J. Japan Statist. Soc.
Vol. 36 No. 1 2006 91–106

DECOMPOSITIONS FOR EXTENDED DOUBLE SYMMETRY MODELS IN SQUARE CONTINGENCY TABLES WITH ORDERED CATEGORIES

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For square contingency tables with ordered categories, Tomizawa (1992) proposed three kinds of double symmetry models, whose each has a structure of both symmetry about the main diagonal and asymmetry about the reverse diagonal of the table. This paper proposes the extensions of those models and gives the decompositions for three kinds of double symmetry models into the extended quasi double symmetry models, the weighted marginal double symmetry models, and the balance models. Those decompositions are applied to two kinds of data on unaided distance vision.

Key words and phrases: Decomposition, double symmetry, marginal double symmetry, ordered category, quasi double symmetry, square contingency table, unaided vision data.

1. Introduction

For an $R \times R$ square contingency table, let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1, 2, \ldots, R$; $j = 1, 2, \ldots, R$). The symmetry (S) model is defined by

$$p_{ij} = \psi_{ij} \qquad (i = 1, 2, \ldots, R;\;\; j = 1, 2, \ldots, R),$$

where $\psi_{ij} = \psi_{ji}$ (Bishop *et al.* (1975), p. 282). This describes a structure of symmetry of the probabilities $\{p_{ij}\}$ with respect to the main diagonal of the table.

Caussinus (1965) considered the quasi symmetry (QS) model, defined by

$$p_{ij} = \mu\alpha_i\beta_j\psi_{ij} \qquad (i = 1, 2, \ldots, R;\;\; j = 1, 2, \ldots, R),$$

where $\psi_{ij} = \psi_{ji}$. A special case of this model with $\{\alpha_i = \beta_i\}$ is the S model. Denote the odds ratio for rows i and j ($> i$), and columns s and t ($> s$) by $\theta_{(i<j;s<t)}$. Thus $\theta_{(i<j;s<t)} = (p_{is}p_{jt})/(p_{js}p_{it})$. Using the odds ratios, the QS model is further expressed as

$$\theta_{(i<j;s<t)} = \theta_{(s<t;i<j)} \qquad (i < j;\;\; s < t).$$

Therefore the QS model has characterization in terms of symmetry of odds ratios.

Received January 17, 2005. Revised April 4, 2005. Accepted April 25, 2005.

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Let X_1 and X_2 denote the row and column variables, respectively. The marginal homogeneity (MH) model is defined by

$$\Pr(X_1 = i) = \Pr(X_2 = i) \quad (i = 1, 2, \ldots, R),$$

namely

$$p_{i\cdot} = p_{\cdot i} \quad (i = 1, 2, \ldots, R),$$

where $p_{i\cdot} = \sum_{t=1}^{R} p_{it}$ and $p_{\cdot i} = \sum_{s=1}^{R} p_{si}$ (Stuart (1955)). This indicates that the row marginal distribution is identical with the column marginal distribution.

Caussinus (1965) pointed out that the S model holds if and only if both the QS and MH models hold.

Wall and Lienert (1976) considered the point-symmetry (PS) model, defined by

$$p_{ij} = \psi_{ij} \quad (i = 1, 2, \ldots, R;\; j = 1, 2, \ldots, R),$$

where $\psi_{ij} = \psi_{i^*j^*}$. The symbol "*" denotes $i^* = R + 1 - i$ through this paper. This model indicates a structure of point-symmetry of the probabilities $\{p_{ij}\}$ with respect to the center point (when R is even) or the center cell (when R is odd) in the square table.

Tomizawa (1985a) considered the double symmetry (DS) model, defined by

$$p_{ij} = \psi_{ij} \quad (i = 1, 2, \ldots, R;\; j = 1, 2, \ldots, R),$$

where $\psi_{ij} = \psi_{ji} = \psi_{i^*j^*}$ $(= \psi_{j^*i^*})$. This indicates that there is a structure of symmetry and point-symmetry of the probabilities $\{p_{ij}\}$ in the square table. The DS model implies each of the S and PS models.

Tomizawa (1992) considered the two parameters double symmetry (2DS) model, defined by

$$p_{ij} = \begin{cases} \delta\phi^{-(i+j)/2}\psi_{ij} & (i + j < R + 1), \\ \phi^{-(i+j)/2}\psi_{ij} & (i + j \geq R + 1), \end{cases}$$

where $\psi_{ij} = \psi_{ji} = \psi_{i^*j^*}$ $(= \psi_{j^*i^*})$. A special case of the 2DS model with $\delta = \phi = 1$ is the DS model. The 2DS model indicates that

$$p_{ij} = p_{ji} \quad (i \neq j)$$

and

$$\frac{p_{ij}}{p_{j^*i^*}} = \delta\phi^{R+1-(i+j)} \quad (i + j < R + 1).$$

Therefore this indicates that (i) the probability that an observation will fall in the (i, j)th cell is equal to the probability that it falls in the (j, i)th cell and (ii) the probability that the observation falls in the (i, j)th cell in the upper left triangle of the table is $\delta\phi^{R+1-(i+j)}$ times higher than the probability that it falls in the (j^*, i^*)th cell [or point-symmetric (i^*, j^*)th cell] in the lower right triangle of the table.

Denote the difference between the average of X_1+X_2 $(= R+1)$ and X_1+X_2 by D. Thus, $D = R + 1 - (X_1 + X_2)$. Under the 2DS model, $p_{ij}/p_{j^*i^*}$ for $i + j < R + 1$ is the odds that an observation falls in a cell (i, j) satisfying the distance $|D| = k$ above the reverse main diagonal of the table, instead of in a cell (j^*, i^*) satisfying the same distance $|D| = k$ below it. The odds depends only on the distance k between the reverse diagonal containing the cell and the reverse main diagonal, and the odds increases (or decreases) monotonically as the distance k increases.

