

## ON BIVARIATE REVERSED HAZARD RATES

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In this paper we discuss various definitions of bivariate reversed hazard rate and their properties. An exponential representation of bivariate distribution using reversed hazard rates is given and we also develop a new family of bivariate distributions using bivariate reversed hazard rate. Finally we give a local dependence measure using bivariate reversed hazard rates and study its properties. Various applications of the models are pointed out.

*Key words and phrases:* Bivariate distribution, bivariate reversed hazard rate, local dependence measure.

### 1. Introduction

Let  $T$  be a non-negative random variable representing lifetime of a component with distribution function  $F(t)$ . The reversed hazard rate (RHR) of  $T$  is defined as

$$(1.1) \quad h(t) = \lim_{\Delta t \rightarrow 0} P(t - \Delta t < T \leq t \mid T \leq t) / \Delta t.$$

$h(t)\Delta t$  is the approximate probability that component fails in the interval  $(t - \Delta t, t]$  given that it failed before time  $t$ . If  $T$  has continuous density function  $f(t)$ , (1.1) becomes

$$(1.2) \quad h(t) = f(t)/F(t) = d \log F(t) / dt.$$

It is easy to show that  $h(t)$  determines the distribution uniquely by the relationship  $F(t) = \exp\{-\int_t^\infty h(u)du\}$ .

Earlier, Keilson and Sumita (1982) discussed the role of reversed hazard rate in the context of stochastic ordering. Block *et al.* (1998) studied properties of RHR and characterized a class of distribution having constant RHR in their interval of support. Recently, Sengupta *et al.* (2004) studied proportional RHR models and their applications in reliability analysis. For more properties and applications of RHR function, one could refer to Kalbfleisch and Lawless (1989), Gupta and Nanda (2001), Chandra and Roy (2001), Gupta and Wu (2001) and Nair *et al.* (2006).

Unlike the univariate set up, one can give more than one definition for RHR in the multivariate setup. Recently, Roy (2002) defined bivariate reversed hazard rate as a vector and studied its properties. Further, Roy (2002) introduced a class of bivariate distributions using RHR vector. In this paper we consider various definitions of RHR in the multivariate setup. We, then, study properties of these

bivariate RHRs. Using bivariate RHRs, a general class of bivariate distributions that extend the result given in Roy (2002) is introduced. Further, we give a local dependence measure using bivariate RHR, which is analogous to the well known cross ratio function given in Oakes (1989). Various applications of the models are given.

The text is organized as follows. In Section 2, we give various definitions of bivariate RHR and discuss their properties. An exponential representation of bivariate distributions using bivariate RHR is given in Section 3. In Section 4, we develop a new family of bivariate distributions and study its properties. Various special cases of the family are pointed out. In Section 5, we introduce a local dependence measure using bivariate RHRs and study its properties. We, then, propose a simple nonparametric estimator of the measure in Section 6. Various applications of the proposed class of models are pointed out in Section 7.

## 2. Bivariate reversed hazard rate

Let  $T = (T_1, T_2)$  be a nonnegative random vector representing lifetimes of two components of a system with an absolutely continuous distribution function  $F(t_1, t_2)$ . Suppose that the probability density function (p.d.f.) of  $T$ ,  $f(t_1, t_2)$  exists. Bismi (2005) defined bivariate reversed hazard rate as a scalar, given by

$$(2.1) \quad m(t_1, t_2) = f(t_1, t_2)/F(t_1, t_2).$$

It is very easy to see that (2.1) is a natural extension of the univariate RHR given in (1.2).  $m(t_1, t_2)\Delta t_1\Delta t_2 + o(\Delta t_1, \Delta t_2)$  can be interpreted as the probability of failure of components 1 and 2 in intervals  $(t_1 - \Delta t_1, t_1]$  and  $(t_2 - \Delta t_2, t_2]$  respectively, given that they failed before  $(t_1, t_2)$ .

Unlike the univariate setup,  $m(t_1, t_2)$  does not determine  $F(t_1, t_2)$  uniquely. However, one can prove that  $m(t_1, t_2) = m_1(t_1)m_2(t_2)$  implies the independence, where  $m_i(t_i)$  is marginal RHR of  $T_i$ ,  $i = 1, 2$ .

**THEOREM 1.** *The variables  $T_1$  and  $T_2$  are independent if and only if*

$$(2.2) \quad m(t_1, t_2) = m_1(t_1)m_2(t_2), \quad \text{all } t_1, t_2 > 0.$$

The proof is given in Appendix.

