec{r},\vec{l},\nu)-S(\vec{r},\vec{l},\nu)] \ , \quad (1)$$

where χ is the opacity (extinction) coefficient and S is the source function defined as the ratio of the emission to extinction coefficient. Eq. (1) is an ordinary first order differential equation *only if* the sources and sinks of photons are known. However, in general, specific RT equations (for each frequency and direction) are coupled by the transport (emission and extinction) coefficients. These coefficients depend on physical properties of the medium which in turn depend on the radiation field itself. Several RT problems can be sorted out according to the form of the transport coefficients, i.e. their dependence on the radiation field.

In the restricted RT problem, i.e. the problem of spectral line formation in a given stellar atmospheric model (with the temperature and pressure stratification fixed), the transport coefficients depend on the atomic level populations which can be significantly affected by the radiation field. The dependence of the internal (excitation and ionization) state of the gas on the radiation field intensity is expressed by statistical equilibrium (SE) equations (also called rate equations) describing the balance between all the transitions (radiative + collisional) that populate and those that depopulate a given atomic level. Radiation field intensity in its turn depends, via the RT process, on the state of the gas over a wide range of distant points. The non-local coupling of the radiation field and the internal state of the gas is carried out mathematically by the simultaneous solution of the corresponding RT and SE equations, which describe the radiation transport on macroscopic and microscopic levels, respectively.

The non-local coupling is linear in the well-known two-level-atom line formation problem. The opacity is known and the line source function depends linearly on the radiation field intensities through the scattering integral (integral of the specific intensities over frequencies and directions). Hence, in general, RT equation has an integro-differential form. Physically, the coupling arises due to the scattering process during which photons absorbed at a certain line frequency and in a certain direction can be re-emitted at other line frequencies and in other directions, getting a chance to travel very large distances before being reabsorbed. The state of the gas at each point of the medium is, therefore, coupled to that at all other points via the radiation field. The system of integro-differential RT equations (for each frequency and direction) can be solved by using either direct or iterative methods.

In more general case of multi-level-atom line formation problem, both opacity and emissivity depend on the radiation filed intensity via microscopic radiative processes populating (depopulating) atomic levels. Due to the coupling of the photons that belong to different line transitions (interlocking effect), the coefficients of each of the two systems (RT and SE) depend on the solution of the other in implicit and strongly non-linear way so that an iterative method is required.

In the global problem of stellar atmospheres modelling, the constraints of momentum conservation and energy balance are also to be satisfied. The values of the temperature and pressure depend, via these constraints, on the radiation field intensities, i.e. on the solution of RT equations. RT equations are then coupled not only through their source and sink terms but also through these constraint equations. Assuming that the energy transport is entirely performed by photons throughout a static atmosphere, the problem of stellar atmospheric modelling consists of the solution of the RT equations under the constraints of radiative, hydrostatic and statistical equilibrium. The global coupling of the photons across the spectrum and the overall structure of the medium through which radiation propagates is implicit and non-linear. The solution of this problem can be obtained either by global (e.g. linearization with iterations) or by structural (i.e. sequential iterative) approach.

During the last decades, high quality observations revealed that real stellar environment is often highly dynamic and inhomogeneous, so that a wide range of new interesting RT problems (time dependent RT, multidimensional RT, RT in expanding media, etc.) was opened. Besides, it is realized that a lack of atomic data often sets the limits on the accuracy of computations. Recently, a huge new project has been undertaken to calculate accurate atomic data for radiative and collisional processes and opacities of astrophysical interest (Mendoza et al. 2001). Considerable progress achieved not only in high resolution observations, but also in computational facilities, availability and accuracy of atomic data and in powerful numerical algorithms, stimulated the efforts to improve the physical reality in stellar atmospheric modelling.

Due to the overwhelming complexity of the subject it is not possible to review all advances achieved in the field. Briefly reviewing the main streams in development of the radiative transfer theory, we shall, therefore, focus on impressive advances in powerful numerical algorithms developed to solve RT problems. Special emphasis will be put on the methods developed in 1990s which use some alternative approaches to the solution of RT problems and seem to be very promising for a number of more complex purposes. Excellent reviews of various numerical methods are given in Kalkofen's books (1984, 1987), as well as in the papers like e.g. those by Rybicki (1991) and by Simonneau and Crivellari (2002). Nice historical surveys can be found in the papers by Mihalas (1994, 2001) who has also recently (Mihalas 2002) prepared an extensive bibliography of books, papers and reports on astrophysical spectroscopy.

