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EXACT AND APPROXIMATE DISTRIBUTIONS FOR THE LINEAR COMBINATION OF INVERTED DIRICHLET COMPONENTS

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It is well known that $X + Y$ has the F distribution when X and Y follow the inverted Dirichlet distribution. In this paper, we derive the exact distribution of the general form $\alpha X + \beta Y$ (involving the Gauss hypergeometric function) and the corresponding moment properties. We also propose approximations and discuss evidence of their robustness based on the powerful Kolmogorov-Smirnov test. The work is motivated by real-life examples in quality and reliability engineering.

Key words and phrases: Dirichlet distribution, Gauss hypergeometric function, linear combinations of random variables.

1. Introduction

For given random variables X and Y, the distribution of linear combinations of the form $\alpha X + \beta Y$ is of interest in problems in quality and reliability engineering. Some examples are:

(i) Quality determination of fresh fruits and vegetables is very important in agricultural production. Despite the numerous techniques developed for non-destructive evaluation of the quality of fruits and vegetables, quality sorting was used which was primarily based on manual decisions and hand labor. Ozer *et al.* (1998) developed procedures and models, for quality sorting of agricultural produce by fusing information from multiple sensors. The quality of fruit was defined as a linear combination of numerous parameters such as: firmness, total soluble solids (TSS), acidity, aroma, color, color uniformity, bruises, scars, cuts, presence of soil, size, shape, insects and/or diseases and sensible parameters that are specific to the individual fruit; such as the force of stem detachment and the visualization of slip development in cantaloupes. Ozer *et al.* (1998) developed a system that can classify fruits based upon several of these parameters by using multi-sensors data acquisition.

(ii) The theory of process capability indices is of fundamental importance in quality technology. In a celebrated paper, Pearn *et al.* (1992) proposed the process capability index C_{pmk} and investigated the statistical properties of its natural estimator for stable normal processes. This index has proved very popular among practitioners of quality technology. In a theoretical

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follow-up study, Chen and Hsu (1995) have shown that, under general conditions, the asymptotic distribution of the natural estimator is often a linear combination of the normal and the folded-normal distributions.

(iii) In the diagnosis of ill-conditioned compliant assemblies, the purpose is to estimate faults in processes based on the measured assembly responses. Several authors have used the well-known least squares approach for this estimation (see, for example, Rong *et al.* (2001)). This statistical method is used to estimate linear combinations of the faults that generate similar fault signatures.

(iv) When dealing with two or more "control" variables, one is often interested in the optimal linear combination. This has been extensively investigated in the literature (see, for example, Glynn and Iglehart (1989)).

The distribution of $\alpha X + \beta Y$ has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Fisher (1935) and Chapman (1950) for Student's t family, Christopeit and Helmes (1979) for normal family, Davies (1980) and Farebrother (1984) for chi-squared family, Ali and Obaidullah (1982) for exponential family, Moschopoulos (1985) and Provost (1989) for gamma family, Dobson *et al.* (1991) for Poisson family, Pham-Gia and Turkkan (1993) and Pham and Turkkan (1994) for beta family, Kamgar-Parsi *et al.* (1995) and Albert (2002) for uniform family, Hitezenko (1998) and Hu and Lin (2001) for Rayleigh family, and Witkovský (2001) for inverted gamma family. However, there is relatively little work of this kind when X and Y are correlated random variables.

In this paper, we consider the distribution of $S = \alpha X + \beta Y$ when X and Y are distributed according to the joint pdf

$$f(x, y) = \frac{\Gamma(a + b + c) x^{a-1} y^{b-1}}{\Gamma(a)\Gamma(b)\Gamma(c)(1 + x + y)^{a+b+c}} \tag{1.1}$$

for $x > 0$, $y > 0$, $a > 0$, $b > 0$ and $c > 0$. This is known as the inverted Dirichlet distribution (see, for example, Kotz *et al.* (2000)). It has received applications in many areas (see, for example, Tiao and Guttman (1965) and Xu (1990)).

