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ESTIMATION OF LINEAR FUNCTIONS OF ORDERED SCALE PARAMETERS OF TWO GAMMA DISTRIBUTIONS UNDER ENTROPY LOSS

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The problem of estimating linear functions of ordered scale parameters of two Gamma distributions is considered under entropy loss. A necessary and sufficient condition for the maximum likelihood estimator (MLE) to dominate the crude unbiased estimator (UE) is given on two non-negative coefficients. Furthermore, improvement on the UE of the reciprocal of each scale parameter is also obtained under entropy loss. Some numerical results are given to illustrate how much improvement is obtained over the UE.

Key words and phrases: Admissible estimator, entropy loss, MLE, reciprocal of scale parameter, unbiased estimator.

1. Introduction

In this paper we discuss the problem of estimating linear functions of scale parameters of two Gamma distributions under entropy loss when shape parameters are known and order restriction is given on the scale parameters. As is mentioned in Chang and Shinozaki (2002), this general estimation problem includes as special cases the one of linear functions of ordered variances in two sample problems, and also the one of variance components in a one-way random effect model (see, for example, Section 3.5 of Lehmann and Casella (1998)).

There has been considerable interest in the estimation of parameters when there are some order restrictions, or more generally linear inequality restrictions among parameters. See for example, Barlow *et al.* (1972), Robertson *et al.* (1988), Silvapulle and Sen (2004) and van Eeden (2006). In various applications we often have some prior knowledge about the parameter values of these forms and it is natural to utilize the knowledge to obtain better estimators than the usual ones. Many papers focus on comparing the maximum likelihood estimator (MLE), which satisfies the order restriction with the unbiased estimator (UE) of normal means, coordinately in terms of mean square error (MSE) (Lee (1981), Kelly (1989)). On the other hand Hwang and Peddada (1994) have discussed the stochastic domination of an improved estimator for order restricted parameters of symmetric distributions.

Rueda and Salvador (1995) have considered the problem of estimating linear

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functions of normal means when two linear inequality constraints are given, and have shown that MLE gives an improvement for any coefficients in terms of MSE. In estimating linear functions of positive normal means, Shinozaki and Chang (1999) have given a necessary and sufficient condition on coefficients so that the linear function of MLE dominates the one of UE in terms of MSE and have shown that MLE dominates UE for any choice of coefficients if and only if the number of means is less than 5. Independently, Fernández *et al.* (2000) also have discussed the same problem under the symmetric unimodal location model. In recent years, Oono and Shinozaki (2005) have considered the estimation problem of linear functions of two order restricted normal means when variances are unknown and possibly unequal.

Other than the estimation of restricted normal means, there are many papers dealing with the estimation of scale parameters under order restriction. Estimation of smaller variance has been discussed by Kushary and Cohen (1989). Kaur and Singh (1991) have considered the estimation of two ordered exponential means with the same sample sizes and have shown that MLE dominates UE coordinately. Vijayasree *et al.* (1995) have also considered the componentwise estimation of ordered parameters of $k(\geq 2)$ exponential populations. Hwang and Peddada (1994) and Kubokawa and Saleh (1994) have discussed the general scale parameter estimation problem under order restriction. See Oono and Shinozaki (2006a, b) for variance estimation under general order restriction. Estimation of linear functions of ordered scale parameters of two Gamma distributions has been discussed by Chang and Shinozaki (2002) under squared error loss. It has been shown that MLE does not always improve upon UE and a necessary and sufficient condition on the ratio of two coefficients is given for MLE to dominate the crude UE.

In estimating scale parameters, squared error loss may not be natural and entropy loss may be pertinent. Dey *et al.* (1987) have discussed the admissibility of best scale invariant estimators of scale parameters, and of their reciprocals of independent Gamma distributions under entropy loss. They have considered simultaneous estimation problems of scale parameters and their reciprocals under entropy loss function when order restriction is not present.

Here we first compare MLE with UE in estimating linear functions of ordered scale parameters of two Gamma distributions under entropy loss. In Section 2, we again show that MLE does not always improve upon UE and give a necessary and sufficient condition on nonnegative coefficients for MLE to dominate UE. In Section 3, we discuss the estimation of reciprocals of scale parameters of two Gamma distributions under entropy loss when order restriction is given. In Section 4, some results with numerical evaluation are given to illustrate the behavior of risk functions of MLE and estimators of reciprocals of scale parameters of two Gamma distributions. We give concluding remarks in Section 5.

2. A necessary and sufficient condition for MLE to dominate UE

Let X_i, $i = 1, 2$ be independent $Gamma(\alpha_i, \lambda_i)$ random variables, having the density

$$f_{\lambda_i}(x_i) = x_i^{\alpha_i - 1} \lambda_i^{-\alpha_i} e^{-x_i/\lambda_i}/\Gamma(\alpha_i), \qquad 0 < x_i < \infty,$$

where $\alpha_i(> 0)$ is known and $\lambda_i(> 0)$ is unknown but satisfying the order restriction $0 < \lambda_1 \le \lambda_2 < \infty$. We note that even if we have more than one observation, we can reduce the case to the one above as stated in Chang and Shinozaki (2002). The MLE of λ_i is given by

$$\hat{\lambda}_i = \frac{X_i}{\alpha_i} + (-1)^i \frac{(\alpha_2 X_1 - \alpha_1 X_2)^+}{\alpha_i(\alpha_1 + \alpha_2)}, \qquad i = 1, 2,$$

where $a^+ = \max(0, a)$ and X_i/α_i is the unbiased estimator(UE) and also the best invariant one of λ_i.