2. A SURVEY OF RADIATIVE TRANSFER PROBLEMS AND SOLUTIONS

It has been a century since the first studies on radiative transfer.[1] The foundation of the stellar atmospheres theory is laid by Schuster, Schwarzschild, Milne and Eddington in the first decades of the 20th century. The equation of radiative transfer is derived and its solution studied in different forms. Certain approximations and very schematic models (including the extreme models of pure absorption and pure scattering) are introduced to describe the spectral line formation. Despite the oversimplified physical assumptions the first analytical solutions were of

[1]The famous work of Schuster (1905) on the transport of radiation through a foggy atmosphere appeared a century ago. However, the integral equation with Λ operator (usually referred to as Schwarzschild's equation) was published much earlier, in 1889, in two independent works of E. Lommel and O.D.Chwolson (Ivanov 1991).

great importance as they enabled the basic theoretical understanding of the problem. They still serve as very useful benchmarks to check the reliability of any new method developed aiming at the solution of more complex problems. Some of the most important early papers on radiative transfer were collected and edited by Menzel (1966).

The assumption of pure absorption, later referred to as the local thermodynamic equilibrium (LTE) approximation, has been extensively used in the astrophysical literature. Under this assumption, the internal state of the gas is described in terms of the equilibrium distribution functions (i.e. atomic level populations are specified by the Saha-Boltzmann law at the local electron temperature and density). In the absence of computational facilities, simplifying physical assumptions had to be made so that the first models of stellar atmospheres were crude: planar, static, grey (frequency independent opacity), LTE and in radiative equilibrium (RE).

In 1940s and 1950s several powerful methods for solving RT problems were developed: the method of discrete ordinates by Chandrasekhar, Ambarzumian's method based on the invariance principle, Sobolev's escape probability method (1957), etc. Their importance is twofold: on one hand, they are the bases of many modern techniques and, on the other, their "exact" solutions to simplified transfer problems serve as a reliable test of accuracy of new numerical methods. The detailed description of the basic methods in transfer problems was given in the books by Kourganoff (1952) and Chandrasekhar (1960).

The numerical phase in the RT theory has begun in the mid-60s, with the development of high-speed electronic computers. The computational facilities and new temperature correction procedures enabled the construction of the so-called *classical* (planar, static, non-grey, LTE, RE) stellar atmospheric models.

The picture of spectrum formation was greatly improved when LTE was abandoned in favor of statistical equilibrium (so-called non-LTE or NLTE problem). Namely, it was realized that in the low density (line forming) regions of an atmosphere radiative rates exceed the collisional ones, so that the radiation field which is not in equilibrium drives the material away from LTE. Non-LTE radiative transfer required new robust methods that could handle the non-linear coupling of all relevant variables in a fully self-consistent manner.

Here we shall focus on some of the most powerful numerical methods developed to solve non-LTE radiative transfer problems.

2.1. Numerical methods for the solution of non-LTE RT problems

Λ iteration

The well-known[2] most straightforward iterative procedure, so-called Λ iteration, solves the problem equations in turn. However, this simple procedure fails to converge in typical non-LTE conditions. Λ iteration follows the process of a particular scattering event and at each iteration step it corrects the solutions only within a unit optical path. Therefore, in scattering dominated media of large optical thickness, the convergence of Λ iteration is extremely slow. More detailed description of its properties is given in Mihalas (1978).

Global approach: Complete linearization

The first fully self-consistent solution of RT problems was achieved by the method of complete linearization (CL) of Auer and Mihalas (1969). This method was developed to solve the most general problem of stellar atmosphere modelling and its simplification to non-LTE spectral synthesis (the solution of the full non-linear multilevel problem). This approach is best explained by a remark of Auer (1971): "...we are trying to solve the transfer equation subject to a set of nonlinear constraints. Real stars solve them all simultaneously and as we shall see if we are to make any progress in non-LTE atmosphere theory we must also". Mutual coupling among all physical variables is accounted for by complete linearization of all the equations over these variables and by their simultaneous solution. Starting with some initial estimates, the system of linearized equations is solved for corrections to the current solution. The linearized equations being tridiagonal in form can be solved by a recursion scheme of Feautrier (1964a) or of Rybicki (1971) type (reformulation of CL method based on Rybicki solution technique is given by Auer and Heasley (1976)). Having the new solutions obtained, the iteration procedure is repeated until the convergence is achieved. Procedure is strongly convergent if the starting solution is not too far from the final one. Although conceptually completely different, the CL method and the Λ iteration method are akin in the sense that both use exact (full) RT operator. The CL method was widely used during 1970s. It enabled the input of improved physics (non-LTE) which led to substantially better fits of the computed spectra to the observed ones. It was the only practical method for reliable computations of non-LTE model atmospheres (especially for hot stars). This global approach does not need any deeper physical analysis than that necessary to formulate the equations representing the physical problem under study. However, as a single mathematical treatment is used for all the equations as a whole, only a global optimization is possible. Moreover, to develop the computer code and to modify it in order to match specific problems take a lot of efforts. Due to a cumbersome matrix structure, destructive numerical instabilities may arise. A great number of discretization (frequency, angle, depth) points needed for a good description of the system under study makes the CL method very time and memory consuming and in practice restricted to simplified atomic models and geometries. Much cheaper, simple and efficient methods were needed to

[2]In RT literature, Λ iteration is introduced by Hopf in 1928.

solve more realistic and, hence, more complex problems.

