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IMPROVEMENT ON THE BEST EQUIVARIANT PREDICTORS UNDER THE ORDERED PARAMETERS

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This paper treats a statistical prediction problem under the ordered parameters. An improvement on the best equivariant predictor is discussed for the location family and the scale family.

Key words and phrases: Equivariant predictor, location family, ordered parameters, scale family, total positivity of order 2.

1. Introduction

There are situations in which some order restrictions are assumed about the parameters of the underlying distribution. Order restrictions may be provided by either prior information about the parameters or the mathematical structure of the problem. Such restrictions may enable us to improve on usual estimation procedures. Methods for improving have been investigated by Cohen and Sackrowiz (1970), Brewster and Zidek (1974), Kushary and Cohen (1989), and Kubokawa and Saleh (1994) and Kubokawa (1994). The relevant estimation problems with restricted parameters have been treated by Kubokawa (2004) and Machand and Strawderman (2005). See also the references in their papers. The most attention in the literature has been given to estimation problems. In this paper we shall consider prediction problems under the ordered parameters.

There are two random variables X and Y whose joint distribution is indexed by an unknown parameter ξ. Based on the value of X, we want to predict the value of Y. We consider such a situation that another random variable Z, whose distribution is indexed by an unknown parameter η with $\eta \geq \xi$, is available to the prediction problem. We shall provide methods for improving on the best equivariant predictor by making use of the value of Z.

A function called totally positive of order 2 (TP_2) plays a fundamental role in deriving the main results. A function $K(x, y)$ is said to be TP_2 if

$$\begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) \\ K(x_2, y_1) & K(x_2, y_2) \end{vmatrix} \geq 0$$

for all $x_1 < x_2$ and $y_1 < y_2$. See Karlin (1968) for the complete treatment of TP_2 and Barlow and Proschan (1975) for its applications to reliability and life testing.

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Kubokawa and Saleh (1994) applied a very useful method called the integrated expression of risk difference (IERD) method to get an improved estimator. We shall also use the IERD method to derive an improved predictor.

In Section 2, assuming a location family, an improvement on the best location equivariant predictor will be considered. Section 3 will treat a scale family.

2. Location family

Suppose that (X, Y, Z) has the joint density

$$f(x-\xi, y-\xi, z-\eta) \tag{2.1}$$

where f is known, and ξ and η are unknown location parameters with $\xi \leq \eta$. We shall consider the problem of improving a predictor

$$\delta_c = X + c \tag{2.2}$$

by

$$\delta_\phi = X + \phi(Z - X) \tag{2.3}$$

where c is a constant and ϕ is a function. We shall assume that the loss function is of the form

$$L(y-d) \tag{2.4}$$

when predicting $Y = y$ by d, and $L(t)$ is strictly decreasing for $t < 0$ and strictly increasing for $t > 0$ with $L(0) = 0$.

Let $U = Z - X$ and $V = Y - X$. From (2.2), (2.3) and (2.4), the risk function of δ_c and δ_ϕ can be written as

$$R(\theta, \delta_c) = E_\theta\{L(V - c)\} \tag{2.5}$$

and

$$R(\theta, \delta_\phi) = E_\theta\{L(V - \phi(U))\} \tag{2.6}$$

where $\theta \in \Theta_L = \{\theta = (\xi, \eta) \mid \xi \leq \eta\}$. From (2.1) the joint density of (U, V) is given by $g(u - \lambda, v)$ where $\lambda = \eta - \xi \geq 0$ and

$$g(u, v) = \int_{-\infty}^{\infty} f(t, t+v, t+u)dt. \tag{2.7}$$

Hence from (2.5) and (2.6) the difference of the risk functions can be expressed as

$$\begin{aligned} &R(\theta, \delta_c) - R(\theta, \delta_\phi) \\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{L(v-c) - L(v - \phi(u))\} g(u-\lambda, v)dudv. \end{aligned} \tag{2.8}$$

The following lemma by Kubokawa and Saleh (1994) is very useful in the subsequent discussion.

LEMMA 2.1. *For positive functions $g(x)$ and $h(x)$, assuming that $h(x)/g(x)$ is non-increasing. If $K(x) < 0$ for $x < x_0$ and $K(x) > 0$ for $x > x_0$, then*

$$\int_{-\infty}^{\infty} K(x)\frac{h(x)}{g(x)}dx \le \frac{h(x_0)}{g(x_0)}\int_{-\infty}^{\infty} K(x)dx$$

where the equality holds if and only if $h(x)/g(x)$ is a constant almost everywhere.