As stated in Dey *et al.* (1987), it follows from Stein (1959) or Brown (1966) that X_i/α_i is an admissible estimator of λ_i under entropy loss function when only X_i is observed. Further, Dey *et al.* (1987) have proved that $(X_1/\alpha_1, X_2/\alpha_2)$ is admissible for estimating (λ_1, λ_2) under the sum of entropy losses when we don't have the order restriction $\lambda_1 \le \lambda_2$ and $\min(\alpha_1, \alpha_2) > 4$.

Let c_1, c_2 be given non-negative constants and we want to estimate $\eta = c_1\lambda_1 + c_2\lambda_2$. We compare two estimators, UE, $\sum_{i=1}^2 c_i X_i/\alpha_i$ and, MLE, $\sum_{i=1}^2 c_i \hat{\lambda}_i$ in terms of their risk functions when the loss function is given by

$$L(\eta, \delta) = \delta\eta^{-1} - \log(\delta\eta^{-1}) - 1,$$

where δ is an estimator of η. We note that if at least one of c_1 and c_2 is negative, UE takes negative values with positive probability and we cannot apply the entropy loss function.

We may expect that MLE will dominate UE for any nonnegative constants c_1 and c_2. However, we will show that this is not the case. We give a necessary and sufficient condition on c_1 and c_2 for MLE to dominate UE. Hereafter, we denote the risk of an estimator δ of η by

$$R(\lambda, \delta) = E\{L(\eta, \delta)\},$$

where $\lambda = (\lambda_1, \lambda_2)$. We give a sufficient condition for the uniform improvement in Theorem 2.1. We show its necessity in Theorem 2.2.

THEOREM 2.1. *The MLE* $\sum_{i=1}^2 c_i \hat{\lambda}_i$ *dominates the UE* $\sum_{i=1}^2 c_i X_i/\alpha_i$ *in terms of risk if*

$$\text{(i)} \quad \frac{c_2}{\alpha_2} \ge \frac{c_1}{\alpha_1}$$

or

$$\text{(ii)} \quad c_1 > 0 \quad \textit{and} \quad c_2 = 0.$$

Proof. The difference between the risks of MLE and UE is given as

$$\Delta R = R\left(\lambda, \sum_{i=1}^{2} c_i \frac{X_i}{\alpha_i}\right) - R\left(\lambda, \sum_{i=1}^{2} c_i \hat{\lambda}_i\right)$$

$$= E\left\{\log\left[1 + \frac{c'_2 - c'_1}{\alpha_1+\alpha_2}\frac{(\alpha_2 X_1 - \alpha_1 X_2)^+}{\sum_{i=1}^2 c'_i X_i}\right] - \frac{c'_2 - c'_1}{\alpha_1+\alpha_2}\frac{(\alpha_2 X_1 - \alpha_1 X_2)^+}{\sum_{i=1}^2 c_i \lambda_i}\right\}, \tag{2.1}$$

where $c'_i = c_i/\alpha_i$, $i = 1, 2$. To evaluate this risk difference, we make the variable transformation

$$W = \frac{X_1}{\lambda_1} + \frac{X_2}{\lambda_2}, \qquad Z = \frac{X_1}{\lambda_1 W}.$$

Then we see that $W \sim Gamma(\alpha_1+\alpha_2, 1)$, $Z \sim Beta(\alpha_1, \alpha_2)$ and they are independently distributed. By using the random variables W and Z, we can express ΔR as

$$\Delta R = E\left\{\log\left[1 + \frac{c'_2 - c'_1}{\alpha_1+\alpha_2}\frac{\{\alpha_2\lambda_1 Z - \alpha_1\lambda_2(1-Z)\}^+}{c'_1\lambda_1 Z + c'_2\lambda_2(1-Z)}\right]\right\}$$
$$- \frac{c'_2 - c'_1}{\alpha_1+\alpha_2} E(W) E\left\{\frac{\{\alpha_2\lambda_1 Z - \alpha_1\lambda_2(1-Z)\}^+}{\sum_{i=1}^2 c_i\lambda_i}\right\}.$$

Since $E(W) = \alpha_1 + \alpha_2$, setting $\beta = \frac{\alpha_2\lambda_1+\alpha_1\lambda_2}{(\alpha_1+\alpha_2)\lambda_2}$ and $\gamma = \frac{\alpha_1\lambda_2}{\alpha_2\lambda_1+\alpha_1\lambda_2}$, we have

$$\Delta R = E\{h(Z)\},$$

where

$$h(z) = \log\left[1 + \frac{(c'_2 - c'_1)\lambda_2\beta(z-\gamma)^+}{c'_1\lambda_1 z + c'_2\lambda_2(1-z)}\right] - \frac{(c'_2 - c'_1)\lambda_2\beta(\alpha_1+\alpha_2)(z-\gamma)^+}{\sum_{i=1}^2 c_i\lambda_i}. \tag{2.2}$$

To show that $h(z) \geq 0$ for $\gamma \leq z < 1$, we first notice that $h(\gamma) = 0$. Thus we need only to show that $h(z)$ is a non-decreasing function for $\gamma \leq z < 1$. For that purpose we differentiate $h(z)$ and have

$$\frac{d}{dz}h(z) = (c'_2 - c'_1)\lambda_2\beta\left\{1 + \frac{(c'_2 - c'_1)\lambda_2\beta(z-\gamma)}{c'_1\lambda_1 z + c'_2\lambda_2(1-z)}\right\}^{-1} g(z),$$

where

$$g(z) = \frac{1}{c'_1\lambda_1 z + c'_2\lambda_2(1-z)}\left\{1 + \frac{(c'_2\lambda_2 - c'_1\lambda_1)(z-\gamma)}{c'_1\lambda_1 z + c'_2\lambda_2(1-z)}\right\}$$
$$- \frac{\alpha_1+\alpha_2}{\sum_{i=1}^2 c_i\lambda_i}\left\{1 + \frac{(c'_2 - c'_1)\lambda_2\beta(z-\gamma)}{c'_1\lambda_1 z + c'_2\lambda_2(1-z)}\right\}.$$

We consider the following two cases (i) and (ii) separately.