At the time, there were two kinds of methods attempting to facilitate the solution of line transfer problem, either by simplifying the use of SE equations or by employing an approximative treatment of RT coupled with the full set of SE equations. The first possibility is exploited within the so-called ETLA (Equivalent-Two-Level-Atom) approach developed by Avrett (1966) and Cuny (1968) and described by Mihalas (1978) and Avrett and Loeser (1987). Only one transition in the model atom at a time is combined with the transfer problem, whereas the coupling of all levels is achieved by iteration over all transitions. It may happen that iterative procedure converges upon inconsistent solutions as the multilevel transition interlocking is treated iteratively. The other approach uses some approximation to simplify the detailed description of the RT process at the cost of having to iterate a few times in order to get an accurate solution. The approximation is based either on the physical study of the transport of photons through the medium (escape probability, core saturation etc.) or on some computational considerations. Such approach and, especially, the two following ideas make the basis of a very broad class of methods known as ALI (Approximate (or Accelerated) Lambda Iteration) methods.

Core saturation

The idea that a modification of the Λ iteration method can be used as a practical method of solution of the RT and SE equations and its first successful realization appeared in the paper by Rybicki (1972). In order to eliminate the cause of the slow convergence of simple Λ iteration (poor conditioning of the equations arising from a large number of scatterings), Rybicki proposed the elimination of scattering events in the line core as they do not contribute much to the transfer process. He used the core saturation assumption (expressing the equality of monochromatic mean intensity of the radiation field and the local source function at the optically thick line core frequencies) in the SE equations. The solution of the preconditioned equations that contain only wing components (active in radiation transport) by direct Λ iteration has shown more rapid convergence than that of the original full equations. The main disadvantage of this method is a need for an adjustable parameter specifying the core-saturation region.

Cannon's method

Another breakthrough in computational RT was made with Cannon's perturbation technique (1973a,b). Cannon was the first to use the idea of 'operator splitting' (well-known in numerical analysis) in radiative transfer computations. He replaced the full (exact) Λ operator by a simplified one (so called approximate Lambda operator (ALO)) and computed a small error term made by this approximation by using a perturbation technique. His approximate operator was a Λ operator evaluated with fewer angular-frequency quadrature points. Cannon's operator perturbation technique was an efficient way to combine computational advantages of approximate solutions with the accuracy of exact methods.

ALI methods

The idea of using certain physical (or computational) approximations (such as core saturation) to accelerate the Λ iteration together with the operator perturbation technique introduced by Cannon to simplify the direct solution, makes the basis of the class of methods known as ALI (Accelerated Lambda Iteration).

The first application of these ideas to the field of stellar atmospheres was given in the paper by Scharmer (1981). Using the core saturation assumption of Rybicki and the Eddington-Barbier relation with the perturbation technique of Cannon, Scharmer obtained two approximate Λ operators for the case of two-level-atom line transfer. Sharmer's global operator, that provides faster convergence, was later applied to the non-LTE problems in atmospheres with velocity fields (Scharmer and Nordlund 1982) and to multilevel line transfer problems (Scharmer and Carlsson 1985, Carlsson 1986). However, the inversions of nearly triangular matrix put practically the same limitations on the method as those of the complete linearization scheme. A progress was made when Scharmer's simpler approximate operator was generalized to multi-level non-LTE line formation problems by Werner and Husfeld (1985). While the CL method was restricted to model atoms with a few line transitions only, with the use of this ALO it became possible to handle up to about 100 atomic energy levels. However, these methods also need linearization of either both transfer and rate equations (Scharmer and Carlsson 1985) or the rate equations alone (Werner and Husfeld 1985). The first model atmosphere calculations with the ALI method were performed by Werner (1986). Hamman (1985) adapted the ALI method to the line formation problem in spherically symmetric, expanding atmospheres. The constraint of RE is additionally taken into account by Hamman and Wessolowski (1990).

An approach to the construction of approximate Λ operator based more on mathematical than physical considerations can be found in the papers by Olson et al. (OAB 1986) and Olson and Kunasz (1987). Olson et al. were the first to point out that the diagonal of the exact Λ matrix itself represents nearly optimum diagonal approximate operator. The basic advantage of OAB operator is that no free parameter controlling the convergence process is required. This operator is used as the local-operator option of the multilevel transfer code MULTI (Carlson 1986, 1991). From a 'short characteristic' solution of the transfer problem Olson and Kunasz (1987) derived a fast method to compute the exact diagonal of the Λ matrix. They also showed that more rapid convergence can be obtained by using the tridiago-

nal (or pentadiagonal) part of the exact Λ operator as the approximate one. However, these non-local operators are computationally more expensive due to the inevitable matrix inversions.