Since (2.1) does not determine  $F(t_1, t_2)$  uniquely, other types of RHR should be considered. Roy (2002) defined bivariate RHR as a vector,  $r(t_1, t_2) = (r_1(t_1, t_2), r_2(t_1, t_2))$ , where

$$(2.3) \quad \begin{aligned} r_i(t_1, t_2) &= \lim_{\Delta t_i \rightarrow 0} P(t_i - \Delta t_i \leq T_i \leq t_i \mid T_1 \leq t_1, T_2 \leq t_2) / \Delta t_i \\ &= \partial \log F(t_1, t_2) / \partial t_i, \quad i = 1, 2. \end{aligned}$$

For  $i = 1$ ,  $r_1(t_1, t_2)\Delta t_1$  is the probability of failure of the first component in the interval  $(t_1 - \Delta t_1, t_1]$  given that it has failed before  $t_1$  and the second has

failed before  $t_2$ . The interpretation for  $r_2(t_1, t_2)$  is similar. From Roy (2002), it follows that  $r_i(t_1, t_2)$  determine  $F(t_1, t_2)$  uniquely by the relationships

$$(2.4) \quad F(t_1, t_2) = \exp \left\{ - \int_{t_1}^{\infty} r_1(u, \infty) du - \int_{t_2}^{\infty} r_2(t_1, u) du \right\}$$

or

$$(2.5) \quad F(t_1, t_2) = \exp \left\{ - \int_{t_1}^{\infty} r_1(u, t_2) du - \int_{t_2}^{\infty} r_2(\infty, u) du \right\},$$

where  $r_1(t_1, \infty) = m_1(t_1)$  and  $r_2(\infty, t_2) = m_2(t_2)$  are the marginal RHR of  $T_1$  and  $T_2$  respectively. Further, from (2.4) and (2.5), we can easily show that  $T_1$  and  $T_2$  are independent if and only if  $r_i(t_1, t_2) = m_i(t_i)$ ,  $i = 1, 2$ .

A third definition of RHR that play vital role in the analysis of dependent data is given by  $r^*(t_1, t_2) = (r_1^*(t_1, t_2), r_2^*(t_1, t_2))$ , where

$$(2.6) \quad r_i^*(t_1, t_2) = \lim_{\Delta t_i \rightarrow 0} P(t_i - \Delta t_i \leq T_i \leq t_i \mid T_i \leq t_i, T_j = t_j) / \Delta t_i \\ = f(t_i \mid T_j = t_j) / F(t_i \mid T_j = t_j), \quad i = 1, 2, \quad i \neq j,$$

with  $f(t_i \mid T_j = t_j)$  as the conditional density function of  $T_i$  given  $T_j = t_j$  and  $F(t_i \mid T_j = t_j)$  as the conditional distribution function of  $T_i$  given  $T_j = t_j$ . Thus the definition (2.6) is nothing but univariate RHR of conditional variable  $T_i$  given  $T_j = t_j$ . Since conditional distributions, in general, does not uniquely determine the joint density, (2.6) does not provide  $F(t_1, t_2)$  uniquely. However, it can be shown that  $T_1$  and  $T_2$  are independent if and only if  $r_i^*(t_1, t_2) = m_i(t_i)$ ,  $i = 1, 2$ . Further, if  $T_1$  and  $T_2$  are independent then  $r_i(t_1, t_2) = r_i^*(t_1, t_2)$ . Thus  $r_i^*(t_1, t_2) / r_i(t_1, t_2)$  can be considered as a measure of association of  $T_1$  and  $T_2$ , which is analogous to the well known cross ratio function of Oakes (1989).

### 3. Exponential representation

From (2.4) and (2.5), it follows that  $F(t_1, t_2)$  can be represented by  $r_i(t_1, t_2)$ ,  $i = 1, 2$  in two different ways. In the following, we give a unique representation for  $F(t_1, t_2)$  in terms of bivariate RHR given in (2.1) and (2.3).

**THEOREM 2.** *The distribution function of  $T = (T_1, T_2)$  can be represented in terms of reversed hazard rates as*

$$(3.1) \quad F(t_1, t_2) = \exp \left\{ - \int_{t_1}^{\infty} m_1(u) du \right\} \exp \left\{ - \int_{t_2}^{\infty} m_2(v) dv \right\} \\ \times \exp \left\{ \int_{t_1}^{\infty} \int_{t_2}^{\infty} (m(u, v) - r_1(u, v)r_2(u, v)) dudv \right\}.$$

**PROOF.** The bivariate distribution function  $F(t_1, t_2)$  of  $T = (T_1, T_2)$  can be written as

$$(3.2) \quad F(t_1, t_2) = F_1(t_1)F_2(t_2) \exp\{A(t_1, t_2)\},$$

where  $A(t_1, t_2) = \log[F(t_1, t_2)/F_1(t_1)F_2(t_2)]$  and  $F_i(t_i) = \exp\{-\int_{t_i}^{\infty} m_i(u)du\}$ .