(i) Case when $c_2/\alpha_2 \geq c_1/\alpha_1$. It is sufficient for us to show that $g(z)$ is non-negative for $\gamma < z < 1$. Noting that $c_2'\lambda_2 - c_1'\lambda_1 \geq (c_2' - c_1')\lambda_2\beta$ since $\beta \leq 1$ and $\lambda_2 \geq \lambda_1$, we have

$$g(z) \geq \left\{ \frac{1}{c_1'\lambda_1 z + c_2'\lambda_2(1-z)} - \frac{\alpha_1+\alpha_2}{\sum_{i=1}^2 c_i\lambda_i} \right\} \left\{ 1 + \frac{(c_2'-c_1')\lambda_2\beta(z-\gamma)}{c_1'\lambda_1 z + c_2'\lambda_2(1-z)} \right\}.$$

Thus we need only to show that

$$\begin{aligned}(2.3)\qquad &\frac{1}{c_1'\lambda_1 z + c_2'\lambda_2(1-z)} - \frac{\alpha_1+\alpha_2}{\sum_{i=1}^2 c_i\lambda_i} \\ &\quad = \frac{\sum_{i=1}^2 c_i\lambda_i - (\alpha_1+\alpha_2)\{c_1'\lambda_1 z + c_2'\lambda_2(1-z)\}}{\{c_1'\lambda_1 z + c_2'\lambda_2(1-z)\}(\sum_{i=1}^2 c_i\lambda_i)}\end{aligned}$$

is nonnegative. If we set the numerator of of the right hand side of (2.3) as $\ell(z)$, we need only to show that $\ell(\gamma) \geq 0$ and $\ell(1) \geq 0$, both of which can be easily verified. This completes the proof for the case when $c_2/\alpha_2 \geq c_1/\alpha_1$.

(ii) Case when $c_1 > 0$ and $c_2 = 0$. We set $c_1 = 1$ for simplicity. Then the derivative of $h(z)$ is given as

$$\begin{aligned}\frac{d}{dz}h(z) = \frac{\lambda_2\beta}{\alpha_1}\left\{1 - \frac{\lambda_2\beta(z-\gamma)}{\lambda_1 z}\right\}^{-1} \Bigg[&\frac{\alpha_1+\alpha_2}{\lambda_1}\left\{1 - \frac{\lambda_2\beta(z-\gamma)}{\lambda_1 z}\right\} \\ &- \frac{\alpha_1}{\lambda_1 z}\left(1 - \frac{z-\gamma}{z}\right)\Bigg].\end{aligned}$$

We can easily show that

$$\frac{\lambda_2\beta}{\alpha_1}\left\{1 - \frac{\lambda_2\beta(z-\gamma)}{\lambda_1 z}\right\}^{-1} \geq 0, \qquad \text{for} \quad \gamma < z < 1.$$

Further we have

$$\begin{aligned}&\frac{\alpha_1+\alpha_2}{\lambda_1}\left\{1 - \frac{\lambda_2\beta(z-\gamma)}{\lambda_1 z}\right\} - \frac{\alpha_1}{\lambda_1 z}\left(1 - \frac{z-\gamma}{z}\right) \\ &\quad = \left(\frac{\alpha_1}{\lambda_1 z}\right)^2 \left\{ \frac{(\lambda_1-\lambda_2)z^2 + \lambda_2 z}{\alpha_1} - \frac{\lambda_1\lambda_2}{\alpha_2\lambda_1 + \alpha_1\lambda_2} \right\} \\ &\quad \geq \left(\frac{\alpha_1}{\lambda_1 z}\right)^2 \left(\frac{\lambda_1 z}{\alpha_1} - \frac{\lambda_1\lambda_2}{\alpha_2\lambda_1 + \alpha_1\lambda_2} \right)\end{aligned}$$

which is nonnegative for $\gamma < z < 1$. This completes the proof.

Now we will show that if $c_1/\alpha_1 > c_2/\alpha_2 > 0$, then MLE does not dominate UE. More specifically we give the following theorem.