The approach that avoids linearization by introducing ALOs directly into SE equations to make them linear from the outset (preconditioning) is given by Rybicki and Hummer (1991). Linearization and preconditioning are the two most widely used approaches for achieving the linearity in the iterative solution of RT problems. Most practical methods represent some kind of their combination. The first approach, which uses complete linearization applying ALI to the resulting linear equations, is more general, but it leads to extremely complicated equations. The second one, using preconditioning, is much simpler but not always applicable (Rybicki 1991).

The common feature of all the above methods is that the choice of ALO is, to a certain degree, optional. Namely, ALO has to be defined to satisfy two contradictory requirements: on one hand, it has to be more simple to invert than the Λ operator and, on the other, it has to be as close physical approximation of the exact Λ operator as possible (i.e. it should retain all the essential physics of RT) to ensure stable and rapid convergence to the exact solution. Thus, the use of diagonal operator that can be easily inverted leads to a simple yet of still slow convergence rate procedure, whereas tri- and higher-order diagonal ALOs introduce the non-locality into equations that thus become more difficult to solve and more unstable. The main disadvantage of some of the ALI methods (e.g. ALI methods which use physical models like Rybicki's core saturation assumption to construct approximate operator) is in the use of certain arbitrary adjustable parameters the values of which have to be found empirically. With the departures from their optimum values, very slow convergence or even divergence may occur. Moreover, almost all of the ALI methods need additional acceleration by some mathematical techniques (Ng acceleration, Overrelaxation method) to achieve high convergence rate.

Therefore, despite important progress made thanks to the ALI methods, the search for more efficient algorithms is still an open field of research.

Structural approach: Sequential iterative procedure

Some of the above mentioned problems (numerical instabilities and high computational costs) can be avoided by a different, so-called *structural* approach to the solution of global astrophysical problems (Simonneau and Crivellari 1994b). The idea of this approach is the following: a global problem can be subdivided into specific physical phenomena (key-problems) represented by individual blocks of a general block diagram. The physics itself determines the optimum structure of the block-diagram. A current solution for each block is determined separately, assuming that the output, in terms of physical information, from all the others is known. Hence, the solution is achieved through a sequential iterative procedure. The most straightforward iteration scheme is very slow. However, by introducing its slight modifications, it is possible to achieve an *extremely high* convergence rate. A kind of revision of the classical Λ iteration aimed at speeding up its convergence is based upon the use of certain quasi-invariant functions, the so-called iteration factors, whose iterative computation leads very quickly to the exact solution. These factors are in fact the input/output of the blocks that constitute the block diagram. They have to be chosen to account for the physics of the blocks and the mathematical nature of their equations and variables. In order to provide rapid convergence it is necessary that these factors are good quasi-invariants, i.e. to change very little from one iteration to another. This can be achieved if they are defined as the ratios of the homologous physical quantities.

The approach using the iteration factors enables the exact solution of the global problem to be obtained inexpensively, within a fast and stable iterative procedure. Moreover, opposite to the global approach, it is possible to optimize each block of a general block diagram independently. Now we shall focus on some efficient numerical methods, developed during 1990s, to solve a block corresponding to RT problems.

Method of iteration factors/profiles

In radiative transfer literature, the idea of iteration factors appeared for the first time in the paper by Feautrier (1964b). He suggested that the use of the ratio of two moments of the radiation field intensity could speed up the stellar atmosphere model computations. The first realization of the idea was the variable (depth-dependent) Eddington factor (VEF) technique developed for the solution of monochromatic transfer problem in plane-parallel (Auer and Mihalas 1970) and in spherical geometry (Hummer and Rybicki 1971). The iteration on VEFs, defined as the ratios of the third to the first angular moments of the radiation field, provides a very fast convergence to the exact solution. VEFs have been applied to the linearization method to reduce the "angular dimensions" of the system under study, but a large number of frequency points still made the numerical computations heavy. The idea of VEFs was later generalized to other physical variables and the corresponding suitable iteration factors were introduced in a mixed iterative procedure with the aim to provide the exact and fast convergent solution without any matrix calculations. At each iteration step the factors are computed from the current solution and then used to update it. The iteration factors method has been developed for several RT problems. In the study on resonance line transfer and the transport of excited atoms Borsenberger, Oxenius and Simonneau (1987) used the iteration factors to solve a system of two kinetic equations coupled by the corresponding source terms. The iteration factors are also used for the computation of LTE stellar atmosphere models in radiative equilibrium (Simonneau and Crivellari 1988) and when the convective transport is taken into ac-