The function  $A(t_1, t_2)$  can be viewed as a measure of dependence between  $T_1$  and  $T_2$  and we can write  $A(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} \varphi(u, v)dudv$ , where  $\varphi(u, v)$  is some bivariate function.

Consider the representation

$$(3.3) \quad F(t_1, t_2) = \exp\{-H(t_1, t_2)\},$$

where

$$(3.4) \quad H(t_1, t_2) = \int_{t_1}^{\infty} m_1(u)du + \int_{t_2}^{\infty} m_2(v)dv - \int_{t_1}^{\infty} \int_{t_2}^{\infty} \varphi(u, v)dudv.$$

The representation (3.3) can be viewed as a generalization of the univariate exponential representation to the bivariate case.

Now consider

$$(3.5) \quad r_i(t_1, t_2) = -\partial H(t_1, t_2)/\partial t_i = m_i(t_i) + \partial A(t_1, t_2)/\partial t_i, \quad i = 1, 2.$$

Differentiating both sides of (3.5) we get

$$\partial^2 A(t_1, t_2)/\partial t_1 \partial t_2 = [f(t_1, t_2)/F(t_1, t_2)] - [\partial \log F(t_1, t_2)/\partial t_1][\partial \log F(t_1, t_2)/\partial t_2],$$

which gives

$$(3.6) \quad \varphi(u, v) = m(u, v) - r_1(u, v)r_2(u, v).$$

Thus from (3.3), (3.4) and (3.6) we obtain (3.1).

*Remark 1.* It may be noted that (3.1) can be written as

$$F(t_1, t_2) = \exp\left\{-\int_{t_1}^{\infty} r_1(u, \infty)du\right\} \exp\left\{-\int_{t_2}^{\infty} r_2(\infty, v)dv\right\} \\ \times \exp\left\{\int_{t_1}^{\infty} \int_{t_2}^{\infty} (m(u, v) - r_1(u, v)r_2(u, v))dudv\right\}.$$

*Remark 2.* If  $m(u, v) - r_1(u, v)r_2(u, v) = -\gamma m_1(u)m_2(v)$ ,  $0 \leq \gamma \leq 1$ , then (3.1) reduces to the model given in Roy (2002),

$$F(t_1, t_2) = F_1(t_1)F_2(t_2) \exp\{-\gamma \log F_1(t_1) \cdot \log F_2(t_2)\}.$$

*Remark 3.* If  $m(u, v) - r_1(u, v)r_2(u, v) = \theta f_1(u)f_2(v)/[1 + \theta(1 - F_1(u))(1 - F_2(v))]^2$ , then (3.1) reduces to Morgenstern (1956)'s family given by

$$F(t_1, t_2) = F_1(t_1)F_2(t_2)[1 + \theta(1 - F_1(t_1))(1 - F_2(t_2))].$$

**4. A new family of bivariate distributions**

On the basis of (3.1), we construct a new class of bivariate distributions.

**THEOREM 3.** *Let  $F(t_1, t_2)$  be a bivariate distribution function defined by exponential representation (3.1). Assume that*

- I)  $\alpha_i > 0, \beta_i \geq 0, i = 1, 2,$
- II)  $\beta_2 \geq \beta_1,$
- III)  $\alpha_i - \beta_2 \geq 0, i = 1, 2,$

and

- IV)  $m(u, v)/r_1(u, v)r_2(u, v) \geq \beta_2/\beta_1, u, v \geq 0.$

Then

$$(4.1) \quad \begin{aligned} F_{\alpha_1, \alpha_2, \beta_1, \beta_2}(t_1, t_2) &= (F_1(t_1))^{\alpha_1} (F_2(t_2))^{\alpha_2} \\ &\quad \times \exp \left\{ \int_{t_1}^{\infty} \int_{t_2}^{\infty} (\beta_1 m(u, v) - \beta_2 r_1(u, v)r_2(u, v)) dudv \right\} \end{aligned}$$

define a class of bivariate distribution function for some lifetimes  $T_1^{(\alpha_1, \alpha_2, \beta_1, \beta_2)}$  and  $T_2^{(\alpha_1, \alpha_2, \beta_1, \beta_2)}$  with marginals  $(F_1(t_1))^{\alpha_1}$  and  $(F_2(t_2))^{\alpha_2}$ , respectively.