In the sequel we will assume that interchange of integral and derivative is permissible whenever necessary. Let

$$G(u,v) = \int_0^{\infty} g(u-t,v)dt. \tag{2.9}$$

Then we have the following result.

THEOREM 2.1. *Assume that*

$$G(u,v) \text{ is } TP_2, \tag{2.10}$$

$$\phi(u) \text{ is non-decreasing and } \lim_{u\to\infty}\phi(u) = c, \tag{2.11}$$

$$\int_{-\infty}^{\infty} L'(v-\phi(u))G(u,v)dv \le 0 \text{ for each } u. \tag{2.12}$$

Then $R(\theta,\delta_\phi) \le R(\theta,\delta_c)$ for any $\theta \in \Theta_L$.

PROOF. Using the IERD method, it follows from (2.8), (2.9) and (2.11) that

$$\begin{aligned} R(\theta,\delta_c) &- R(\theta,\delta_\phi) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{L(v-\phi(u+\infty)) - L(v-\phi(u))\}g(u-\lambda,v)dudv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^{\infty}\left\{\frac{d}{dt}L(v-\phi(t+u))\right\}dt\right)g(u-\lambda,v)dudv \\ &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^{\infty}L'(v-\phi(t+u))\phi'(t+u)dt\right)g(u-\lambda,v)dudv \\ &= -\int_{-\infty}^{\infty}\phi'(u)\left(\int_{-\infty}^{\infty}L'(v-\phi(u))G(u-\lambda,v)dv\right)du. \end{aligned} \tag{2.13}$$

Since

$$\begin{aligned} \int_{-\infty}^{\infty} L'(v-\phi(u))G(u-\lambda,v)dv &= \int_{-\infty}^{\infty} L'(v-\phi(u))G(u,v)\frac{G(u-\lambda,v)}{G(u,v)}dv \\ &\le \frac{G(u-\lambda,\phi(u))}{G(u,\phi(u))}\int_{-\infty}^{\infty} L'(v-\phi(u))G(u,v)dv \end{aligned}$$

by (2.10) and Lemma 2.1, it follows from (2.11) and (2.12) that the right hand side of (2.13) is nonnegative. Hence

$$R(\theta,\delta_c) \ge R(\theta,\delta_\phi),$$

which completes the proof.

From (2.9)

$$G(u,v)=\int_{-\infty}^{u} g(t,v)dt. \tag{2.14}$$

If $g(u,v)$ is TP_2, then $G(u,v)$ is TP_2. See Barlow and Proschan (1975, p. 105). Furthermore, let

$$h(v)=\lim_{u\to\infty} G(u,v). \tag{2.15}$$

Then from (2.14) $h(v)$ is the density of V.

Suppose that $\int_{-\infty}^{\infty} L(v-c)G(u,v)dv$ is not a monotone function of c for each u and $G(u,x-y)$ is TP_2 in x and y for each u. Then there exists $\phi_0(u)$ uniquely such that

$$\int_{-\infty}^{\infty} L'(v-\phi_0(u))G(u,v)dv=0 \tag{2.16}$$

for each u. See Barlow and Proschan (1975, p. 93). We also suppose that $\int_{-\infty}^{\infty} L(v-c)h(v)dv$ is not a monotone functions of c. It follows from (2.15) that there exists c_0 uniquely such that

$$\int_{-\infty}^{\infty} L'(v-c_0)h(v)dv=0 \tag{2.17}$$

and the best location equivariant predictor based on X is given by

$$\delta_{c_0}=X+c_0.$$

Corollary 2.1. *Assume*

$$G(u,v) \text{ is } TP_2, \tag{2.18}$$

$$G(u,x-y) \text{ is } TP_2 \text{ in } x \text{ and } y \text{ for each } u. \tag{2.19}$$

Then $R(\theta,\delta_{\phi_0})\le R(\theta,\delta_{c_0})$ *for any* $\theta\in\Theta_L$.

Proof. It suffices to show that (2.11) is satisfied. Suppose that there exist $u_1<u_2$ such that $\phi_0(u_1)>\phi_0(u_2)$. Let $c_1=\phi_0(u_1)$ and $c_2=\phi_0(u_2)$. It follows from (2.18) and (2.19) that

$$\frac{G(u_1,v+c_1)}{G(u_2,v+c_1)}\frac{G(u_2,v+c_1)}{G(u_2,v+c_2)}$$

is non-increasing in v. Hence from (2.16) and Lemma 2.1

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} L'(v-c_1)G(u_1,v)dv \\
&= \int_{-\infty}^{\infty} L'(v)G(u_1,v+c_1)dv \\
&= \int_{-\infty}^{\infty} L'(v)\frac{G(u_1,v+c_1)}{G(u_2,v+c_1)}\frac{G(u_2,v+c_1)}{G(u_2,v+c_2)}G(u_2,v+c_2)dv \\
&< \frac{G(u_1,c_1)}{G(u_2,c_2)}\int_{-\infty}^{\infty} L'(v)G(u_2,v+c_2)dv=0
\end{aligned}$$