THEOREM 2.2. *Suppose that $c_1/\alpha_1 > c_2/\alpha_2 > 0$. If λ_1 and λ_2 satisfy the condition $c_2\lambda_2/\alpha_2 > c_1\lambda_1/\alpha_1$, then*

$$R\left(\lambda, \sum_{i=1}^2 c_i \frac{X_i}{\alpha_i}\right) < R\left(\lambda, \sum_{i=1}^2 c_i \hat{\lambda}_i\right).$$

PROOF. We show that if λ_1 and λ_2 satisfy the condition, then $h(z)$ given by (2.2) is negative. To show this we first note the following inequality:

$$\log(1+x) \le x, \quad \text{for} \quad x > -1.$$

Applying this inequality to (2.2), we have

$$h(z) \le \{(c_1' - c_2')\lambda_2\beta(z-\gamma)^+\}\left\{\frac{\alpha_1+\alpha_2}{\sum_{i=1}^2 c_i\lambda_i} - \frac{1}{c_1'\lambda_1 z + c_2'\lambda_2(1-z)}\right\}. \tag{2.4}$$

The first factor of the right-hand side of (2.4) is nonnegative since $c_1' > c_2'$. Thus we need only to show that

$$(\alpha_1+\alpha_2)\{c_1'\lambda_1 z + c_2'\lambda_2(1-z)\} - \sum_{i=1}^2 c_i\lambda_i \equiv s(z)$$

is say, negative for $\gamma < z < 1$. We see that $s(z)$ is a strictly decreasing function since $c_2'\lambda_2 > c_1'\lambda_1$ and

$$s(\gamma) = \frac{\alpha_1\alpha_2(c_2'\lambda_2 - c_1'\lambda_1)(\lambda_1-\lambda_2)}{\alpha_2\lambda_1 + \alpha_1\lambda_2}$$

is not positive since $\lambda_1 \le \lambda_2$ and $c_2'\lambda_2 > c_1'\lambda_1$. This completes the proof.

From Theorems 2.1 and 2.2 we see that MLE dominates UE if and only if (i) $c_2/\alpha_2 \ge c_1/\alpha_1$ or (ii) $c_1 > 0$ and $c_2 = 0$. We see that if (c_1, c_2) satisfies the condition, then positive multiple of (c_1, c_2) also satisfies it. This is also clear from (2.1) which is invariant even if we replace (c_1, c_2) with its positive multiple. We also note that the range of c_1/c_2 for which MLE dominates UE becomes larger if α_1 gets larger or α_2 gets smaller.

3. Estimation of reciprocals of ordered scale parameters

Here we consider the estimation of reciprocals of Gamma scale parameters, $\theta_i = \lambda_i^{-1}$, $i = 1, 2$. Order restriction is reversed and we have $\theta_1 \ge \theta_2$. If the shape parameter is an integer, the Gamma distribution is a sum of independent exponential random variables, and reciprocals of scale parameters represent failure rates in statistical theory of reliability.

As stated in Dey *et al.* (1987), we see from Stein (1959) or Brown (1966) that UE (and also the best invariant estimator), $(\alpha_i - 1)X_i^{-1}$, is an admissible estimator of θ_i based solely on X_i under the entropy loss function when $\alpha_i > 1$. Further, Dey *et al.* (1987) have shown that when we estimate θ_1 and θ_2 simultaneously under the sum of entropy loss functions and we don't have the order restriction, $((\alpha_1-1)X_1^{-1}, (\alpha_2-1)X_2^{-1})$ is an admissible estimator when $\min(\alpha_1, \alpha_2) > 5$.

We first consider the estimation of θ_1. If $\lambda_1 = \lambda_2 \equiv \lambda$, $X_1 + X_2$ is distributed as $Gamma(\alpha_1+\alpha_2, \lambda)$, and $(\alpha_1+\alpha_2-1)/(X_1+X_2)$ is the best invariant UE of λ^{-1}. Thus, similarly to $\hat{\lambda}_i$, we may consider

$$\max\left(\frac{\alpha_1-1}{X_1}, \frac{\alpha_1+\alpha_2-1}{X_1+X_2}\right)$$

is a natural estimator of θ_1 and is a candidate which dominates $(\alpha_1 - 1)/X_1$. Actually we give a slightly larger class of estimators which dominate $(\alpha_1 - 1)/X_1$ in the following theorem.

THEOREM 3.1. *If two constants a_1 and b_1 satisfy the conditions $0 < b_1 \leq \alpha_2$ and $0 < b_1 a_1 \leq \alpha_2$, then*

$$\hat{\theta}_1(X_1, X_2) = \frac{\alpha_1 - 1}{X_1} + \frac{a_1\{b_1 X_1 - (\alpha_1 - 1)X_2\}^+}{X_1(X_1 + X_2)}$$

dominates $(\alpha_1 - 1)X_1^{-1}$ in terms of risk.

Note. If we choose $b_1 = \alpha_2$ and $a_1 = 1$, then we have $\max\{(\alpha_1 - 1)X_1^{-1}, (\alpha_1 + \alpha_2 - 1)(X_1 + X_2)^{-1}\}$. If we take b_1 smaller, we can make a_1 larger. If we set $b_1 = \alpha_2 - 1$ and $a_1 = 1$, then a_1 and b_1 satisfy the conditions and we have $\max\{(\alpha_1 - 1)X_1^{-1}, (\alpha_1 + \alpha_2 - 2)(X_1 + X_2)^{-1}\}$.