**PROOF.** Condition (IV) is just a stronger version of  $\varphi(u, v) \geq 0$  on a parental distribution  $F(t_1, t_2)$ . When  $\beta_1 = \beta_2$ , condition (IV) reduces to the condition  $\varphi(u, v) \geq 0$ . Obviously, due to conditions (I) to (IV) of the theorem, the corresponding boundary conditions trivially hold. Thus,  $F_{\alpha_1, \alpha_2, \beta_1, \beta_2}(-\infty, t_2) = F_{\alpha_1, \alpha_2, \beta_1, \beta_2}(t_1, -\infty) = F_{\alpha_1, \alpha_2, \beta_1, \beta_2}(-\infty, -\infty) = 0$ .

To check the non-negativity of the joint probability density function, we differentiate (4.1) twice. This leads to

$$(4.2) \quad \begin{aligned} f_{\alpha_1, \alpha_2, \beta_1, \beta_2}(t_1, t_2) &= F_{\alpha_1, \alpha_2, \beta_1, \beta_2}(t_1, t_2) \\ &\quad \times \left[ \alpha_1 m_1(t_1) - \int_{t_2}^{\infty} (\beta_1 m(t_1, v) - \beta_2 r_1(t_1, v)r_2(t_1, v)) dv \right] \\ &\quad \times \left[ \alpha_2 m_2(t_2) - \int_{t_1}^{\infty} (\beta_1 m(u, t_2) - \beta_2 r_1(u, t_2)r_2(u, t_2)) du \right] \\ &\quad + F_{\alpha_1, \alpha_2, \beta_1, \beta_2}(t_1, t_2) [\beta_1 m(t_1, t_2) - \beta_2 r_1(t_1, t_2)r_2(t_1, t_2)]. \end{aligned}$$

Now using assumptions (I) to (IV), we get

$$(4.3) \quad \begin{aligned} &\left[ \alpha_1 m_1(t_1) - \int_{t_2}^{\infty} (\beta_1 m(t_1, v) - \beta_2 r_1(t_1, v)r_2(t_1, v)) dv \right] \\ &\geq \left[ \alpha_1 m_1(t_1) - \beta_2 \int_{t_2}^{\infty} (m(t_1, v) - r_1(t_1, v)r_2(t_1, v)) dv \right] \\ &= [\alpha_1 m_1(t_1) + \beta_2 (r_1(t_1, t_2) - m_1(t_1))] \\ &= [(\alpha_1 - \beta_2)m_1(t_1) + \beta_2 r_1(t_1, t_2)] \geq 0. \end{aligned}$$

Similarly,

$$(4.4) \quad \left[ \alpha_2 m_2(t_2) - \int_{t_1}^{\infty} (\beta_1 m(u, t_2) - \beta_2 r_1(u, t_2) r_2(u, t_2)) du \right] \geq [(\alpha_2 - \beta_2) m_2(t_2) + \beta_2 r_2(t_1, t_2)] \geq 0.$$

Now substituting (4.3), (4.4) and the assumption (IV) in equation (4.2), we get  $f_{\alpha_1, \alpha_2, \beta_1, \beta_2}(t_1, t_2) \geq 0$ , which completes the proof.

*Remark 4.* For  $i = 1, 2$ ,  $\alpha_i = \alpha$ , and  $\beta_i = \beta$  the model (4.1) reduces to

$$F_{\alpha, \alpha, \beta, \beta}(t_1, t_2) = (F_1(t_1))^\alpha (F_2(t_2))^\alpha \exp\{\beta \cdot A(t_1, t_2)\}.$$

*Remark 5.* If  $\alpha_i = \beta_i = \alpha$ ,  $i = 1, 2$ , conditions (I) to (IV) reduce to  $\alpha > 0$  and  $\varphi(u, v) > 0$ .

*Remark 6.* If  $\alpha_i = \beta_i = k$  and  $m(u, v) - r_1(u, v)r_2(u, v) = -\gamma m_1(u)m_2(v)$  for  $0 \leq \gamma \leq 1$ , then (4.1) reduces to the characterized extended bivariate model of Roy (2002) given by

$$F_{k, k, k, k}(t_1, t_2) = (F_1(t_1))^k (F_2(t_2))^k \exp\left\{-\left(\frac{\gamma}{k}\right) \log(F_1(t_1))^k \cdot \log(F_2(t_2))^k\right\}.$$

*Remark 7.* Let the dependence structure of the parental distribution function  $F(t_1, t_2)$  be such that  $r_i^*(t_1, t_2) = (1 + \theta)r_i(t_1, t_2)$ , for  $\theta > 0$ . Then, the bivariate distribution function is

$$(4.5) \quad F(t_1, t_2) = [(F_1(t_1))^{-\theta} + (F_2(t_2))^{-\theta} - 1]^{-(1/\theta)}.$$

Now,

$$(4.6) \quad \begin{aligned} \beta_1 m(u, v) - \beta_2 r_1(u, v)r_2(u, v) &= m(u, v)[\beta_1 - (\beta_2/(1 + \theta))] \\ &= [(1 + \theta)/\theta][\beta_1 - (\beta_2/(1 + \theta))]\varphi(u, v) \\ &= \beta\varphi(u, v), \end{aligned}$$

where  $\beta = [(1 + \theta)/\theta][\beta_1 - (\beta_2/(1 + \theta))]$ . Using condition (IV),  $(\beta_1(1 + \theta) - \beta_2) \geq 0$ .