The one parameter double symmetry (1DS) model (Tomizawa (1992)) is defined by

$$p_{ij} = \phi^{-(i+j)/2}\psi_{ij} \qquad (i = 1, 2, \ldots, R;\;\; j = 1, 2, \ldots, R),$$

where $\psi_{ij} = \psi_{ji} = \psi_{i^*j^*}$ $(= \psi_{j^*i^*})$. The constant parameter double symmetry (CDS) model (Tomizawa (1992)) is defined by

$$p_{ij} = \begin{cases} \delta\psi_{ij} & (i + j < R + 1), \\ \psi_{ij} & (i + j \geq R + 1), \end{cases}$$

where $\psi_{ij} = \psi_{ji} = \psi_{i^*j^*}$ $(= \psi_{j^*i^*})$. The 1DS and CDS models are special cases of the 2DS model.

Consider two sets of data on unaided distance vision of (i) 3242 men aged 30–39 employed in Royal Ordnance factories in Britain from 1943 to 1946 (Table 1), analyzed first by Stuart (1953), and (ii) 3168 pupils comprising nearly equal number of boys and girls aged 6–12 at elementary schools in Tokyo, Japan, examined in June 1984 (Table 2), analyzed first by Tomizawa (1985b). Each of the S model and its decomposed models, i.e., the QS and MH models, fits the data in Tables 1 and 2 well (see Table 4). However, when these models are applied to the data in Tables 1 and 2, the 69% and 88%, respectively, of information on the observations, which are the proportions of the observations on the main diagonal to the total observations, are not utilized because none of the S, QS, and MH models depend on the main diagonal elements of the table; indeed, the results for the main diagonal cells of Tables 1 and 2 are always theoretically zero without depending on the values of observations. So, we should utilize the information on the main diagonal.

Hence, we shall apply some models depending on the main diagonal to the data in Tables 1 and 2. Each of the PS and DS models fits these data poorly (see Table 4). Moreover, the 2DS model fits the data in Table 1 poorly but fits the data in Table 2 well, and each of the 1DS and CDS models fits the data in Tables 1 and 2 poorly (see Table 4). Therefore, we are now interested in seeing the reason why the 2DS model fits the data in Table 1 poorly and why each of the 1DS and CDS models fits the data in Table 2 poorly.

In this paper, we (i) give the decompositions for the 2DS, 1DS and CDS models into some new models, and (ii) analyze the data in more details using the decomposed models. Section 2 proposes some new models, Section 3 gives the decompositions, and Section 4 analyzes further the data in Tables 1 and 2.

2. Models

2.1. Quasi double symmetry models

Consider a model defined by

$$p_{ij} = \mu\alpha_i\beta_j\psi_{ij} \quad (i = 1, 2, \ldots, R;\ j = 1, 2, \ldots, R), \tag{2.1}$$

where

$$\psi_{ij} = \psi_{ji} \quad (i \neq j),$$
$$\frac{\psi_{ij}}{\psi_{j^*i^*}} = \gamma \quad (i + j < R + 1),$$

with $\prod_{i=1}^{R} \alpha_i = \prod_{j=1}^{R} \beta_j = 1$ and $\prod_{i=1}^{R} \psi_{it} = \gamma^{3(R+1)/2-t}$. Note that the 2DS model is a special case of (2.1). We will refer to (2.1) as the quasi two parameters double symmetry (Q2DS) model.

Using the odds ratios, the Q2DS model can also be expressed as

$$\theta_{(i<j;j<k)} = \theta_{(j<k;i<j)} \quad (i < j < k), \tag{2.2}$$

and

$$\gamma\theta_{(i<j;k^*<j^*)} = \theta_{(j<k;j^*<i^*)} \tag{2.3}$$
$$= \theta_{(j^*<i^*;j<k)} \quad (i < j < k). \tag{2.4}$$

We note that (i) (2.2) indicates the symmetry of odds ratios with respect to the main diagonal in the table and (ii) (2.3) and (2.4) indicate the asymmetry of odds ratios with respect to the reverse main diagonal or center point in the table. A special case of the Q2DS model with $\gamma = 1$ is the quasi double symmetry (QDS) model (Tomizawa (1985a)).

From (2.2) and (2.3), the Q2DS model may also be expressed as

$$Q_{ijk} = Q_{kji} \quad (i < j < k), \tag{2.5}$$

and

$$Q^*_{ijk} = \gamma Q^*_{kji} \quad (i < j < k), \tag{2.6}$$

where

$$Q_{stu} = p_{st}p_{tu}p_{us}, \qquad Q^*_{stu} = p_{st^*}p_{tu^*}p_{us^*}.$$

Denote the conditional probability that (X_1, X_2) for an observation takes value (i, j) conditional on $(X_1, X_2) = (i, j)$ or (j, i), by p^c_{ij}. Thus

$$p^c_{ij} = \frac{p_{ij}}{p_{ij} + p_{ji}} \quad (i \neq j).$$

Also, denote the conditional probability that (X_1, X_2) for an observation takes value (i, j^*) conditional on $(X_1, X_2) = (i, j^*)$ or (j, i^*), by $p^{c*}_{ij^*}$. Thus

$$p^{c*}_{ij^*} = \frac{p_{ij^*}}{p_{ij^*} + p_{ji^*}} \quad (i \neq j).$$

Table 1. Unaided distance vision of 3242 men aged 30–39 employed in Royal Ordnance factories in Britain from 1943 to 1946; from Stuart (1953). (The parenthesized values from above are the MLEs of expected frequencies under the Q2DS, M2DS-I and M2DS-II models, respectively.)