which shows a contradiction. So $\phi_0(u)$ is non-decreasing, and hence $\lim_{u\to\infty}\phi_0(u)$ exists. From (2.15) and (2.16)

$$0=\lim_{u\to\infty}\int_{-\infty}^{\infty}L'(v-\phi_0(u))G(u,v)dv=\int_{-\infty}^{\infty}L'\left(v-\lim_{u\to\infty}\phi_0(u)\right)h(v)dv,$$

which yields

$$\lim_{u\to\infty}\phi_0(u)=c_0$$

from the uniqueness of c_0.

Example 2.1. Suppose that (X,Y,Z) has a multivariate normal distribution with unknown mean vector (ξ,ξ,η) and known covariance matrix and $\xi\le\eta$. Then (U,V) has the bivariate normal distribution with mean vector $(\eta-\xi,0)$ and known covariance matrix

$$\begin{pmatrix}\sigma^2 & \rho\sigma\tau\\ \rho\sigma\tau & \tau^2\end{pmatrix}$$

where σ^2 and τ^2 are the variances of U and V, and ρ is the correlation coefficient. We suppose that $\rho>0$. Then it is well known that $g(u,v)$ is TP_2, so that (2.18) is satisfied. The straightforward calculation shows that

$$G(u,v)=\frac{1}{\tau}\phi\left(\frac{v}{\tau}\right)\Phi\left(\frac{u-\frac{\rho\sigma}{\tau}v}{\sqrt{\sigma^2(1-\rho^2)}}\right)\tag{2.20}$$

where Φ is the standard normal distribution function and ϕ is its density. It can be shown that (2.19) is satisfied.

For example, let L be the squared error loss. Then it follows from (2.16) and (2.17) that the best location equivariant predictor is given by $\delta_{c_0}=X$ with $c_0=0$ and its improved predictor becomes $\delta_{\phi_0}=X+\phi_0(Z-X)$ where

$$\phi_0(u)=\frac{\int_{-\infty}^{\infty}vG(u,v)dv}{\int_{-\infty}^{\infty}G(u,v)dv}.\tag{2.21}$$

From (2.14),

$$\int_{-\infty}^{\infty}G(u,v)dv=\int_{-\infty}^{u}\left(\int_{-\infty}^{\infty}g(t,v)dv\right)dt=\Phi(u/\sigma)\tag{2.22}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}vG(u,v)dv&=\int_{-\infty}^{u}\left(\int_{-\infty}^{\infty}vg(t,v)dv\right)dt\\&=\int_{-\infty}^{u}\frac{\rho\tau}{\sigma}\frac{t}{\sigma}\phi(t/\sigma)dt=-\rho\tau\phi(u/\sigma).\end{aligned}\tag{2.23}$$

Substituting (2.22) and (2.23) into (2.21),

$$\phi_0(u)=-\frac{\rho\tau\phi(u/\sigma)}{\Phi(u/\sigma)}.$$

3. Scale family

Suppose that (X, Y, Z) has the joint density

$$\xi^{-2}\eta^{-1}f(x/\xi, y/\xi, z/\eta), \qquad x > 0,\ y > 0,\ z > 0$$

where f is known, and ξ and η are unknown scale parameters with $0 < \xi \le \eta$. We shall consider the problem of improving a predictor $\delta_c = cX$ by

$$\delta_\phi = \phi(Z/X)X$$

relative to the loss function $L(d/y)$ when predicting $Y = y$ by d, where c is a positive constant and ϕ is a positive function. We assume that $L(t)$ $(t > 0)$ is strictly decreasing for $t < 1$ and strictly increasing for $t > 1$ with $L(1) = 0$.