count (Crivellari and Simonneau 1991). Fieldus et al. (1990) generalized the method to include spherically extended line blanketed model atmospheres. The idea of VEFs is generalized in the solution of the well-known paradigm case: the two-level-atom line transfer problem in Simonneau and Atanacković-Vukmanović (1991), Atanacković-Vukmanović (1991) and Atanacković-Vukmanović and Simonneau (1994). At each iteration step, angle and frequency averaged depth-dependent factors defined as the ratios of the relevant intensity moments are computed from the current values of the radiation field and then used to close the system of the RT equation moments. The major advantage of this method arises from the fact that it does not require any matrix operation, so that only small memory storage and computational time are required. [An extremely fast convergence is achieved at the cost of a small additional effort, with respect to the Λ iteration, of iterative computation of these factors and the solution of just one second-order differential equation.]

The idea of iteration factors is also used for the solution of multilevel non-LTE line transfer with partial frequency redistribution (PRD) in the paper by Hubeny and Lites (1995). The iterations on the ratios of the emission to the absorption profile coefficients (Hubeny 1985) are performed within a global scheme, split in two parts, so that these ratios obtained from the transition-by-transition solution of the PRD problems (within ETLA approach) are then used (held fixed) in the solution of the general multilevel coupling performed by means of CL method.

The method of iteration factors is problem dependent i.e. the iteration factors have to be chosen according to the physics of the transfer problem itself in order to achieve best convergence properties. In certain cases (e.g. in the problems where RT equations are not coupled by a single scalar quantity) it is hard or even impossible to apply directly the idea of the iteration factors. In such cases, its generalization to the idea of iteration profiles has proved to be very efficient. The distribution functions (usually the unknowns of the problem) are to be split into the product of the integral of the function itself over the relevant parameters and the profile that accounts for the dependence of the distribution function on the relevant parameters (Crivellari and Simonneau 1995). This is the basic idea of a factorization technique that allows to develop an efficient mixed iterative procedure in which the existing robust methods, developed to solve simple paradigm problems, can be easily applied. At each iteration step the profiles are computed from the current solution (results of the previous iteration) and then used to update it. Quasi-invariant iteration profiles are more flexible and less expensive alternative to the linearization technique. The idea is for the first time applied to the factorization of the Planck function in the computation of LTE stellar atmosphere models (Simonneau and Crivellari 1994a). It is also applied to the case of the spectral line formation when the partial redistribution effects are taken into account (Crivellari and Simonneau 1995).

'Local' implicit methods

A different approach to the solution of RT problems of extraordinary ease of use and effectiveness has been developed within both direct (Implicit Integral Method) and iterative (Forth-and-Back Implicit Λ Iteration) methods. It uses the fact that even if the values of the radiation field are unknown, its physical behavior can be easily represented. Namely, similar to the elimination scheme intrinsic to the differential methods (Feautrier 1964) this approach uses the fact that it is possible to derive a linear relation between the unknown radiation intensities at one depth point and those at the following one. On the other hand, similar to *'global'* integral formalism (Avrett and Loeser 1969), the used *'local'* implicit treatment of RT problem is based on a functional representation of the source function between each two successive depth points. Owing to an *implicit local* scheme introduced within a natural two-stream representation of the radiation field, matrix formalism is completely avoided enabling considerable savings of both computational time and memory storage. More detailed discussion on these methods is given in the paper by Simonneau and Crivellari (2002) and in the papers cited hereinafter.

Implicit Integral Method (IIM)

Global integral methods solve simultaneously all the relevant equations at all the points of the medium by expressing implicitly the solution at any given point as a function of the so far unknown solutions at all the other points. However, huge dimensions of the system under study make the problem unmanageable by means of customary discrete methods. The basic idea of the implicit integral method (IIM) is to retain the reliability intrinsic to global integral methods but with no need for storing and inverting matrices.

Implicit description of the evolution of the specific intensities incoming to a layer from the neighboring ones enables to link point by point the two families of initial conditions via a proper set of scalar coefficients. The global problem is thus reduced to a series of one-layer two-point boundary problems. For each layer a linear relationship between the scattering integrals (and their derivatives) at two boundary points and the out-going specific intensities at the lower boundary of a layer is derived. The coefficients of this relation are set up layer by layer and stored in a forward elimination process for further use in the back-substitution. In parallel, the implicit linear representation of the in-going specific intensities in terms of the out-going ones is derived as the upper initial condition for the next layer. By an implicit use of the unknown out-going specific intensities IIM is a simple elimination method to solve the problem layer by layer. Among integral methods IIM is the only one that avoids the inversion of the Λ operator.