Right eye grade	Left eye grade: Best (1)	Second (2)	Third (3)	Worst (4)	Total
Best (1)	821	112	85	35	1053
	(812.49)	(124.67)	(80.40)	(35.44)	
	(821.19)	(110.12)	(83.68)	(37.62)	
	(820.90)	(109.54)	(81.29)	(37.94)	
Second (2)	116	494	145	27	782
	(120.35)	(485.49)	(148.97)	(27.19)	
	(118.04)	(493.99)	(145.19)	(28.79)	
	(118.66)	(494.24)	(145.05)	(30.36)	
Third (3)	72	151	583	87	893
	(76.60)	(147.03)	(591.51)	(77.86)	
	(73.17)	(150.80)	(583.01)	(98.49)	
	(75.36)	(150.95)	(582.79)	(97.75)	
Worst (4)	43	34	106	331	514
	(42.56)	(33.81)	(98.12)	(339.51)	
	(40.21)	(32.00)	(94.90)	(330.80)	
	(39.92)	(30.63)	(95.50)	(331.12)	
Total	1052	791	919	480	3242

Then the Q2DS model may further be expressed as

$$p^c_{ij}p^c_{jk}p^c_{ki} = p^c_{ji}p^c_{kj}p^c_{ik} \quad (i < j < k), \tag{2.7}$$

and

$$p^{c*}_{ij^*}p^{c*}_{jk^*}p^{c*}_{ki^*} = \gamma p^{c*}_{ji^*}p^{c*}_{kj^*}p^{c*}_{ik^*} \quad (i < j < k). \tag{2.8}$$

Here, (2.7) indicates that the probability that one of three observations satisfies $X_1 < X_2$ conditional on $(X_1, X_2) = (i, j)$ or (j, i), another satisfies $X_1 < X_2$ conditional on $(X_1, X_2) = (j, k)$ or (k, j), but the other satisfies $X_1 > X_2$ conditional on $(X_1, X_2) = (i, k)$ or (k, i), is equal to the probability that one of three observations satisfies $X_1 > X_2$ conditional on $(X_1, X_2) = (i, j)$ or (j, i), another satisfies $X_1 > X_2$ conditional on $(X_1, X_2) = (j, k)$ or (k, j), but the other satisfies $X_1 < X_2$ conditional on $(X_1, X_2) = (i, k)$ or (k, i). In such a sense, (2.5) or (2.7) indicates a structure of symmetry.

Also, (2.8) indicates that the probability that one of three observations satisfies $D > 0$ conditional on $(X_1, X_2) = (i, j^*)$ or (j, i^*), another satisfies $D > 0$ conditional on $(X_1, X_2) = (j, k^*)$ or (k, j^*), but the other satisfies $D < 0$ conditional on $(X_1, X_2) = (i, k^*)$ or (k, i^*), is γ times higher than the probability that one of three observations satisfies $D < 0$ conditional on $(X_1, X_2) = (i, j^*)$ or (j, i^*), another satisfies $D < 0$ conditional on $(X_1, X_2) = (j, k^*)$ or (k, j^*), but the other satisfies $D > 0$ conditional on $(X_1, X_2) = (i, k^*)$ or (k, i^*). In such a

Table 2. Unaided distance vision of 3168 pupils comprising nearly equal number of boys and girls aged 6–12 at elementary schools in Tokyo, Japan, examined in June 1984; from Tomizawa (1985b). (The parenthesized values from above are the MLEs of expected frequencies under the 2DS, QDS, Q2DS, M2DS-I and M2DS-II models, respectively.)

Right eye grade	Left eye grade: Best (1)	Second (2)	Third (3)	Worst (4)	Total
Best (1)	2470	126	21	10	2627
	(2469.90)	(111.10)	(14.19)	(11.00)	
	(2468.63)	(128.67)	(18.74)	(10.96)	
	(2470.13)	(128.59)	(17.22)	(11.06)	
	(2470.03)	(109.06)	(19.32)	(9.26)	
	(2470.33)	(118.85)	(17.91)	(8.67)	
Second (2)	96	138	33	5	272
	(111.10)	(140.53)	(37.50)	(7.31)	
	(93.21)	(139.49)	(35.48)	(3.82)	
	(93.16)	(138.13)	(35.64)	(5.07)	
	(113.63)	(137.97)	(35.41)	(5.79)	
	(102.50)	(139.45)	(37.88)	(7.09)	
Third (3)	10	42	75	15	142
	(14.19)	(37.50)	(72.47)	(15.40)	
	(15.12)	(39.52)	(73.51)	(13.85)	
	(13.78)	(39.36)	(74.87)	(13.99)	
	(10.96)	(39.33)	(75.03)	(15.84)	
	(11.97)	(37.20)	(73.58)	(17.57)	
Worst (4)	12	7	16	92	127
	(11.00)	(7.31)	(15.40)	(92.10)	
	(11.04)	(5.32)	(17.27)	(93.37)	
	(10.93)	(6.92)	(17.27)	(91.88)	
	(13.04)	(6.15)	(15.21)	(91.97)	
	(14.18)	(5.47)	(13.68)	(91.67)	
Total	2588	313	145	122	3168

sense, (2.6) or (2.8) indicates a structure of asymmetry. When $\gamma > 1$, in such a sense, two of three observations tend to have $X_1 + X_2$ being less than the average of $X_1 + X_2$.