Now (4.5) can be generalized using (4.6) as

$$(4.7) \quad \begin{aligned} F_{\alpha_1, \alpha_2, \beta_1, \beta_2}(t_1, t_2) \\ = (F_1(t_1))^{\alpha_1 - \beta} (F_2(t_2))^{\alpha_2 - \beta} [(F_1(t_1))^{-\theta} + (F_2(t_2))^{-\theta} - 1]^{-(\beta/\theta)}. \end{aligned}$$

The bivariate model (4.5) is analogous to the Clayton copula model (Clayton and Cuzick (1985)) based on cross ratio function.

*Remark 8.* Generally, the evaluation of the Fisher information matrix for the family (4.1) is not an easy task as  $m(t_1, t_2)$ ,  $r_1(t_1, t_2)$  and  $r_2(t_1, t_2)$  involves

parameters other than  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$  and  $\beta_2$ . However, for specific distributions, one can obtain the Fisher information matrix and correlation coefficient. For example, the bivariate inverse exponential distribution with distribution function

$$(4.8) \quad F(x_1, x_2) = \exp\left(\frac{-\alpha_1}{x_1} - \frac{\alpha_2}{x_2} - \frac{\beta}{x_1 x_2}\right)$$

is a member of the family (4.1) with  $\beta_1 = \beta_2 = \beta$ . One can evaluate the Fisher information and correlation coefficient using the relationship between (4.8) and the Gumbel's (1960) bivariate exponential distribution.

## 5. A local dependence measure

The dependence among variables plays a vital role in many practical situations. There are various measures of dependence in literature such as correlation coefficient, Kendal's tau, Spearman's rho etc. Clayton (1978) and Oakes (1989) defined an association measure known as cross ratio function, as

$$p(t_1, t_2) = S(t_1, t_2)f(t_1, t_2)/[\partial S(t_1, t_2)/\partial t_1][\partial S(t_1, t_2)/\partial t_2],$$

where  $S(t_1, t_2) = P[T_1 > t_1, T_2 > t_2]$  is the survival function of  $(T_1, T_2)$ . Oakes (1989) interpreted the  $p(t_1, t_2)$  as the ratio of the hazard rate of the conditional distribution of  $T_1$  given  $T_2 = t_2$ , to that of  $T_1$ , given  $T_2 > t_2$ .

Recently, Gupta (2003) has extensively studied relationships among hazard rate and cross ratio function in the bivariate setup. Further Gupta (2003) and Finkelstein (2003) have obtained a class of bivariate distributions for which the cross ratio function is constant.

Analogous to the cross ratio function  $p(t_1, t_2)$ , we define a new local dependence measure in terms of reversed hazard rates, given by

$$\lambda(t_1, t_2) = F(t_1, t_2)f(t_1, t_2)/[\partial F(t_1, t_2)/\partial t_1][\partial F(t_1, t_2)/\partial t_2],$$

which can be expressed as

$$(5.1) \quad \lambda(t_1, t_2) = m(t_1, t_2)/r_1(t_1, t_2)r_2(t_1, t_2).$$

Obviously  $\lambda(t_1, t_2) > 0$  for all  $t_1, t_2 > 0$ .  $\lambda(t_1, t_2)$  can be interpreted as the ratio of the reversed hazard rate of the conditional distribution of  $T_1$  given  $T_2 = t_2$  to that of  $T_1$  given  $T_2 < t_2$ . By symmetry, a similar interpretation holds with  $(T_1, T_2)$  interchanged. Thus

$$\lambda(t_1, t_2) = r_i^*(t_1, t_2)/r_i(t_1, t_2), \quad i = 1, 2.$$

One can easily see that  $\lambda(t_1, t_2) = \theta + 1$ , where  $\theta$  is the parameter in the model (4.5), governing the association between  $T_1$  and  $T_2$ .

Now we study various properties of  $\lambda(t_1, t_2)$ .

**THEOREM 4.**  $\lambda(t_1, t_2) = 1$  if and only if  $T_1$  and  $T_2$  are independent.

PROOF. If  $T_1$  and  $T_2$  are independent, then  $F(t_1, t_2) = F_1(t_1)F_2(t_2)$ , which leads to  $\lambda(t_1, t_2) = 1$ . Conversely, if  $\lambda(t_1, t_2) = 1$ , then  $m(t_1, t_2) = r_1(t_1, t_2)r_2(t_1, t_2)$ .