The necessary condition for applying the IIM is that the same integral couples all the RT equations. This powerful non-matrical direct method was first developed for the two-level-atom line formation

problem with complete frequency redistribution by Simonneau and Crivellari (1993). IIM is applied to the RT problems in spherical geometry by Gros et al. (1997). A series of tests confirmed high reliability, stability and effectiveness of this method.

In order to solve other more general problems in which the necessary condition for the application of IIM is not fulfilled, IIM can be applied within an iterative procedure by using suitable auxiliary functions - iteration profiles. The implicit integral method has been successfully developed to solve problems such as the self-consistent determination of the temperature distribution when computing an LTE stellar atmosphere model (Crivellari and Simonneau 1994), the line transfer with partial frequency redistribution (Crivellari and Simonneau 1995) and the multi-level-atom line formation problem (Crivellari et al. 2002). It proved to be very efficient method in the solution of various RT problems mainly because its algorithm follows the physics of the RT process.

Forth-and-Back Implicit Lambda Iteration (ILI)

Another simple and efficient method developed to accelerate the convergence of the classical Λ-iteration while retaining its straightforwardness is a forth-and-back implicit Λ iteration (ILI). It uses an implicit representation of the source function in the computation of both the in-going and the out-going intensities of the radiation field that are treated separately within a forth-and-back approach. Namely, using the integral form of the RT equation and assuming polynomial (e.g. piecewise parabolic) behavior for the source function, one can write linear relations for the specific intensities as functions of the unknown values of the source function and its derivatives. According to the idea of iteration factors, iterative computations of the coefficients of these *implicit* relations (implicit, as the source function is a priori unknown), rather than of the unknown functions themselves, greatly accelerate the convergence of the direct iterative scheme.

Proceeding from the upper boundary condition and using an initial estimate of the source function (or that known from the previous iteration step) one can compute and store, at all optical depths, the coefficients of the linear relation representing implicitly the values of the in-going mean intensities in terms of yet unknown values of the source function. The aim is to obtain the coefficients of the linear relation between the 'full' mean intensities and the source function which, when substituted into the statistical equilibrium equations, easily leads to the new source function. In order to do this, the coefficients of the corresponding implicit relation for the out-going mean intensities are needed. Proceeding from the lower boundary condition we determine them simultaneously with the new source function when sweeping layer by layer back to the surface. The process is iterated to convergence. A negligible additional computational effort made within ILI method with respect to the classical Lambda iteration results in a very fast convergence to the exact solution. The convergence is even more accelerated when the amount of information transferred from one iteration to the next is reduced. In the case of the minimum amount of information carried from the previous step, in the form of the ratio between the non-local part of the inward mean intensities and the current source function, the rate of convergence becomes extraordinarily high.

The method is developed for the non-LTE line formation problem in Atanacković-Vukmanović (1991) and in Atanacković-Vukmanović, Crivellari and Simonneau (1997). The efficiency and the accuracy of the ILI method have been checked on several problems: the two-level-atom line transfer problem with complete redistribution (CRD), its generalization to the case of partial frequency redistribution, as well as the multi-level-atom line formation problem with CRD. The exact solutions are obtained in a very small number of iterations even under extreme non-LTE conditions. The generalization of the method to spherically symmetric media is given by Atanacković-Vukmanović (2003).

Due to high accuracy and considerable savings of computational time and memory storage (that grows only linearly with the dimension of the problem) this method seems to be a far-reaching tool to deal with more complex problems when the radiative transfer has to be coupled with other physical phenomena.

3. CONCLUDING REMARKS

The theory of radiative transfer in stellar atmospheres has experienced great progress during the last decades. Thanks to the impressive advances in computational methods, many of RT problems have been solved with high accuracy. High quality observations and progress made in powerful computational algorithms stimulated the efforts to improve the physical realism in stellar atmospheric modelling. The fits to the observed spectra became much better thanks to the improved physical input, more complete and accurate atomic data and more efficient algorithms and modelling techniques.

However, despite considerable efforts and significant progress made in the last decades, RT is still one of the most important open problems in stellar atmospheres theory. Many astrophysical phenomena remain to be treated in a more complete way. They often require the solution of 2D or 3D radiative transfer, coupling between radiation and hydrodynamic flow, hence the application of time dependent, non-linear hydrodynamics (e.g. the improved theory of stellar winds, modelling of supernovae explosions, treatment of interacting or contact binaries, ordinary novae, convection in stellar atmospheres, etc). Their solution by the widely used present techniques is expensive, so that there is still a great need for more simple and faster iteration schemes. In this respect, the use of new 'local' implicit methods seems very promising. Moreover, due attention should be paid not only to the input of improved physics but also to

alternative approaches and new formulations of the existing physical problems. As an example, let us mention very important re-formulation of the spectral line formation theory made in the paper by Oxenius and Simonneau (1994), where a new formulation of the standard redistribution theory is introduced within the semi-classical kinetic approach.