PROOF. Putting $\theta = (\theta_1, \theta_2)$, the difference between the risks of two estimators of θ_1, $(\alpha_1 - 1)X_1^{-1}$ and $\hat{\theta}_1(X_1, X_2)$, is given by

$$\Delta R = R\left(\theta, \frac{\alpha_1 - 1}{X_1}\right) - R(\theta, \hat{\theta}_1(X_1, X_2))$$

$$= E\left\{\log\left[1 + \frac{a_1\{b_1 X_1 - (\alpha_1 - 1)X_2\}^+}{(\alpha_1 - 1)(X_1 + X_2)}\right] - \frac{a_1\{b_1 X_1 - (\alpha_1 - 1)X_2\}^+}{X_1(X_1 + X_2)\theta_1}\right\}. \tag{3.1}$$

Again we make the variable transformation

$$W = \frac{X_1}{\lambda_1} + \frac{X_2}{\lambda_2} = \theta_1 X_1 + \theta_2 X_2, \qquad Z = \frac{X_1}{\lambda_1 W} = \frac{\theta_1 X_1}{W}. \tag{3.2}$$

Then we can express the risk difference as

$$\Delta R = E\left\{\log\left[1 + \frac{a_1\{b_1 Z\theta_2 - (\alpha_1 - 1)(1 - Z)\theta_1\}^+}{(\alpha_1 - 1)\{\theta_2 Z + \theta_1(1 - Z)\}}\right]\right\} - E\left(\frac{1}{W}\right) E\left\{\frac{a_1\{b_1 Z\theta_2 - (\alpha_1 - 1)(1 - Z)\theta_1\}^+}{Z\{\theta_2 Z + \theta_1(1 - Z)\}}\right\}.$$

Since $E(1/W) = 1/(\alpha_1 + \alpha_2 - 1)$, we have

$$\Delta R = E\{h(Z)\},$$

where

$$h(z) = \log\left[1 + \frac{a_1\xi(z - \rho)^+}{(\alpha_1 - 1)\{\theta_2 z + \theta_1(1 - z)\}}\right] - \frac{a_1\xi(z - \rho)^+}{(\alpha_1 + \alpha_2 - 1)z\{\theta_2 z + \theta_1(1 - z)\}},$$

$\xi = b_1\theta_2 + (\alpha_1 - 1)\theta_1$ and $\rho = (\alpha_1 - 1)\theta_1/\xi$. It is clear that $h(\rho) = 0$, and we need only to show that $h(z)$ is a non-decreasing function of z for $\rho \leq z < 1$. The derivative of $h(z)$ is given as

$$\frac{d}{dz}h(z) = a_1\xi\left[1 + \frac{a_1\xi(z - \rho)^+}{(\alpha_1 - 1)\{\theta_2 z + \theta_1(1 - z)\}}\right]^{-1} g(z),$$

where

$$\begin{aligned} g(z) = &\frac{1}{(\alpha_1 - 1)\{\theta_2 z + \theta_1(1-z)\}}\left\{1 + \frac{(\theta_1 - \theta_2)(z-\rho)}{\theta_2 z + \theta_1(1-z)}\right\} \\ &- \frac{1}{(\alpha_1+\alpha_2-1)z\{\theta_2 z + \theta_1(1-z)\}}\left[1 + \frac{a_1\xi(z-\rho)^+}{(\alpha_1-1)\{\theta_2 z + \theta_1(1-z)\}}\right] \\ &\times \left\{\frac{\rho}{z} + \frac{(\theta_1-\theta_2)(z-\rho)}{\theta_2 z + \theta_1(1-z)}\right\}. \end{aligned}$$

For $z \geq \rho$ we have

$$\begin{aligned} g(z) \geq &\left\{\frac{1}{\alpha_1 - 1} - \frac{1}{(\alpha_1+\alpha_2-1)z}\left[1 + \frac{a_1\xi(z-\rho)^+}{(\alpha_1-1)\{\theta_2 z + \theta_1(1-z)\}}\right]\right\} \\ &\times \frac{1}{\theta_2 z + \theta_1(1-z)}\left\{1 + \frac{(\theta_1-\theta_2)(z-\rho)}{\theta_2 z + \theta_1(1-z)}\right\}. \end{aligned} \tag{3.3}$$

Let the first factor of the right-hand side of (3.3) be $t(z)$, then we show that $t(z) \geq 0$ for $\rho \leq z \leq 1$. It is easily seen that we only need to examine the two endpoints $z = \rho$ and $z = 1$. We have

$$t(\rho) = \frac{1}{\alpha_1 - 1} - \frac{1}{(\alpha_1+\alpha_2-1)\rho} = \frac{\alpha_2\theta_1 - b_1\theta_2}{(\alpha_1-1)(\alpha_1+\alpha_2-1)\theta_1}$$

which is non-negative if $b_1 \leq \alpha_2$. We also have

$$t(1) = \frac{\alpha_2 - a_1 b_1}{(\alpha_1-1)(\alpha_1+\alpha_2-1)}$$

which is non-negative if $a_1 b_1 \leq \alpha_2$. This completes the proof.

We give a class of estimators of θ_2 which improve upon $(\alpha_2 - 1)X_2^{-1}$ in the following theorem. The proof is very similar to the one of Theorem 3.1 and is omitted.

THEOREM 3.2. *If two constants a_2 and b_2 satisfy the conditions $b_2 \geq \alpha_1$ and $0 < a_2 \leq 1$, then*

$$\hat{\theta}_2(X_1, X_2) = \frac{\alpha_2 - 1}{X_2} - \frac{a_2\{(\alpha_2-1)X_1 - b_2 X_2\}^+}{X_2(X_1+X_2)}$$

dominates $(\alpha_2 - 1)X_2^{-1}$ in terms of risk.

If we choose $b_2 = \alpha_1$ and $a_2 = 1$, we have $\min\{(\alpha_2-1)X_2^{-1}, (\alpha_1+\alpha_2-1)(X_1+X_2)^{-1}\}$. The estimator $\min\{(\alpha_2-1)X_2^{-1}, (\alpha_1+\alpha_2-2)(X_1+X_2)^{-1}\}$ corresponds to the choice of $b_2 = \alpha_1 - 1$ and $a_2 = 1$, which do not satisfy the condition.