2.2. Marginal double symmetry models

Consider a model defined by

$$p^{-}_{i\cdot}(\Gamma,\Psi) = p^{-}_{\cdot i}(\Gamma,\Psi) = p^{-}_{i^*\cdot}(\Gamma,\Psi) = p^{-}_{\cdot i^*}(\Gamma,\Psi) \qquad (i = 1, 2, \ldots, R), \tag{2.9}$$

where

$$p^{-}_{i\cdot}(\Gamma,\Psi) = \sum_{t=1}^{i^*} p_{it} + \sum_{t=i^*+1}^{R} \Gamma\Psi^{R+1-(i^*+t^*)} p_{it},$$

$$p_{\cdot i}^{-}(\Gamma, \Psi) = \sum_{t=1}^{i^*} p_{ti} + \sum_{t=i^*+1}^{R} \Gamma \Psi^{R+1-(i^*+t^*)} p_{ti}.$$

This indicates that (i) the row marginal totals summed by multiplying the probabilities for the cells with a distance k ($k = 1, 2, \ldots, R-1$) below the reverse main diagonal in the table by a common weight $\Gamma\Psi^k$ (> 0) are equal to the column marginal totals summed by the same way, and (ii) the row (column) marginal totals are symmetric with respect to the midpoint of the row (column) categories. We shall refer (2.9) to the marginal two parameters double symmetry model I (M2DS-I). We see that the 2DS model implies the M2DS-I model (see Figure 1).

From Γ and Ψ under the M2DS-I model, it may be difficult to obtain the direct interpretation of the model, however, these would be useful for seeing how the structure of probabilities is departure from the marginal double symmetry. If $\Gamma\Psi^k > 1$ for every k ($k = 1, 2, \ldots, R-1$), then $\Pr(X_1 \le i) > \Pr(X_2 \ge i^*)$ and $\Pr(X_2 \le i) > \Pr(X_1 \ge i^*)$; thus $\Pr(X_1 \le i) + \Pr(X_2 \le i) > \Pr(X_1 \ge i^*) + \Pr(X_2 \ge i^*)$ for $i = 1, 2, \ldots, R-1$. Therefore Γ and Ψ under the M2DS-I model may be useful for inferring the structure of marginal asymmetry.

We shall now refer (i) (2.9) with $\Gamma = 1$ to the marginal one parameter double symmetry model I (M1DS-I), and (ii) (2.9) with $\Psi = 1$ to the marginal constant parameter double symmetry model I (MCDS-I). Note that (2.9) with $\Gamma = \Psi = 1$ is the marginal double symmetry (MDS) model, defined by Tomizawa (1985a).

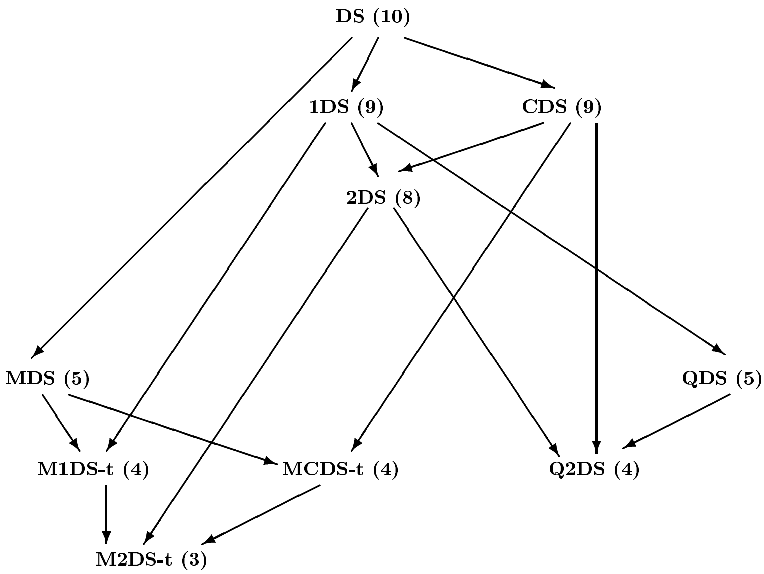

Figure 1. Relationships among various double symmetry models ("A → B" indicates that model A implies model B and "t" indicates that t = I and II). The parenthesized values indicate the numbers of degrees of freedom for the corresponding models applied to the data in Tables 1 and 2.

Next, consider a model defined by

$$p^+_{i\cdot}(\Omega,\Phi) = p^+_{\cdot i}(\Omega,\Phi) = p^+_{i^*\cdot}(\Omega,\Phi) = p^+_{\cdot i^*}(\Omega,\Phi) \qquad (i = 1, 2, \ldots, R), \tag{2.10}$$

where

$$p^+_{i\cdot}(\Omega,\Phi) = \sum_{t=1}^{i^*-1} \Omega\Phi^{R+1-(i+t)} p_{it} + \sum_{t=i^*}^{R} p_{it},$$
$$p^+_{\cdot i}(\Omega,\Phi) = \sum_{t=1}^{i^*-1} \Omega\Phi^{R+1-(i+t)} p_{ti} + \sum_{t=i^*}^{R} p_{ti}.$$

This indicates that (i) the row marginal totals summed by multiplying the probabilities for the cells with a distance k ($k = 1, 2, \ldots, R-1$) above the reverse main diagonal in the table by a common weight $\Omega\Phi^k$ (> 0) are equal to the column marginal totals summed by the same way, and (ii) the row (column) marginal totals are symmetric with respect to the midpoint of the row (column) categories. We shall refer (2.10) to the M2DS-II model. Further we shall refer (i) (2.10) with $\Omega = 1$ to the M1DS-II model, and (ii) (2.10) with $\Phi = 1$ to the MCDS-II model.

2.3. Balance models

Let

$$\gamma = \sum_{i<j<k} Q^*_{ijk} \Bigg/ \sum_{i<j<k} Q^*_{kji}. \tag{2.11}$$

First, consider a model defined by

$$\gamma = \Gamma, \tag{2.12}$$

where Γ satisfies

$$p^-_{1\cdot}(\Gamma,\Psi) - p^-_{R\cdot}(\Gamma,\Psi) = 0,$$
$$p^-_{2\cdot}(\Gamma,\Psi) - p^-_{R-1\cdot}(\Gamma,\Psi) = 0.$$

This indicates that γ in the Q2DS model is equal to Γ in the M2DS-I model when both the Q2DS and M2DS-I models hold. We shall refer (2.12) to the two parameters balance model I (2BA-I). It may be not meaningful to apply only the 2BA-I model for the data, but the 2BA-I model would be useful to consider the decompositions for the 2DS model (see Section 3).