It is probably true that "these days, the subject of stellar atmospheres seems, to some people, rather old fashioned compared to 'exciting' parts of modern astrophysics" (Mihalas 2001). However, the observations every day open new questions and make that the RT in stellar environment is still an open and interesting field of research. At last, as Hunger (1991) noticed there is one good and simple reason to pursue such kind of research "...It was about 10 years ago, I was asked why we were sitting on stellar atmospheres and not turning to extragalactic astronomy, a field that became popular in those days. My answer was that as long as one wants to understand spectra whose emitters are optically thick in at least one frequency, one is confronted with the problem of radiative transfer. The physics, mathematics, and numerics behind it is so complex that alone the program codes belong to the most comprehensive codes known. If one has given up this expertise, it would be difficult to regain it in a short time."

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REFERENCES

Atanacković-Vukmanović, O.: 1991, Ph.D. Thesis, University of Belgrade.
Atanacković-Vukmanović, O.: 2003, *Serb. Astron. J.*, **167**, 27.
Atanacković-Vukmanović, O. and Simonneau, E.: 1994, *J. Quant. Spectrosc. Radiat. Transfer*, **51**, 525.
Atanacković-Vukmanović, O., Crivellari, L. and Simonneau, E.: 1997, *Astrophys. J.*, **487**, 735.
Auer, L.H.: 1971, *J. Quant. Spectrosc. Radiat. Transfer*, **11**, 573.
Auer, L.H. and Heasley, J.: 1976, *Astrophys. J.*, **205**, 165.
Auer, L.H. and Mihalas, D.: 1969, *Astrophys. J.*, **158**, 641.
Auer, L.H. and Mihalas, D.: 1970, *Mon. Not. R. Astron. Soc.*, **149**, 65.
Avrett, E.H.: 1966, *Astrophys. J.*, **144**, 59.
Avrett, E.H. and Loeser, R.: 1969, *SAO Spec. Rep.* No.303.
Avrett, E.H. and Loeser, R.: 1987, in Numerical Radiative transfer, ed. by W. Kalkofen, Cambridge University Press, 135.
Borsenberger, J., Oxenius, J. and Simonneau, E.: 1987, *J. Quant. Spectrosc. Radiat. Transfer*, **37**, 331.
Cannon, C.J.: 1973a, *J. Quant. Spectrosc. Radiat. Transfer*, **13**, 627.
Cannon, C.J.: 1973b, *Astrophys. J.*, **185**, 621.
Carlsson, B.G.: 1986, A Computer Program for Solving Multi-level Non-LTE Radiative Transfer Problems in Moving or Static Atmospheres, Uppsala Astron. Observatory Report No. 33.
Carlsson, B.G.: 1991, in Stellar Atmospheres Beyond Classical Models, L. Crivellari, I. Hubeny and D.G. Hummer eds., NATO ASI Series No. 341, Dordrecht, Boston and London, Kluwer Academic Press, 39.
Chandrasekhar, S.: 1950, Radiative Transfer, Oxford University Press; New York: Dover Publications edition, 1960.
Crivellari, L. and Simonneau, E.: 1991, *Astrophys. J.*, **367**, 612.
Crivellari, L. and Simonneau, E.: 1994, *Astrophys. J.*, **429**, 331.
Crivellari, L. and Simonneau, E.: 1995, *Astrophys. J.*, **451**, 328.
Crivellari, L., Cardona, O. and Simonneau, E.: 2002, *Astrophysics* **45**, 587.
Cuny, Y.: 1968, in Resonance Lines in Astrophysics, ed. R. Athay, J. Mathis and A. Skumanich (Boulder: National Center for Atmospheric Research)
Feautrier, P.: 1964a, *C. R. Acad. Sci.*, Paris, **258**, 3189.
Feautrier, P.: 1964b, in Proceedings of the First Harvard-Smithsonian Conference on Stellar Atmospheres, SAO Spec. Rep. No. 167, Cambridge, Mass., p.108.
Fieldus, M.S., Lester, J.B. and Rogers, C.: 1990, *Astron. Astrophys.*, **230**, 371.
Gros, M., Crivellari, L. and Simonneau, E.: 1997, *Astrophys. J.*, **489**, 331.
Hamman, W.-R.: 1985, *Astron. Astrophys.*, **148**, 364.
Hamman, W.-R. and Wessolowski, U.: 1990, *Astron. Astrophys.*, **227**, 171.
Hubeny, I,: 1985, *Bull. Astron. Inst. Czechoslovakia*, **36**, 1.
Hubeny, I. and Lites, B.W.: 1995, *Astrophys. J.*, **455**, 376.
Hummer, D.G. and Rybicki, G.B.: 1971, *Mon. Not. R. Astron. Soc.*, **152**, 1.
Hunger, K.: 1991, in Stellar Atmospheres Beyond Classical Models, L. Crivellari, I. Hubeny and D.G. Hummer eds., NATO ASI Series No. 341, Dordrecht, Boston and London, Kluwer Academic Press, 449.
Ivanov, V.V.: 1991, in Stellar Atmospheres Beyond Classical Models, L. Crivellari, I. Hubeny and D.G. Hummer eds., NATO ASI Series No. 341, Dordrecht, Boston and London, Kluwer Academic Press, 81.
Kalkofen, W., (Ed.): 1984, Methods in Radiative Transfer, Cambridge University Press.
Kalkofen, W., (Ed.): 1987, Numerical Radiative Transfer, Cambridge University Press.