Substituting the above in (3.1), we get  $F(t_1, t_2) = F_1(t_1)F_2(t_2)$ , which completes the proof.

THEOREM 5.  $\lambda(t_1, t_2) = r_1^*(t_1, t_2)/r_1(t_1, t_2) = r_2^*(t_1, t_2)/r_2(t_1, t_2)$ .

PROOF. Consider  $\partial F(t_1, t_2)/\partial t_2 = \int_0^{t_1} f(u, t_2)du = f_2(t_2)F(t_1 | T_2 = t_2)$ , which gives

$$(5.2) \quad F(t_1 | T_2 = t_2) = (\partial F(t_1, t_2)/\partial t_2)/f_2(t_2).$$

Now substituting (5.2) in (2.6), we get

$$(5.3) \quad r_1^*(t_1, t_2) = f(t_1 | T_2 = t_2)f_2(t_2)/(\partial F(t_1, t_2)/\partial t_2) = m(t_1, t_2)/r_2(t_1, t_2).$$

Thus from (5.3) and (5.1) we get

$$(5.4) \quad r_1^*(t_1, t_2)/r_1(t_1, t_2) = \lambda(t_1, t_2).$$

Similarly we can obtain

$$(5.5) \quad r_2^*(t_1, t_2)/r_2(t_1, t_2) = \lambda(t_1, t_2),$$

which completes the proof.

From the discussions in the above sections, we have following theorem, whose proof is direct.

THEOREM 6. *The following statements are equivalent:*

- I)  $\lambda(t_1, t_2) = 1$ .
- II)  $r_1^*(t_1, t_2) = r_1(t_1, t_2)$ .
- III)  $r_2^*(t_1, t_2) = r_2(t_1, t_2)$ .
- IV)  $r_1(t_1, t_2) = m_1(t_1)$ .
- V)  $r_2(t_1, t_2) = m_2(t_2)$ .
- VI)  $r_1^*(t_1, t_2) = m_1(t_1)$ .
- VII)  $r_2^*(t_1, t_2) = m_2(t_2)$ .
- VIII)  $m(t_1, t_2) = r_1(t_1, t_2)r_2(t_1, t_2)$ .
- IX)  $m(t_1, t_2) = m_1(t_1)m_2(t_2)$ .
- X)  $p(t_1, t_2) = 1$ .
- XI)  $T_1$  and  $T_2$  are independent.

THEOREM 7. *The bivariate RHRs and  $\lambda(t_1, t_2)$  are related by*

$$\begin{aligned} r_1(t_1, t_2)r_2(t_1, t_2)[\lambda(t_1, t_2) - 1] &= r_2(t_1, t_2)[r_1^*(t_1, t_2) - r_1(t_1, t_2)] \\ &= r_1(t_1, t_2)[r_2^*(t_1, t_2) - r_2(t_1, t_2)]. \end{aligned}$$

PROOF. The proof follows from (5.4) and (5.5).



**6. Estimation of  $\lambda(t_1, t_2)$**

In Section 5, we proved that  $\lambda(t_1, t_2) = 1$  ( $\theta = 0$ ) if and only if  $T_1$  and  $T_2$  are independent. In order to use  $\lambda(t_1, t_2)$  as test of independence, we first need to find an estimate of  $\lambda(t_1, t_2)$  from the sample. Based on samples  $(T_{1i}, T_{2i})$ ,  $i = 1, 2, \dots, n$  from the bivariate distribution (4.5), we propose a non-parametric estimator of  $\lambda(t_1, t_2)$  using Kendall's (1962) coefficient of concordance. For  $1 \leq i < j \leq n$ , define  $X_{ij} = 1$  or  $X_{ij} = -1$  according as  $T_{1i} < T_{1j}$  or  $T_{1i} > T_{1j}$ . Define  $Y_{ij}$  similarly for  $T_{2i}$  and  $T_{2j}$  and let  $Z_{ij} = X_{ij}Y_{ij}$ . Set  $U = \frac{\sum_{i=1}^n \sum_{j>i}^n Z_{ij}}{\binom{n}{2}}$ . Since the probability of concordance is  $\lambda(t_1, t_2)/\lambda(t_1, t_2) + 1$ ,  $U$  is an unbiased estimator of  $(\lambda(t_1, t_2) - 1)/(\lambda(t_1, t_2) + 1)$ . Thus, we propose a nonparametric estimator of  $\lambda(t_1, t_2)$  by  $\hat{\lambda}(t_1, t_2) = (1 + U)/(1 - U)$ . The asymptotic normality of  $U$  follows from the results of Hoeffding (1948).