Secondly, consider a model defined by

$$\frac{1}{\gamma} = \Omega, \tag{2.13}$$

where Ω satisfies

$$p^+_{1\cdot}(\Omega,\Phi) - p^+_{R\cdot}(\Omega,\Phi) = 0,$$
$$p^+_{2\cdot}(\Omega,\Phi) - p^+_{R-1\cdot}(\Omega,\Phi) = 0.$$

This indicates that γ^{-1} in the Q2DS model is equal to Ω in the M2DS-II model when both the Q2DS and M2DS-II models hold. We shall refer (2.13) to the 2BA-II model.

Thirdly, consider a model defined by

$$\gamma = \Gamma, \tag{2.14}$$

where Γ satisfies

$$p^{-}_{1\cdot}(\Gamma, 1) - p^{-}_{R\cdot}(\Gamma, 1) = 0.$$

This indicates that γ in the Q2DS model is equal to Γ in the MCDS-I model when both the Q2DS and MCDS-I models hold. We shall refer (2.14) to the constant parameter balance model I (CBA-I).

Lastly, consider a model defined by

$$\frac{1}{\gamma} = \Omega, \tag{2.15}$$

where Ω satisfies

$$p^{+}_{1\cdot}(\Omega, 1) - p^{+}_{R\cdot}(\Omega, 1) = 0.$$

This indicates that γ^{-1} in the Q2DS model is equal to Ω in the MCDS-II model when both the Q2DS and MCDS-II models hold. We shall refer (2.15) to the CBA-II model.

Assume that a multinomial distribution applies to the $R \times R$ table. The maximum likelihood estimates (MLEs) of expected frequencies under each model could be obtained using the Newton-Raphson method to the log-likelihood equations or using the iterative procedures, for example, the general iterative procedure for log-linear models of Darroch and Ratcliff (1972). Each model can be tested

Table 3. Numbers of degrees of freedom for various double symmetry models applied to the $R \times R$ table.

Models	When R is even Degrees of freedom	When R is odd Degrees of freedom
DS	$R(3R-2)/4$	$(R-1)(3R+1)/4$
CDS (1DS)	$(3R^2-2R-4)/4$	$(3R-5)(R+1)/4$
2DS	$(3R+4)(R-2)/4$	$(3R^2-2R-9)/4$
QDS	$(R-2)(3R-2)/4$	$(R-1)(3R-5)/4$
Q2DS	$R(3R-8)/4$	$(3R^2-8R+1)/4$
MDS	$(3R-2)/2$	$3(R-1)/2$
MCDS-t (M1DS-t)	$(3R-4)/2$	$(3R-5)/2$
M2DS-t	$3(R-2)/2$	$(3R-7)/2$
2BA-t (CBA-t)	1	1

Note: "t" indicates t = I and II.

for goodness-of-fit by the likelihood ratio chi-squared statistic (denoted by G^2) with the corresponding degrees of freedom. The numbers of degrees of freedom for models are given in Table 3.

3. Decompositions for the two parameters double symmetry model

We shall consider the decompositions for the 2DS, 1DS and CDS models. We obtain the following theorem.

THEOREM 1. *For t = I and II, the 2DS model holds if and only if all the Q2DS, M2DS-t and 2BA-t models hold. The number of degrees of freedom for the 2DS model is equal to the sum of those for the Q2DS, M2DS-t and 2BA-t models.*

PROOF. For t = I and II, if the 2DS model holds, then the Q2DS, M2DS-t and 2BA-t models hold. Assume that all the Q2DS, M2DS-t and 2BA-t models hold, and then we shall show that the 2DS model holds.

Consider the case of t = I. Since the Q2DS and M2DS-I models hold, we obtain

$$p_{i\cdot}^{-}(\Gamma,\Psi) = p_{\cdot i}^{-}(\Gamma,\Psi) \qquad (i = 1, 2, \ldots, R), \tag{3.1}$$

where

$$\begin{aligned}
p_{i\cdot}^{-}(\Gamma,\Psi) &= \mu\alpha_i A_i,\\
p_{\cdot i}^{-}(\Gamma,\Psi) &= \mu\beta_i B_i,\\
A_i &= \sum_{t=1}^{i^*} \beta_t\psi_{it} + \sum_{t=i^*+1}^{R} \Gamma\Psi^{R+1-(i^*+t^*)}\beta_t\psi_{it},\\
B_i &= \sum_{t=1}^{i^*} \alpha_t\psi_{it} + \sum_{t=i^*+1}^{R} \Gamma\Psi^{R+1-(i^*+t^*)}\alpha_t\psi_{it}.
\end{aligned}$$

Since (3.1), we see

$$\alpha_i = \beta_i h_i \qquad (i = 1, 2, \ldots, R), \tag{3.2}$$

where

$$h_i = \frac{B_i}{A_i}.$$

By substituting (3.2) in B_i, we obtain

$$f = Wf,$$

where

$$f = (h_1, h_2, \ldots, h_R)'$$

and "$'$" denotes the transpose and the (i,t)th element of the $R \times R$ matrix W is given by

$$(W)_{it} = \begin{cases} \dfrac{1}{A_i}\beta_t\psi_{it} & (t \le i^*),\\[2ex] \dfrac{1}{A_i}\Gamma\Psi^{R+1-(i^*+t^*)}\beta_t\psi_{it} & (t > i^*). \end{cases}$$