The paper is organized as follows. In Sections 2 and 3, we derive exact expressions for the pdf and moments of $S = \alpha X + \beta Y$, involving the Gauss hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!}$$

(where $(c)_k = c(c+1) \cdots (c+k-1)$ denotes the ascending factorial), the properties of which can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000). In Section 4, we propose approximations for the distribution of S and discuss evidence of their robustness based on the powerful Kolmogorov-Smirnov test. These approximations will be useful especially to the practitioners of the inverted Dirichlet distribution. Section 5 concludes the paper and discusses possible future work.

2. PDFS

The constants α and β in $S = \alpha X + \beta Y$ can be positive and negative; so, four cases are possible: $\alpha > 0$, $\beta > 0$; $\alpha < 0$, $\beta > 0$; $\alpha > 0$, $\beta < 0$; and, $\alpha < 0$, $\beta < 0$. But, by symmetry, it is sufficient to consider the two cases: $\alpha > 0$, $\beta > 0$; and, $\alpha < 0$, $\beta > 0$. Theorems 1 and 2 below derive the pdfs of S for these two cases.

THEOREM 1. *If X and Y are jointly distributed according to* (1.1) *then $S = \alpha X + \beta Y$ for $\alpha > 0$ and $\beta > 0$ has the pdf*

$$\begin{aligned} f_S(s) = & \frac{\beta^{a+c}\Gamma(a+b+c)}{\alpha^a \Gamma(a+b)\Gamma(c)} \frac{s^{a+b-1}}{(\beta+s)^{a+b+c}} \\ & \times {}_2F_1\left(a, a+b+c; a+b; \frac{(\alpha-\beta)s}{\alpha(\beta+s)}\right) \end{aligned} \tag{2.1}$$

for $0 < s < \infty$.

PROOF. Using (1.1), one can write

$$\begin{aligned} f_S(s) &= \frac{1}{\beta}\int_0^{s/\alpha} f\left(x, \frac{s-\alpha x}{\beta}\right) dx \\ &= \frac{\Gamma(a+b+c)}{\beta\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^{s/\alpha} x^{a-1}\left(\frac{s-\alpha x}{\beta}\right)^{b-1}\left(1+x+\frac{s-\alpha x}{\beta}\right)^{-(a+b+c)} dx \\ &= \frac{\alpha^{b-1}\beta^{a+c}\Gamma(a+b+c)}{(\beta-\alpha)^{a+b+c}\Gamma(a)\Gamma(b)\Gamma(c)} \\ &\quad \times \int_0^{s/\alpha} x^{a-1}\left(\frac{s}{\alpha}-x\right)^{b-1}\left(\frac{\beta+s}{\beta-\alpha}+x\right)^{-(a+b+c)} dx. \end{aligned} \tag{2.2}$$

By equation (2.2.6.15) in Prudnikov *et al.* (1986, volume 1), the integral in (2.2) can be calculated as

$$\begin{aligned} & \int_0^{s/\alpha} x^{a-1}\left(\frac{s}{\alpha}-x\right)^{b-1}\left(\frac{\beta+s}{\beta-\alpha}+x\right)^{-(a+b+c)} dx \\ & \quad = B(a,b)\left(\frac{s}{\alpha}\right)^{a+b-1}\left(\frac{\beta+s}{\beta-\alpha}\right)^{-(a+b+c)} \\ & \quad\quad \times {}_2F_1\left(a, a+b+c; a+b; \frac{(\alpha-\beta)s}{\alpha(\beta+s)}\right). \end{aligned} \tag{2.3}$$

The result in (2.1) follows by combining (2.2) and (2.3). □

THEOREM 2. *If X and Y are jointly distributed according to* (1.1) *then $S = \alpha X + \beta Y$ for $\alpha < 0$ and $\beta > 0$ has the pdf*

$$\begin{aligned} f_S(s) = & \frac{c\beta^{a+c}\Gamma(a+b+c)}{(-\alpha)^a \Gamma(a+c+1)\Gamma(b)} \frac{s^{a+b-1}}{(\beta+s)^{a+b+c}} \\ & \times {}_2F_1\left(a, a+b+c; a+c+1; 1+\frac{(\beta-\alpha)s}{\alpha(\beta+s)}\right) \end{aligned} \tag{2.4}$$

for $s > 0$, *and*

$$f_S(s) = \frac{c(-\alpha)^{b+c}\Gamma(a+b+c)}{\beta^b\Gamma(b+c+1)\Gamma(a)} \frac{(-s)^{a+b-1}}{(-\alpha-s)^{a+b+c}} \times {}_2F_1\left(b, a+b+c; b+c+1; 1-\frac{(\beta-\alpha)s}{\beta(\alpha+s)}\right) \tag{2.5}$$

for $s < 0$.