As stated earlier, Dey *et al.* (1987) have shown that if $\min(\alpha_1, \alpha_2) > 5$, $((\alpha_1 - 1)/X_1, (\alpha_2-1)/X_2)$ is an admissible estimator of (θ_1, θ_2) under the sum of entropy loss. However, from Theorems 3.1 and 3.2 we see that $(\hat{\theta}_1(X_1, X_2), \hat{\theta}_2(X_1, X_2))$ dominates $((\alpha_1 - 1)/X_1, (\alpha_2 - 1)/X_2)$ when we have the restriction $\theta_1 \geq \theta_2$, provided that the conditions given in the theorems are satisfied.

The condition $b_2 \geq \alpha_1$ given in Theorem 3.2 implies that $\hat{\theta}_2(x_1, x_2) = (\alpha_2 - 1)/x_2$ at least in the region $x_2 > (\alpha_2 - 1)x_1/\alpha_1$. If the estimator $\hat{\theta}_1(X_1, X_2)$ is to satisfy the order restriction $\hat{\theta}_1(X_1, X_2) \geq \hat{\theta}_2(X_1, X_2)$ for such an estimator $\hat{\theta}_2(X_1, X_2)$, we need $b_1 x_1 > (\alpha_1 - 1)x_2$ for any (x_1, x_2) such that $(\alpha_1 - 1)/x_1 < (\alpha_2 - 1)/x_2 < \alpha_1/x_1$. Thus we have $b_1 \geq \alpha_2 - 1$. In this case the region where $\hat{\theta}_1(x_1, x_2)$ is not equal to $(\alpha_1 - 1)/x_1$ is different from the one where $\hat{\theta}_2(x_1, x_2)$ is not equal to $(\alpha_2 - 1)/x_2$. This makes it technically difficult to discuss the general problem of improving the UE, $c_1(\alpha_1 - 1)/X_1 + c_2(\alpha_2 - 1)/X_2$, of $c_1\theta_1 + c_2\theta_2$ for arbitrarily fixed non-negative constants c_1 and c_2. We have not succeeded in giving any definite result.

Note. Figure 1 shows the regions where $\hat{\theta}_1(X_1, X_2)$ and $\hat{\theta}_2(X_1, X_2)$ differ from UE.

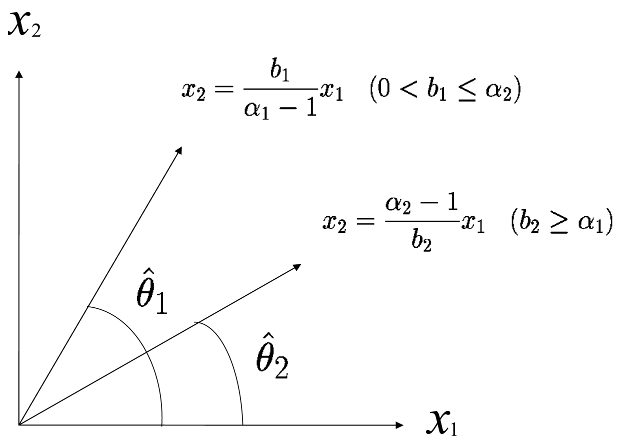

Figure 1. Regions where $\hat{\theta}_1(X_1, X_2)$ and $\hat{\theta}_2(X_1, X_2)$ differ from UE.

4. Numerical results

In this section we give some results based on numerical evaluations to illustrate the extent of the risk improvement. We have made numerical evaluations of risk functions of the estimators by first making a variable transformation given by (2.2), and then by performing numerical integration using Mathematica.

We first consider the estimation of $c_1\lambda_1 + c_2\lambda_2$ where c_1 and c_2 are non-negative constants. We define the relative risk of MLE $c_1\hat{\lambda}_1 + c_2\hat{\lambda}_2$ compared to

UE $c_1X_1/\alpha_1 + c_2X_2/\alpha_2$ as

$$\frac{R(\lambda, c_1\hat{\lambda}_1 + c_2\hat{\lambda}_2)}{R\left(\lambda, c_1\dfrac{X_1}{\alpha_1} + c_2\dfrac{X_2}{\alpha_2}\right)}.$$

We have made calculations for 12 cases of (α_1, α_2): $(1, 1)$, $(1, 3)$, $(1, 5)$, $(3, 1)$, $(3, 3)$, $(3, 5)$, $(5, 1)$, $(5, 3)$, $(5, 5)$, $(9, 1)$, $(9, 3)$ and $(9, 5)$. We have chosen (c_1, c_2) as $(1, 0)$, $(0, 1)$ and $(1, 2)$, which correspond to the estimation of λ_1, λ_2 and $\lambda_1 + 2\lambda_2$ respectively. We note that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ always dominate UE and that $\hat{\lambda}_1 + 2\hat{\lambda}_2$ dominates UE except for the cases $(\alpha_1, \alpha_2) = (1, 3)$ and $(1, 5)$. Without loss of generality we fix $\lambda_1 = 1$ and have calculated risk estimates for $\lambda_2 = 1(1)10$. For each choice of (α_1, α_2) relative risks for three choices of (c_1, c_2) are shown in one figure (Fig. 2). We note that relative risk of $\hat{\lambda}_1$ gets smaller when α_1 becomes smaller or α_2 larger. On the contrary the relative risk of $\hat{\lambda}_2$ gets smaller when α_1 becomes larger or α_2 becomes smaller. The relative risk of $\hat{\lambda}_2$ is quite small in some cases, but that of $\hat{\lambda}_1$ is not small. As a function of λ_2, the minimum seems