Let $U = Z/X$ and $V = Y/X$. Then the risk functions of δ_c and δ_ϕ can be written as

$$R(\theta, \delta_c) = E_\theta\{L(c/V)\}$$

and

$$R(\theta, \delta_\phi) = E_\theta\{L(\phi(U)/V)\}$$

where $\theta \in \Theta_L = \{\theta = (\xi, \eta) \mid 0 < \xi \le \eta\}$. The joint density of (U, V) is given by $\lambda^{-1}g(u/\lambda, v)$ where $\lambda = \eta/\xi \ge 1$ and

$$g(u, v) = \int_0^\infty t^2 f(t, tv, tu)dt.$$

Let

$$G(u, v) = u\int_1^\infty t^{-2}g(u/t, v)dt. \tag{3.1}$$

THEOREM 3.1. *Assume that*

$$G(u, v) \text{ is } TP_2, \tag{3.2}$$

$$\phi(u) \text{ is non-decreasing and } \lim_{u\to\infty}\phi(u) = c, \tag{3.3}$$

$$\int_0^\infty v^{-1}L'(\phi(u)/v)G(u, v)dv \ge 0 \text{ for each } u. \tag{3.4}$$

Then $R(\theta, \delta_\phi) \le R(\theta, \delta_c)$ *for any* $\theta \in \Theta_L$.

PROOF. Using (3.3), the difference of the risk functions is expressed as

$$\begin{aligned} &R(\theta, \delta_c) - R(\theta, \delta_\phi) \\ &= \int_0^\infty\int_0^\infty \{L(c/v) - L(\phi(u)/v)\}\lambda^{-1}g(u/\lambda, v)dudv \\ &= \int_0^\infty\int_0^\infty \left(\int_1^\infty \left\{\frac{d}{dt}L(\phi(tu)/v)\right\}dt\right)\lambda^{-1}g(u/\lambda, v)dudv \\ &= \int_0^\infty\int_0^\infty \left(\int_1^\infty \frac{\phi'(tu)u}{v}L'\left(\frac{\phi(tu)}{v}\right)dt\right)\lambda^{-1}g(u/\lambda, v)dudv \end{aligned} \tag{3.5}$$

$$= \int_0^\infty \int_0^\infty \int_1^\infty \frac{\phi'(u)u}{t^2 v} L'\left(\frac{\phi(u)}{v}\right) \lambda^{-1} g(u/t\lambda, v) dt du dv$$
$$= \int_0^\infty \int_0^\infty \frac{\phi'(u)}{v} L'\left(\frac{\phi(u)}{v}\right) G\left(\frac{u}{\lambda}, v\right) du dv$$
$$= \int_0^\infty \phi'(u) \left(\int_0^\infty v^{-1} L'\left(\frac{\phi(u)}{v}\right) G(u,v) \frac{G(u/\lambda, v)}{G(u,v)} dv\right) du.$$

It follows from (3.2) that $G(u/\lambda, v)/G(u,v)$ is non-increasing in v for each u. Hence from Lemma 2.1 and (3.4)

$$\int_0^\infty v^{-1} L'\left(\frac{\phi(u)}{v}\right) G(u,v) \frac{G(u/\lambda, v)}{G(u,v)} dv$$
$$\geq \frac{G(u/\lambda, \phi(u))}{G(u, \phi(u))} \int_0^\infty v^{-1} L'\left(\frac{\phi(u)}{v}\right) G(u,v) dv$$
$$\geq 0,$$

so that (3.3) and (3.5) yield the result.

It follows from (3.1) that

$$G(u,v) = \int_0^u g(t,v) dt.$$

Hence if $g(u,v)$ is TP_2, then $G(u,v)$ becomes TP_2. Let

$$h(v) = \lim_{u \to \infty} G(u,v). \tag{3.6}$$

Then $h(v)$ is the density of V. Suppose that $G(u, y/x)$ is TP_2 in x and y for each u and $\int_0^\infty L(c/v)G(u,v)dv$ is not a monotone function of c for each u. Then there exists $\phi_0(u)$ uniquely such that

$$\int_0^\infty v^{-1} L'(\phi_0(u)/v) G(u,v) dv = 0 \tag{3.7}$$

for each u. We also suppose that $\int_0^\infty L(c/v)h(v)dv$ is not a monotone function of c. Then there exists $c_0 (> 0)$ uniquely such that

$$\int_0^\infty v^{-1} L'(c_0/v) h(v) dv = 0 \tag{3.8}$$

and the best scale equivariant predictor based on X is given by

$$\delta_{c_0} = c_0 X.$$

Corollary 3.1. *Assume that*

(3.9) $G(u,v)$ *is* TP_2,

(3.10) $G(u, y/x)$ *is* TP_2 *in* x *and* y *for each* u.

Then $R(\theta, \delta_{\phi_0}) \le R(\theta, \delta_{c_0})$ *for any* $\theta \in \Theta_L$.