Kourganoff, V.: 1952, Basic Methods in Transfer Problems, Oxford University Press; New York: Dover Publications edition, 1963.
Mendoza, C., Nahar, S., Pradhan, A., Seaton, M., Zeippen, C.: 2001, American Physical Society, DAMOP Meeting Cannada Bulletin of the APS, **46**, No.3.
Menzel, D., (Ed.): 1966, Selected Papers on the Transfer of Radiation, New York: Dover Publications.
Mihalas, D.: 1978, Stellar Atmospheres, 2nd ed., Freeman, San Francisco.
Mihalas, D.: 1994, ASP Conference Series, Vol. 60, eds. Corbally, C.J., Gray, R.O. and Garrison, R.F., 41.
Mihalas, D.: 2001, *Publ. Astron. Soc. Aust.*, **18**, 311.
Mihalas, D.: 2002, A Guide to the Literature on Quantitative Spectroscopy in Astrophysics, `http://www.galax.obspm.fr/IAU/docu.html`
Olson, G.L., Auer, L.H. and Buchler, J.R.: 1986, *J. Quant. Spectrosc. Radiat. Transfer*, **35**, 431.
Olson,G.L. and Kunasz,P.: 1987, *J. Quant. Spectrosc. Radiat. Transfer*, **38**, 325.
Oxenius, J. and Simonneau, E.: 1994, *Annals of Physics*, **234**, 60.
Rybicki, G.B.: 1971, *J. Quant. Spectrosc. Radiat. Transfer*, **11**, 589.
Rybicki, G.B.: 1972, in Line Formation in the Presence of Magnetic Fields, (eds. R.G. Athay, L.L. House, G. Newkirk, Jr.), High Altitude Observatory, Boulder, p.145.
Rybicki, G.: 1991, in Stellar Atmospheres Beyond Classical Models, L. Crivellari, I. Hubeny and D.G. Hummer eds., NATO ASI Series No. 341, Dordrecht, Boston and London, Kluwer Academic Press, 1.
Rybicki, G. and Hummer, D.: 1991, *Astron. Astrophys.* **245**, 171.
Scharmer, G.B.: 1981, *Astrophys. J.*, **249**, 720.
Scharmer, G.B. and Carlsson, M.: 1985, *J. Comput. Phys.*, **59**, 56.
Scharmer, G.B. and Nordlund, A.: 1982, *Stockholm Obs. Rep.*, **19**.
Schuster, A.: 1905, *Astrophys. J.*, **21**, 1.
Simonneau, E. and Atanacković-Vukmanović, O.: 1991, in NATO ASI Series C (Dordrecht: Kluwer), 105.
Simonneau, E. and Crivellari, L.: 1988, *Astrophys. J.*, **330**, 415.
Simonneau, E. and Crivellari, L.: 1993, *Astrophys. J.*, **409**, 830.
Simonneau, E. and Crivellari, L.: 1994a, *Astrophys. J.*, **428**, 753.
Simonneau, E. and Crivellari, L.: 1994b, Structural Algorithms to Solve Radiative Transfer Problems, IAC Monograph No.1.
Simonneau, E. and Crivellari, L.: 2002, EAS Publications Series, Vol. 5, Proceedings of "Radiative Transfer and Hydrodynamics in Astrophysics", 31.
Sobolev, V.V.: 1957, *Astron. Zh.*, **34**, 694.
Werner, K.: 1986, *Astron. Astrophys.* **161**, 177.
Werner, K. and Husfeld, D.: 1985, *Astron. Astrophys.* **148**, 417.

О ПРОБЛЕМИМА ПРЕНОСА ЗРАЧЕЊА И ЊИХОВОМ РЕШАВАЊУ

O. Atanacković-Vukmanović

Department of Astronomy, Faculty of Mathematics, University of Belgrade
Studentski trg 16, 11000 Belgrade, Serbia and Montenegro

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Прегледни рад по позиву

У раду је дат приказ основних етапа у решавању проблема преноса зрачења, са посебним освртом на приступе и ефикасне нумеричке методе који су недавно развијени за решавање проблема преноса зрачења у условима не-ЛТР.