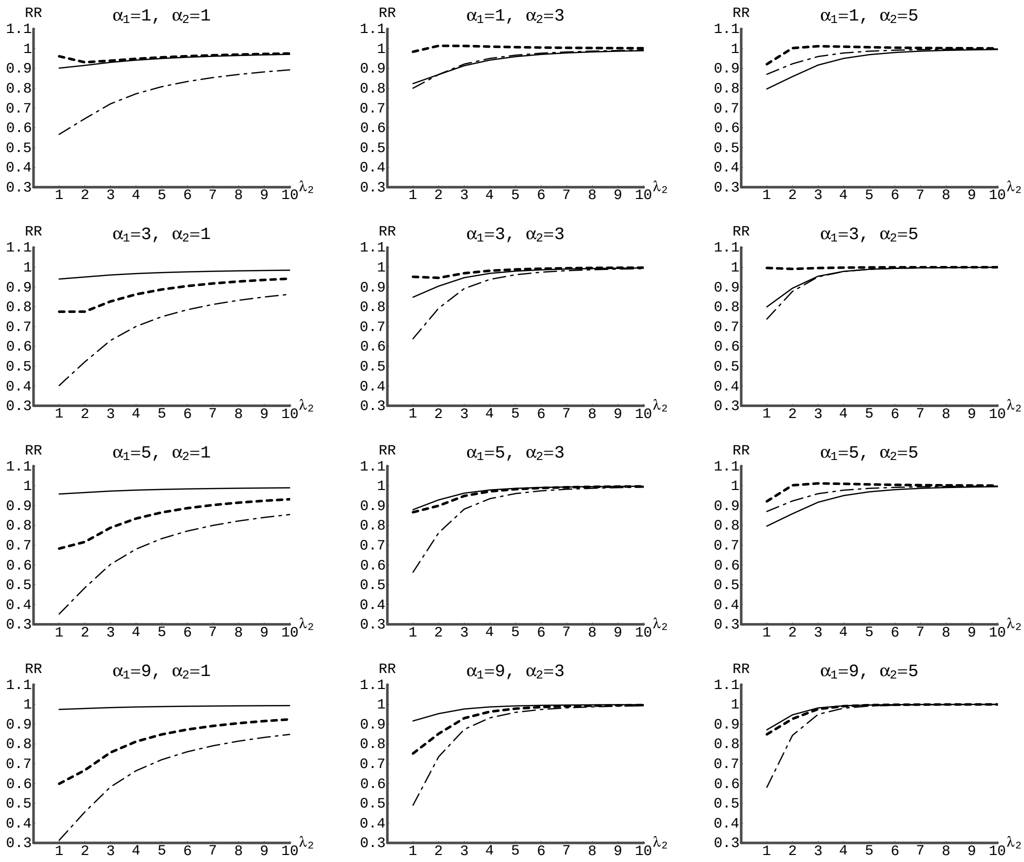

Figure 2. Relative risks of MLE, dashed line for $c_1 = 1$, $c_2 = 2$, solid line for $c_1 = 1$, $c_2 = 0$, and dotted line for $c_1 = 0$, $c_2 = 1$.

to be attained when $\lambda_2 = \lambda_1$ for both $\hat{\lambda}_1$ and $\hat{\lambda}_2$. The relative risk of $\hat{\lambda}_1 + 2\hat{\lambda}_2$ is larger than those of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ except for some cases where $\alpha_1 = 1$.

Next we consider the estimation of $\theta_1 = 1/\lambda_1$ and $\theta_2 = 1/\lambda_2$. We set $\hat{\theta}_1 = \max\{(\alpha_1 - 1)X_1^{-1}, (\alpha_1 + \alpha_2 - 1)(X_1 + X_2)^{-1}\}$ and $\hat{\theta}_2 = \min\{(\alpha_2 - 1)X_2^{-1}, (\alpha_1 + \alpha_2 - 1)(X_1 + X_2)^{-1}\}$ and define the relative risk of $\hat{\theta}_i$ compared to $(\alpha_i - 1)X_i^{-1}$ as

$$\frac{R(\theta, \hat{\theta}_i)}{R(\theta, (\alpha_i - 1)X_i^{-1})}, \qquad i = 1, 2.$$

We have made calculations for $\alpha_1 = 2(1)5$, 7, 10, and for $\alpha_2 = 2$, 3, 4, 5. Again we fix $\theta_1 = 1$ and have calculated risk estimates for $\theta_2 = 1$, 1/2, 1/4, 1/8. The relative risks are given in Table 1. We see that risk improvement can be quite substantial in some cases for both $\hat{\theta}_1$ and $\hat{\theta}_2$, especially when the value of θ_2 is close to that of θ_1. The relative risk of $\hat{\theta}_1$ gets smaller when α_1 becomes smaller or α_2 becomes larger. On the contrary the relative risk of $\hat{\theta}_2$ gets smaller when α_1 becomes larger or α_2 becomes smaller.