All elements of W are positive and satisfy $WJ_R = J_R$ where $J_R = (1, 1, \ldots, 1)'$ is a vector of order R whose components are all unity. Therefore, noting that (3.2) with $\{h_i > 0\}$ and $\prod_{i=1}^{R} \alpha_i = \prod_{j=1}^{R} \beta_j = 1$ in the Q2DS model, we obtain

$$h_1 = h_2 = \cdots = h_R = 1.$$

Namely, $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, R$. We also obtain

$$p_{i\cdot}^{-}(\Gamma, \Psi) = p_{i^*\cdot}^{-}(\Gamma, \Psi) \qquad (i = 1, 2, \ldots, R), \tag{3.3}$$

where

$$\begin{aligned}
p_{i\cdot}^{-}(\Gamma, \Psi) &= \mu\alpha_i C_i, \\
p_{i^*\cdot}^{-}(\Gamma, \Psi) &= \mu\alpha_{i^*} D_i, \\
C_i &= \sum_{t=1}^{i^*} \alpha_t \psi_{it} + \sum_{t=i^*+1}^{R} \Gamma\Psi^{R+1-(i^*+t^*)}\alpha_t \psi_{it}, \\
D_i &= \sum_{t=1}^{i} \alpha_t \psi_{i^*t} + \sum_{t=i+1}^{R} \Gamma\Psi^{R+1-(i+t^*)}\alpha_t \psi_{i^*t}.
\end{aligned}$$

Note that $D_i = C_{i^*}$. Since (3.3), we see

$$\alpha_i = \alpha_{i^*}\Psi^{(R+1)/2-i} k_i \qquad (i = 1, 2, \ldots, R), \tag{3.4}$$

where

$$\begin{aligned}
k_i &= \frac{E_i}{C_i}, \\
E_i &= \frac{D_i}{\Psi^{(R+1)/2-i}}.
\end{aligned} \tag{3.5}$$

By substituting (3.4) in D_i, we see

$$\begin{aligned}
D_i &= \sum_{t=1}^{i} \alpha_{t^*}\Psi^{(R+1)/2-t}\psi_{i^*t}k_t + \sum_{t=i+1}^{R} \Gamma\Psi^{R+1-(i+t^*)}\alpha_{t^*}\Psi^{(R+1)/2-t}\psi_{i^*t}k_t \\
&= \sum_{t=1}^{i-1} \alpha_{t^*}\Psi^{(R+1)/2-t}\psi_{i^*t}k_t + \alpha_{i^*}\Psi^{(R+1)/2-i}\psi_{i^*i}k_i \\
&\quad + \Gamma\Psi^{(R+1)/2-i}\sum_{t=i+1}^{R} \alpha_{t^*}\psi_{i^*t}k_t.
\end{aligned}$$

Since the Q2DS model holds, we obtain

$$\begin{aligned}
D_i &= \sum_{t=1}^{i-1} \gamma\Psi^{(R+1)/2-t}\alpha_{t^*}\psi_{it^*}k_t + \alpha_{i^*}\Psi^{(R+1)/2-i}\psi_{ii^*}k_i \\
&\quad + \Gamma\Psi^{(R+1)/2-i}\gamma^{-1}\sum_{t=i+1}^{R} \alpha_{t^*}\psi_{it^*}k_t.
\end{aligned}$$

In addition, since the 2BA-I model holds, i.e., $\gamma = \Gamma$, we see

$$\begin{aligned} D_i &= \sum_{t=1}^{i-1} \Gamma \Psi^{(R+1)/2-t} \alpha_{t^*} \psi_{it^*} k_t + \alpha_{i^*} \Psi^{(R+1)/2-i} \psi_{ii^*} k_i \\ &\quad + \Psi^{(R+1)/2-i} \sum_{t=i+1}^{R} \alpha_{t^*} \psi_{it^*} k_t \\ &= \Psi^{(R+1)/2-i} \sum_{s=1}^{i^*} \alpha_s \psi_{is} k_{s^*} + \sum_{s=i^*+1}^{R} \Gamma \Psi^{(R+1)/2-s^*} \alpha_s \psi_{is} k_{s^*}. \end{aligned}$$

Thus,

$$E_i = \sum_{s=1}^{i^*} \alpha_s \psi_{is} k_{s^*} + \sum_{s=i^*+1}^{R} \Gamma \Psi^{R+1-(i^*+s^*)} \alpha_s \psi_{is} k_{s^*}.$$

Therefore, since (3.5), we see

$$g = Hg,$$

where

$$g = (k_1, k_2, \ldots, k_R)',$$

and the (i, s^*)th element of the $R \times R$ matrix H is given by

$$(H)_{is^*} = \begin{cases} \dfrac{1}{C_i} \alpha_s \psi_{is} & (s \le i^*), \\ \dfrac{1}{C_i} \Gamma \Psi^{R+1-(i^*+s^*)} \alpha_s \psi_{is} & (s > i^*). \end{cases}$$

All elements of H are positive and satisfy $HJ_R = J_R$. Therefore, noting that (3.4) with $\{k_i > 0\}$ and $\prod_{i=1}^{R} \alpha_i = 1$, we obtain

$$k_1 = k_2 = \cdots = k_R = 1.$$

Thus,

$$\frac{\alpha_i}{\alpha_{i^*}} = \Psi^{(R+1)/2-i} \qquad (i = 1, 2, \ldots, R).$$

Therefore, we see

$$p_{ij} = p_{ji} \qquad (i \ne j)$$

and

$$\frac{p_{ij}}{p_{j^*i^*}} = \Gamma \Psi^{R+1-(i+j)} \qquad (i + j < R + 1).$$

Namely, the 2DS model holds. The case of t = II can be proved in a similar way as the case of t = I. The proof on degrees of freedom is easily shown from Table 3. The proof is completed.