**7. Applications**

The class of models (4.1) can be used to represent the lifetime of a parallel system in reliability analysis. Suppose that there are  $k$ -identical systems, each has two components. Let  $T_i = (T_{1i}, T_{2i})$  be the lifetime vector of the  $i$ -th system,  $i = 1, 2, \dots, k$ . Consider a bivariate parallel combination as a collection of two parallel connections, the first one with all the first components and the second one with all the second components. Thus the bivariate lifetime vector of the parallel combination is given by  $U = (U_1, U_2)$ , where  $U_1 = \max(T_{1i})$  and  $U_2 = \max(T_{2i})$ ,  $i = 1, 2, \dots, k$ . When the distribution of  $T = (T_1, T_2)$  is of the form (4.1), the distribution of  $U = (U_1, U_2)$  is obtained as

$$F^*(t_1, t_2) = F_1(t_1)^{\alpha_1 k} F_2(t_2)^{\alpha_2 k} e^{\int_{t_1}^{\infty} \int_{t_2}^{\infty} (\beta_1 k m(u,v) - \beta_2 k r_1(u,v) r_2(u,v)) du dv}$$

Thus, we have a closure property of the model (4.1) under a bivariate parallel combination.

The bivariate lifetime model (4.5) can be used as a lifetime model induced by frailties in the following way.

Suppose that  $T = (T_1, T_2)$  represents the lifetime of a two component system. Suppose that there exists a positive random variable  $W$  such that the conditional distribution function of  $T_i$  given  $W = w$  is

$$(7.1) \quad P(T_i < t_i | W = w) = F_i(t_i)^w, \quad i = 1, 2$$

and that given  $W = w$ ,  $T_1$  and  $T_2$  are conditionally independent. Then (7.1) can be considered as a frailty model in the univariate setup. Then a bivariate frailty model is given by  $F(t_1, t_2) = \int (F_1(t_1)F_2(t_2))^w dG(w)$ , where  $F_1(t_1)$  and  $F_2(t_2)$  are some baseline distribution functions and  $G(\cdot)$  is the distribution function of  $W$ . (7.1) is equivalent to a proportional reversed hazards model of Sengupta *et al.* (2004). If  $F_i(t_i)$  is a distribution function, so also is  $F_i^*(t_i) = \exp[-\{\frac{1}{F_i(t_i)}\}^\theta + 1]$ ,  $i = 1, 2$ .

A random effects interpretation of the model (4.5) can be given in terms of  $F_i^*(t_i)$ ,  $i = 1, 2$ . Let  $W$  have a gamma density with p.d.f.  $g(w) \propto e^{-w} w^{(1/\theta)-1}$

and suppose that conditionally on  $W = w$ ,  $T_1$  and  $T_2$  are independent with distribution functions  $F_1^*(t_1)^w$  and  $F_2^*(t_2)^w$ , respectively. Then it is easy to see that, unconditionally,  $T_1$  and  $T_2$  have joint distribution function (4.5). This gives another interpretation of the model (4.5). Further, this representation gives a convenient method for simulating  $T_1$  and  $T_2$ . As  $\theta \rightarrow 0$ ,  $F(t_1, t_2) \rightarrow F_1(t_1)F_2(t_2)$  corresponding to independence between  $T_1$  and  $T_2$ .

The joint p.d.f. of  $T_1$  and  $T_2$  is

$$f(t_1, t_2) = [(1 + \theta)f_1(t_1)f_2(t_2)K(t_1, t_2)^{-((1/\theta)+2)}]/[(F_1(t_1)F_2(t_2))^{(1+\theta)}]$$

where  $K(t_1, t_2) = (F_1(t_1))^{-\theta} + (F_2(t_2))^{-\theta} - 1$ .

One can generalize (4.5) by considering the distribution of  $T_i$ , given  $W = w$  as  $F_i(t_i)^{\gamma_i} F_i^*(t_i)^w$ ,  $i = 1, 2$  where  $\gamma_i$  is an additional parameter and the distribution of  $W$  is gamma with shape parameter  $\beta/\theta$ . In this case, the joint distribution of  $T_1$  and  $T_2$  will be of the form (4.7) with  $\alpha_i - \beta = \gamma_i$ ,  $i = 1, 2$ .

Thus, when two observed lifetimes  $T_1$  and  $T_2$  each depend on the same unobserved frailty via a proportional reversed hazards model, this common dependence induces an association between the observed times.