PROOF. If $s > 0$ then one can write

$$\begin{aligned} f_S(s) &= \frac{1}{\beta}\int_0^\infty f\left(x, \frac{s-\alpha x}{\beta}\right) dx \\ &= \frac{\Gamma(a+b+c)}{\beta\Gamma(a)\Gamma(b)\Gamma(c)}\int_0^\infty x^{a-1}\left(\frac{s-\alpha x}{\beta}\right)^{b-1}\left(1+x+\frac{s-\alpha x}{\beta}\right)^{-(a+b+c)} dx \\ &= \frac{(-\alpha)^{b-1}\beta^{a+c}\Gamma(a+b+c)}{(\beta-\alpha)^{a+b+c}\Gamma(a)\Gamma(b)\Gamma(c)} \\ &\quad \times \int_0^\infty x^{a-1}\left(x-\frac{s}{\alpha}\right)^{b-1}\left(\frac{\beta+s}{\beta-\alpha}+x\right)^{-(a+b+c)} dx. \end{aligned} \tag{2.6}$$

By equation (2.2.6.24) in Prudnikov *et al.* (1986, volume 1), the integral in (2.6) can be calculated as

$$\begin{aligned} &\int_0^\infty x^{a-1}\left(x-\frac{s}{\alpha}\right)^{b-1}\left(\frac{\beta+s}{\beta-\alpha}+x\right)^{-(a+b+c)} dx \\ &= B(a, 1+c)\left(-\frac{s}{\alpha}\right)^{a+b-1}\left(\frac{\beta+s}{\beta-\alpha}\right)^{-(a+b+c)} \\ &\quad \times {}_2F_1\left(a, a+b+c; 1+a+c; 1+\frac{(\beta-\alpha)s}{\alpha(\beta+s)}\right). \end{aligned} \tag{2.7}$$

The result in (2.4) follows by combining (2.6) and (2.7). The proof of (2.5) is similar. □

The following corollary notes one special case where (2.4) and (2.5) reduce to elementary forms.

COROLLARY 1. *If* $b = 1$ *then* (2.4) *reduces to*

$$f_S(s) = \frac{c\beta^{a+c}}{(\beta-\alpha)^a(\beta+s)^{1+c}} \tag{2.8}$$

for $s > 0$. *If* $a = 1$ *then* (2.5) *reduces to*

$$f_S(s) = \frac{c(-\alpha)^{b+c}}{(\beta-\alpha)^b(-\alpha-s)^{1+c}} \tag{2.9}$$

for $s < 0$.