PROOF. It suffices to show that (3.3) is satisfied. Suppose that there exist $u_1 < u_2$ such that $\phi_0(u_1) > \phi_0(u_2)$. Let $c_1 = \phi_0(u_1)$ and $c_2 = \phi_0(u_2)$. Then

$$\begin{aligned} 0 &= \int_0^\infty v^{-1} L'(c_1/v) G(u_1, v) dv \\ &= \int_0^\infty v^{-1} L'(v) G(u_1, c_1/v) dv \\ &= \int_0^\infty v^{-1} L'(v) \frac{G(u_1, c_1/v)}{G(u_2, c_1/v)} \frac{G(u_2, c_1/v)}{G(u_2, c_2/v)} G(u_2, c_2/v) dv. \end{aligned}$$

It follows from (3.9) and (3.10) that

$$\frac{G(u_1, c_1/v)}{G(u_2, c_1/v)} \frac{G(u_2, c_1/v)}{G(u_2, c_2/v)}$$

is non-decreasing. By Lemma 2.1

$$\begin{aligned} &\int_0^\infty v^{-1} L'(v) \frac{G(u_1, c_1/v)}{G(u_2, c_1/v)} \frac{G(u_2, c_1/v)}{G(u_2, c_2/v)} G(u_2, c_2/v) dv \\ &\quad > \frac{G(u_1, c_1)}{G(u_2, c_2)} \int_0^\infty v^{-1} L'(v) G(u_2, c_2/v) dv = 0, \end{aligned}$$

which shows a contradiction. So $\phi_0(u)$ is non-decreasing and hence $\lim_{u\to\infty} \phi_0(u)$ exists. From (3.7) and (3.8)

$$0 = \lim_{u\to\infty} \int_0^\infty v^{-1} L'(\phi_0(u)/v) G(u, v) dv = \int_0^\infty v^{-1} L' \left(\lim_{u\to\infty} \phi_0(u)/v \right) h(v) dv,$$

so that the uniqueness of c_0 implies

$$\lim_{u\to\infty} \phi_0(u) = c_0.$$

The proof is completed.

Example 3.1. Suppose that W_1 and W_2 are independent random variables according to the exponential distribution with density $\xi^{-1} e^{-x/\xi}$, $x > 0$, $\xi > 0$. Let $X = \min(W_1, W_2)$ and $Y = \max(W_1, W_2)$. The problem of predicting the value of Y is considered. We want to improve a predictor based on X by utilizing Z which is independent of W_1 and W_2 and is distributed according to the exponential distribution with density $\eta^{-1} e^{-x/\eta}$, $x > 0$, $\eta > 0$ and $\eta \ge \xi$. The straightforward calculation shows that

$$g(u, v) = 4(1 + u + v)^{-3}, \qquad u > 0, \ v \ge 1$$

and

$$G(u, v) = \frac{2u(2 + u + 2v)}{(1 + v)^2 (1 + u + v)^2}, \qquad u > 0, \ v \ge 1. \tag{3.11}$$

It is easy to see that $g(u,v)$ is TP_2, so that $G(u,v)$ is TP_2. It can be shown that $G(u,y/x)$ is TP_2 in x and y for each u. Hence from Corollary 3.1 the best scale equivariant predictor δ_{c_0} can be improved by δ_{ϕ_0}.

For example, let $L(t)=(t-1)^2$. Then the best scale equivariant predictor is given from (3.8) by

$$\delta_{c_0}=c_0X$$

where

$$\begin{aligned} c_0 &= \frac{\int_0^\infty v^{-1}h(v)dv}{\int_0^\infty v^{-2}h(v)dv} = \frac{\displaystyle\int_1^\infty \frac{dv}{v(1+v)^2}}{\displaystyle\int_1^\infty \frac{dv}{v^2(1+v)^2}} \\ &= \frac{2\log 2-1}{3-4\log 2}. \end{aligned}$$

The improved predictor is written from (3.7) and (3.11) as

$$\delta_{\phi_0}=\phi_0(Z/X)X$$

where

$$\begin{aligned} \phi_0(u) &= \frac{\int_0^\infty v^{-1}G(u,v)dv}{\int_0^\infty v^{-2}G(u,v)dv} \\ &= \frac{\displaystyle\int_1^\infty \frac{dv}{v(1+v)^2} - \int_1^\infty \frac{dv}{v(1+u+v)^2}}{\displaystyle\int_1^\infty \frac{dv}{v^2(1+v)^2} - \int_1^\infty \frac{dv}{v^2(1+u+v)^2}} \\ &= \frac{2\log 2-1-2(1+u)^{-2}\log(2+u)+2(1+u)^{-1}(2+u)^{-1}}{3-4\log 2-2(3+u)(1+u)^{-2}(2+u)^{-1}+4(1+u)^{-3}\log(2+u)}. \end{aligned}$$

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