We obtain the following corollaries from Theorem 1:

COROLLARY 1. *For $t = I$ and II, the* 1*DS model holds if and only if both the QDS and M*1*DS-t models hold. The number of degrees of freedom for the* 1*DS model is equal to the sum of those for the QDS and M*1*DS-t models.*

COROLLARY 2. *For $t = I$ and II, the CDS model holds if and only if all the Q*2*DS, MCDS-t and CBA-t models hold. The number of degrees of freedom for the CDS model is equal to the sum of those for the Q*2*DS, MCDS-t and CBA-t models.*

4. Analysis of vision data using decompositions

4.1. Analysis of Table 1

The 2DS model fits the data in Table 1 poorly (see Table 4). By using the decompositions for the 2DS model, we shall consider the reason why the 2DS model fits these data poorly.

Each of the Q2DS, M2DS-t (t = I, II) models fits these data very well, but each of 2BA-t (t = I, II) models fits these data poorly. So, the poor fit of the 2DS model is caused by the influence of the lack of structure of the 2BA-t (t = I, II) models.

Under the Q2DS model, from (2.2), a man's right eye vision is symmetric to his left eye vision with respect to the odds ratios. In addition, since the MLE of γ is $\hat{\gamma} = 1.516$, from (2.4), if the odds that a man's right eye grade is i instead of j $(i < j)$ is $\hat{\theta}_{(i<j;k^*<j^*)}$ times higher when his left eye grade is k^* than when it is j^* $(j < k)$, then the odds that his right eye grade is i^* instead of j^* is $1.516 \times \hat{\theta}_{(i<j;k^*<j^*)}$ times higher when his left eye grade is k than when it is j. Namely, a man's right eye vision is not point-symmetric to his left eye vision with respect to the odds ratios.

As another interpretation, from (2.7), under the Q2DS model, the probability that one of three men has the right eye grade X_1 being better than the left eye grade X_2 conditional on $(X_1, X_2) = (i, j)$ or (j, i), another man has X_1 being better than X_2 conditional on $(X_1, X_2) = (j, k)$ or (k, j), but the other man has X_1 being worse than X_2 conditional on $(X_1, X_2) = (i, k)$ or (k, i), is estimated to be equal to the probability that one of three men has the right eye grade X_1 being worse than the left eye grade X_2 conditional on $(X_1, X_2) = (i, j)$ or (j, i), another man has X_1 being worse than X_2 conditional on $(X_1, X_2) = (j, k)$ or (k, j), but the other man has X_1 being better than X_2 conditional on $(X_1, X_2) = (i, k)$ or (k, i). In such a sense, the right eye vision is symmetric to the left eye vision.

In addition, define a man's total vision of both eyes by the sum of the right and left eye grades (i.e., the total vision for a man that takes $(X_1, X_2) = (i, j)$ is $i + j$), and also define the average vision by the average of sum of the right and left eye grades (i.e., it is equal to $R + 1 = 5$ for these data).

Then, from (2.8), under the Q2DS model, the probability that one of three men has the total vision of both eyes being better than the average vision, i.e., $X_1 + X_2 < 5$, conditional on $(X_1, X_2) = (i, j^*)$ or (j, i^*), another man has the total vision being better than the average vision conditional on $(X_1, X_2) = (j, k^*)$ or (k, j^*), but the other man has the total vision being worse than the average vision conditional on $(X_1, X_2) = (i, k^*)$ or (k, i^*), is estimated to be $\hat{\gamma} = 1.516$ times higher than the probability that one of three men has the total vision being worse than the average vision conditional on $(X_1, X_2) = (i, j^*)$ or (j, i^*), another man has the total vision being worse than the average vision conditional on $(X_1, X_2) = (j, k^*)$ or (k, j^*), but the other man has the total vision being better than the average vision conditional on $(X_1, X_2) = (i, k^*)$ or (k, i^*). In such a sense, the total vision of both eyes is not symmetric with respect to the average vision. Also, in such a sense, two for three men tend to have the total visions being better than the average vision because $\hat{\gamma} = 1.516$ is greater than 1.

Under the M2DS-I model, the MLEs of Γ and Ψ are $\hat{\Gamma} = 0.378$ and $\hat{\Psi} = 1.901$. Thus, $\hat{\Gamma}\hat{\Psi} = 0.719 < \hat{\Gamma}\hat{\Psi}^2 = 1.367 < \hat{\Gamma}\hat{\Psi}^3 = 2.599$. Since $\hat{\Gamma}\hat{\Psi} < 1$, $\hat{\Gamma}\hat{\Psi}^2 > 1$ and $\hat{\Gamma}\hat{\Psi}^3 > 1$, unfortunately from only the values of these parameters, we cannot make the inference about the point-asymmetry of the marginal distributions. However, from the MLEs of expected frequencies obtained under the M2DS-I model, we see that for $i = 1, 2, 3$, the probability that a man's right (left) eye grade is i or below is estimated to be greater than the probability that it is i^* or above. Thus, a man's vision of right (left) eye tends to be better than the average grade $(R + 1)/2 = 2.5$ for right (left) eye grade. A similar interpretation can also be obtained under the M2DS-II model with $\hat{\Omega} = 2.793$ and $\hat{\Phi} = 0.492$ though the detail is omitted.

Finally, we note that as described above, the MLE of γ under the Q2DS model is $\hat{\gamma} = 1.516$ and that of Γ (Ω) under the M2DS-I (M2DS-II) model is $\hat{\Gamma} = 0.378$ $(\hat{\Omega} = 2.793)$. Obviously, $\hat{\gamma}$ is not close to $\hat{\Gamma}$ $(\hat{\Omega}^{-1})$. So, it is natural that the 2BA-t (t=I, II) models fit the data in Table 1 poorly.