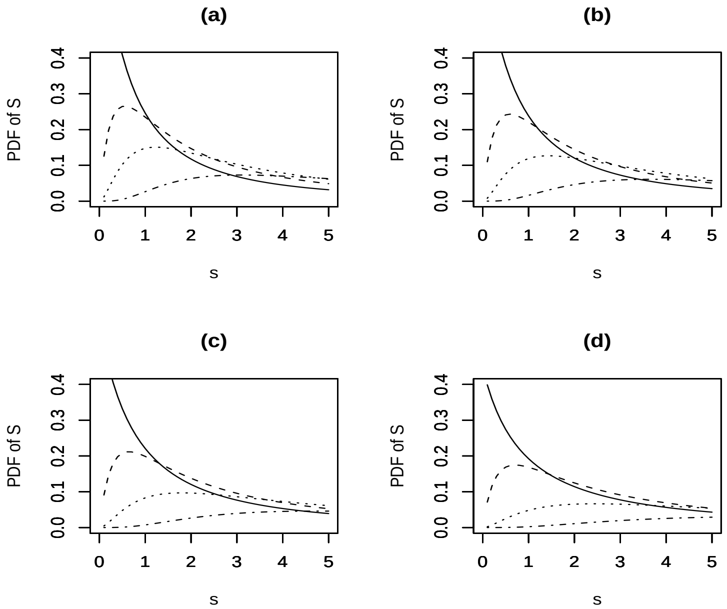

Figure 1. The pdf (2.1) for $\alpha = 1$, $c = 1$ and (a): $\beta = 1.5$; (b): $\beta = 2$; (c): $\beta = 3$; (d): $\beta = 5$. The four curves in each plot are: the solid curve ($a = 0.5$, $b = 0.5$), the curve of dots ($a = 1.5$, $b = 0.5$), the curve of lines ($a = 1.5$, $b = 1.5$), and the curve of dots and lines ($a = 3$, $b = 3$).

Proof. Both (2.8) and (2.9) follow from Theorem 2 by using the property of the Gauss hypergeometric function that ${}_2F_1(a, b; b; x) = (1 - x)^{-a}$. $\square$

Figures 1 and 2 illustrate possible shapes of the pdfs (2.1), (2.4) and (2.5) for selected values of a, b, c, α and β (the pdfs computed by using the hypergeom ([·, ·], [·], ·) function in MAPLE). The four curves in each plot correspond to selected values of a and b. The effect of the parameters is evident: the densities become flatter with increasing values of a and b and with increasing values of β.

3. Moments

Here, we derive the moments of $S = \alpha X + \beta Y$ when X and Y are distributed according to (1.1).

Theorem 3. *If X and Y are jointly distributed according to* (1.1) *then*

$$E(S^n) = \frac{\alpha^n \Gamma(c-n)}{\Gamma(a)\Gamma(b)\Gamma(c)} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\beta}{\alpha}\right)^k \Gamma(a+n-k)\Gamma(b+k) \tag{3.1}$$

for $n \geq 0$ provided that $c > n$. In particular, the first two moments of S are

$$E(S) = \frac{a\alpha + b\beta}{c-1}$$

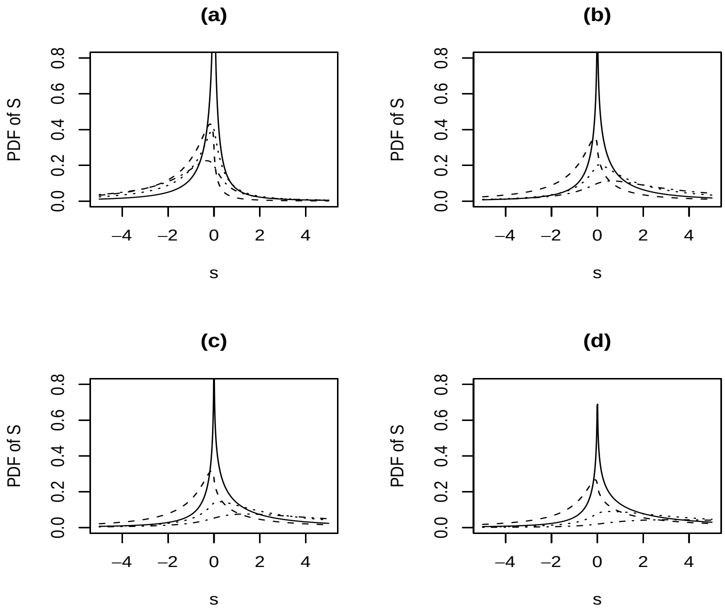

Figure 2. The pdf (2.4)–(2.5) for $\alpha = -1$, $c = 1$ and (a): $\beta = 0.5$; (b): $\beta = 2$; (c): $\beta = 3$; (d): $\beta = 5$. The four curves in each plot are: the solid curve ($a = 0.5$, $b = 0.5$), the curve of dots ($a = 1.5$, $b = 0.5$), the curve of lines ($a = 1.5$, $b = 1.5$), and the curve of dots and lines ($a = 3$, $b = 3$).

(*provided* $c > 1$) *and*

$$E(S^2) = \frac{a^2\alpha^2 + b^2\beta^2 + a\alpha^2 + b\beta^2 + 2ab\alpha\beta}{(c-1)(c-2)}$$

(*provided* $c > 2$).