**Appendix A: Proof of Theorem 1**

When  $T_1$  and  $T_2$  are independent, we can have  $f(t_1, t_2) = f_1(t_1)f_2(t_2)$  and  $F(t_1, t_2) = F_1(t_1)F_2(t_2)$ , where  $f_i(t_i)$  and  $F_i(t_i)$  respectively denotes the marginal density and distribution functions of  $T_i$ ,  $i = 1, 2$ . Thus from (2.1) we can easily obtain (2.2).

To prove the converse, from (2.2) and (3.2) we have

$$F(t_1, t_2)/F_1(t_1)F_2(t_2) = f(t_1, t_2)/f_1(t_1)f_2(t_2) = \exp\{A(t_1, t_2)\},$$

which gives

$$(A.1) \quad f(t_1, t_2) = \exp\{A(t_1, t_2)\}f_1(t_1)f_2(t_2).$$

Now differentiating (3.2) with respect to  $t_1$ , we get

$$(A.2) \quad \partial F(t_1, t_2)/\partial t_1 = m_1(t_1)F(t_1, t_2) + F(t_1, t_2)\partial A(t_1, t_2)/\partial t_1.$$

Also we have

$$(A.3) \quad \partial F(t_1, t_2)/\partial t_1 = \int_0^{t_2} f(t_1, v)dv.$$

Now using (A.1) in (A.3) we get

$$(A.4) \quad \partial F(t_1, t_2)/\partial t_1 = m_1(t_1)F(t_1, t_2) - \int_0^{t_2} m_1(t_1)F(t_1, v)(\partial A(t_1, v)/\partial v)dv.$$

Equating (A.2) and (A.4), we get

$$(A.5) \quad F(t_1, t_2)\partial A(t_1, t_2)/\partial t_1 = -m_1(t_1) \int_0^{t_2} F(t_1, v)(\partial A(t_1, v)/\partial v)dv.$$

Similarly we get

$$(A.6) \quad F(t_1, t_2)\partial A(t_1, t_2)/\partial t_2 = -m_2(t_2) \int_0^{t_1} F(u, t_2)(\partial A(u, t_2)/\partial u)du.$$

Substituting (3.5) in (A.5) and (A.6) we get

$$(A.7) \quad F(t_1, t_2)r_1(t_1, t_2) = m_1(t_1) \int_0^{t_2} F(t_1, v)m_2(v)dv$$

and

$$(A.8) \quad F(t_1, t_2)r_2(t_1, t_2) = m_2(t_2) \int_0^{t_1} F(u, t_2)m_1(u)du.$$

Dividing (A.7) by (A.8) we get

$$(A.9) \quad \begin{aligned} m_1(t_1)r_2(t_1, t_2) \int_0^{t_2} F(t_1, v)m_2(v)dv \\ = m_2(t_2)r_1(t_1, t_2) \int_0^{t_1} F(u, t_2)m_1(u)du. \end{aligned}$$

Now using (3.5) in (A.9) we get

$$(A.10) \quad \begin{aligned} F(t_1, t_2)[m_1(t_1)r_2(t_1, t_2) - m_2(t_2)r_1(t_1, t_2)] \\ = m_1(t_1)r_2(t_1, t_2) \int_0^{t_2} F(t_1, v)(\partial A(t_1, v)/\partial v)dv \\ - m_2(t_2)r_1(t_1, t_2) \int_0^{t_1} F(u, t_2)(\partial A(u, t_2)/\partial u)du. \end{aligned}$$

The equation (A.10) is satisfied for all  $t_1, t_2$  if either

$$(A.11) \quad \begin{aligned} F(t_1, t_2) &= \int_0^{t_2} F(t_1, v)(\partial A(t_1, v)/\partial v)dv \\ &= \int_0^{t_1} F(u, t_2)(\partial A(u, t_2)/\partial u)du \end{aligned}$$

or

$$(A.12) \quad \partial A(t_1, v)/\partial v = \partial A(u, t_2)/\partial u = 0.$$

If possible (A.11) is true, then differentiating with respect to  $t_2$ , we get  $r_2(t_1, t_2) = \partial A(t_1, t_2)/\partial t_2 = r_2(t_1, t_2) - m_2(t_2)$ , which gives  $m_2(t_2) = 0$  and similarly if we differentiate (A.11) with respect to  $t_1$  we get  $m_1(t_1) = 0$ . But this is not possible. So the condition (A.11) is not true.

Now, if (A.12) is true, then  $r_2(t_1, t_2) = m_2(t_2)$  and  $r_1(t_1, t_2) = m_1(t_1)$ . Thus from (2.2), we obtain

$$(A.13) \quad m(t_1, t_2) = r_1(t_1, t_2)r_2(t_1, t_2).$$

Substituting (A.13) in (3.1), we get  $F(t_1, t_2) = F_1(t_1)F_2(t_2)$ , which completes the proof.

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