5. Concluding remarks

In Section 2, the estimation problem of linear functions of ordered scale parameters of two Gamma distributions is discussed under the entropy loss function. We have given a necessary and sufficient condition on two non-negative coefficients for MLE to dominate UE. The same problem has been dealt with under squared error loss functions in Chang and Shinozaki (2002), and a necessary and sufficient condition is given on two coefficients for MLE to dominate UE. A necessary and sufficient condition is also given on coefficients for modified MLE (which we can obtain by replacing α_i by $\alpha_i + 1$ in MLE) to dominate $\sum_{i=1}^{2} c_i x_i/(\alpha_i + 1)$. Although negative coefficients are allowed in the case of squared error loss, we restrict analysis to the case with non-negative coefficients and compare the conditions.

For the case when $c_1 > 0$ and $c_2 = 0$, similar to the case of the entropy loss function, MLE and modified MLE dominate their competitors under mild conditions. Thus we need only to compare the conditions when $c_1 \geq 0$ and $c_2 > 0$. Then we notice that three conditions are all of the form $c_1/c_2 \leq c(\alpha_1, \alpha_2)$, where $c(\alpha_1, \alpha_2)$ is a constant which depends on α_1 and α_2. As is given in Theorem 2.1, $c(\alpha_1, \alpha_2) = \alpha_1/\alpha_2$ for the entropy loss function. We notice that this is comparable to the result $c(\alpha_1, \alpha_2) = (\alpha_1 + 1)/(\alpha_2 + 1)$ when modified MLE is to dominate its competitor under squared error loss. An expression is given for $c(\alpha_1, \alpha_2)$ in terms of an incomplete Beta function when MLE is to dominate UE under squared error loss in Chang and Shinozaki (2002). The values of $c(\alpha_1, \alpha_2)$ are numerically evaluated and the results show that they are much smaller than those of α_1/α_2 and $(\alpha_1 + 1)/(\alpha_2 + 1)$. Thus the obtained result for the entropy loss function given in Section 2 is quite similar to the one for the squared error loss function and for modified MLE.

Table 1. Relative risks of $\hat{\theta}_1$ and $\hat{\theta}_2$ when $\theta_1 = 1$.

		$\alpha_2 = 2$		$\alpha_2 = 3$		$\alpha_2 = 4$		$\alpha_2 = 5$	
α_1	θ_2	$\frac{R(\theta,\hat{\theta}_1)}{R\left(\theta,\frac{\alpha_1-1}{x_1}\right)}$	$\frac{R(\theta,\hat{\theta}_2)}{R\left(\theta,\frac{\alpha_2-1}{x_2}\right)}$	$\frac{R(\theta,\hat{\theta}_1)}{R\left(\theta,\frac{\alpha_1-1}{x_1}\right)}$	$\frac{R(\theta,\hat{\theta}_2)}{R\left(\theta,\frac{\alpha_2-1}{x_2}\right)}$	$\frac{R(\theta,\hat{\theta}_1)}{R\left(\theta,\frac{\alpha_1-1}{x_1}\right)}$	$\frac{R(\theta,\hat{\theta}_2)}{R\left(\theta,\frac{\alpha_2-1}{x_2}\right)}$	$\frac{R(\theta,\hat{\theta}_1)}{R\left(\theta,\frac{\alpha_1-1}{x_1}\right)}$	$\frac{R(\theta,\hat{\theta}_2)}{R\left(\theta,\frac{\alpha_2-1}{x_2}\right)}$
2	1	0.738	0.635	0.680	0.705	0.643	0.756	0.618	0.792
	0.5	0.796	0.699	0.768	0.792	0.753	0.847	0.745	0.881
	0.25	0.894	0.805	0.901	0.906	0.909	0.948	0.915	0.968
	0.125	0.959	0.889	0.972	0.967	0.981	0.988	0.986	0.995
3	1	0.817	0.603	0.760	0.661	0.719	0.709	0.689	0.746
	0.5	0.866	0.676	0.842	0.767	0.829	0.826	0.822	0.864
	0.25	0.938	0.792	0.944	0.899	0.952	0.946	0.958	0.968
	0.125	0.978	0.882	0.987	0.966	0.992	0.989	0.995	0.996
4	1	0.858	0.585	0.805	0.634	0.766	0.677	0.735	0.712
	0.5	0.899	0.662	0.879	0.752	0.868	0.813	0.863	0.853
	0.25	0.955	0.785	0.961	0.896	0.968	0.945	0.973	0.969
	0.125	0.985	0.879	0.992	0.966	0.995	0.989	0.997	0.996
5	1	0.884	0.574	0.836	0.614	0.798	0.654	0.768	0.687
	0.5	0.919	0.654	0.901	0.742	0.892	0.804	0.888	0.846
	0.25	0.965	0.781	0.970	0.893	0.976	0.944	0.981	0.969
	0.125	0.988	0.877	0.994	0.965	0.997	0.989	0.998	0.996
7	1	0.915	0.559	0.875	0.590	0.842	0.622	0.814	0.652
	0.5	0.942	0.644	0.928	0.729	0.921	0.792	0.918	0.836
	0.25	0.976	0.776	0.979	0.890	0.984	0.943	0.988	0.969
	0.125	0.992	0.874	0.996	0.965	0.998	0.989	0.999	0.996
10	1	0.939	0.548	0.908	0.568	0.880	0.594	0.856	0.618
	0.5	0.959	0.636	0.949	0.719	0.943	0.782	0.941	0.828
	0.25	0.983	0.771	0.986	0.887	0.989	0.943	0.992	0.969
	0.125	0.995	0.872	0.997	0.964	0.999	0.989	0.999	0.997

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