4.2. Analysis of Table 2

The 2DS model fits the data in Table 2 well, however, the CDS and 1DS models do not fit these data well (see Table 4). By using the decompositions for the CDS and 1DS models, we shall consider the reason why the CDS and 1DS models fit these data poorly.

The Q2DS model fits these data very well, but each of the MCDS-t (t = I, II) and CBA-t (t = I, II) models fits these data very poorly (see Table 4). So, the poor fit of the CDS model is caused by the influence of the lack of structures of the MCDS-t (t = I, II) and CBA-t (t = I, II) models.

Also, the QDS model fits these data very well, but the M1DS-t (t = I, II) models fit these data poorly (see Table 4). So, the poor fit of the 1DS model is caused by the influence of the lack of structure of the M1DS-t (t = I, II) models.

For the comparison between the QDS and Q2DS models, the difference between the G^2 values for the QDS and Q2DS models is 1.13 with 1 degree of freedom. Therefore, the QDS model may be preferable to the Q2DS model for these data. The QDS model shows that the relationship between a pupil's right and left eye visions is symmetric and point-symmetric with respect to the odds ratios.

Moreover, the 2DS model and its decomposed models fit these data very well (see Table 4). In addition, we shall consider the comparisons between the 2DS

Table 4. Likelihood ratio statistic G^2 for models applied to the data in Tables 1 and 2.

Applied models	Degrees of freedom	For Table 1 G^2	For Table 2 G^2
DS	10	274.02*	2958.74*
CDS	9	174.76*	233.44*
1DS	9	91.54*	20.86*
2DS	8	63.33*	10.56
QDS	5	11.80*	3.94
Q2DS	4	4.49	2.81
MDS	5	262.37*	2947.82*
MCDS-I	4	170.57*	107.14*
MCDS-II	4	170.67*	102.67*
M1DS-I	4	80.48*	14.87*
M1DS-II	4	79.65*	13.67*
M2DS-I	3	3.36	6.42
M2DS-II	3	3.99	5.38
2BA-I	1	45.23*	1.23
2BA-II	1	44.19*	1.35
CBA-I	1	4.27*	53.03*
CBA-II	1	4.04*	52.53*
PS	8	271.64*	2950.94*
S	6	4.77	9.69
QS	3	1.09	2.81
MH	3	3.68	6.87

*means significant at 5% level.

model and each of the Q2DS, M2DS-t (t = I, II) and 2BA-t (t = I, II) models. Then, by the tests based on the difference between the G^2 values, the 2DS model is preferable to each of the decomposed models.

Under the 2DS model, the MLEs of γ and ϕ are $\hat{\gamma} = 0.521$ and $\hat{\phi} = 3.719$, and thus $1 < \hat{\gamma}\hat{\phi} = 1.939 < \hat{\gamma}\hat{\phi}^2 = 7.212 < \hat{\gamma}\hat{\phi}^3 = 26.819$. Therefore, under this model, (i) the probability that a pupil's right and left eye grades are i and j $(> i)$, respectively, is estimated to be equal to the probability that they are j and i, respectively, and (ii) the probability that a pupil's right and left eye grades are i and j, respectively, with the total vision $i+j$ being better than the average vision, i.e., $i+j < 5$, is estimated to be $0.521 \times (3.719)^{5-(i+j)}$ (> 1) times higher than the probability that they are j^* and i^* [or i^* and j^*], respectively, with the total vision being worse than the average vision, i.e., $j^* + i^* > 5$; so, a pupil's total vision tends to be better than the average vision.

Finally we note that although the M2DS-t (t = I, II) models may not be preferable to the 2DS model, but the M2DS-t (t = I, II) models fit the data in Table 2 well. Under the M2DS-I model, the MLEs of Γ and Ψ are $\hat{\Gamma} = 0.472$ and $\hat{\Psi} = 3.849$. Thus, $\hat{\Gamma}\hat{\Psi} = 1.818 < \hat{\Gamma}\hat{\Psi}^2 = 6.997 < \hat{\Gamma}\hat{\Psi}^3 = 26.932$. Since $\hat{\Gamma}\hat{\Psi}^k > 1$ $(k = 1, 2, 3)$ under the M2DS-I model, for $i = 1, 2, 3$, the sum of the probability that a pupil's right eye grade is i or below and the probability that the pupil's left eye grade is i or below, is estimated to be greater than the sum of the probability that the pupil's right eye grade is i^* $(= 5 - i)$ or above and the probability that the pupil's left eye grade is i^* or above. Thus, a pupil's vision of both eyes tends to be better than the average vision. In addition, from the MLEs of expected frequencies obtained under the M2DS-I model, we see that for $i = 1, 2, 3$, the probability that a pupil's right (left) eye grade is i or below is estimated to be greater than the probability that it is i^* or above. Thus, a pupil's vision of right (left) eye tends to be better than the average grade $(R + 1)/2 = 2.5$ for right (left) eye grade. A similar interpretation is also obtained under the M2DS-II model with $\hat{\Omega} = 2.087$ and $\hat{\Phi} = 0.259$, though the detail is omitted.

5. Concluding remarks

In Theorem 1, Corollaries 1 and 2, we have given the decompositions for the 2DS, 1DS and CDS models. For the unaided vision data in Table 1, we have seen the reason why the 2DS model does not fit well by using the decompositions for the 2DS model. Also, for the unaided vision data in Table 2, we have seen the reason why the 1DS (CDS) model does not fit well by using the decompositions for the 1DS (CDS) model. Generally, for a given data, when a model fits poorly, the decompositions for the model would be useful for seeing the reason why the model fits the data poorly.

Acknowledgements

The authors would like to express their sincere thanks to Dr. N. Miyamoto, Tokyo University of Science, for her helpful suggestions and encouragement. They are grateful to the referees for helpful comments.

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