Proof. By writing

$$\begin{aligned} E(S^n) &= E((\alpha X + \beta Y)^n) \\ &= \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \beta^k E(X^{n-k} Y^k) \end{aligned} \tag{3.2}$$

and by applying the product moment formula for the inverted Dirichlet distribution

$$E(X^m Y^n) = \frac{\Gamma(a+m)\Gamma(b+n)\Gamma(c-m-n)}{\Gamma(a)\Gamma(b)\Gamma(c)}$$

(for $c > m + n$) to each expectation in (3.2), one can obtain (3.1). □

Note that the proof of this theorem does not require the knowledge of the distribution of S.

4. Approximations

In this section, we propose approximations for the distribution of $S = \alpha X + \beta Y$. Since $1/(1 \pm S) \mid S \gtrless 0$ (meaning $1/(1+S) \mid S > 0$ or $1/(1-S) \mid S < 0$) is a monotonic function of S and has support in the interval $[0, 1]$, we are motivated to approximate its distribution by a suitable member of the two-parameter beta family of distributions:

$$f(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)} \tag{4.1}$$

for $0 < x < 1$, $p > 0$ and $q > 0$. The idea of approximating distributions with complicated formulas with the beta distribution is very well established in the statistics literature. It goes back to the 1968 *Sankhyā* paper by Professor Das Gupta (see also more recent papers by Sculli and Wong (1985), Fan (1991), and Johannesson and Giri (1995)). The purpose of doing this is not because one cannot compute complicated formulas as those given by (2.1), (2.4) or (2.5). The purpose is to give simple approximations in terms of the beta distribution so that one can use the known procedures for inference, prediction, etc.

There are infinitely many monotonic transformations that could convert the data on S to the unit interval $(0, 1)$. We have chosen the transformation $1/(1 \pm S) \mid S \gtrless 0$ above. There are two main reasons for choosing this over others:

(i) it is the earliest and the simplest known transformation to convert data on the real line to $(0, 1)$.

(ii) the pdfs in (2.1), (2.4) and (2.5) all take the form of an inverted beta pdf multiplied by a Gauss hypergeometric term. Thus, the choice of $1/(1 \pm S) \mid S \gtrless 0$ is most natural because if S (respectively, $-S$) will have an exact inverted beta distribution then $1/(1+S)$ (respectively, $1/(1-S)$) will have an exact beta distribution. In fact, (2.1) reduces to an exact inverted beta pdf if $\alpha = \beta$; (2.4) reduces to an exact inverted beta pdf if either $b = 1$ (see equation (2.8)) or $\beta \downarrow 0$; and, (2.5) reduces to an exact inverted beta pdf if either $a = 1$ (see equation (2.9)) or $\alpha \uparrow 0$. One could think of the Gauss hypergeometric term as an "error term" perturbing the inverted beta pdf.

This is the first time that approximations of the form (4.1) have been proposed for correlated inverted beta random variables. The choice of the beta parameters p and q is made using the method of moments. Equating the first two moments of $1/(1 \pm S) \mid S \gtrless 0$ with those of the beta distribution, we have

$$E\left(\frac{1}{1 \pm S} \;\middle|\; S \gtrless 0\right) = \frac{p}{p+q}$$

and

$$E\left(\frac{1}{(1 \pm S)^2} \;\middle|\; S \gtrless 0\right) = \frac{p(p+1)}{(p+q)(p+q+1)}$$

which we must solve simultaneously to find the beta parameters p and q. After some algebraic manipulation, we find the solutions as

$$p = E\left(\frac{1}{1 \pm S} \;\middle|\; S \gtrless 0\right) \tag{4.2}$$

Table 1. Check on robustness of the approximation for $1/(1+S) \mid S > 0$ when $\alpha = 1$ and $c = 1$.

β	a	b	I_1	I_2	p	q	p-value
1.5	0.5	0.5	0.465	0.299	0.925	1.066	0.802
1.5	1.5	0.5	0.315	0.153	0.962	2.091	0.118
1.5	1.5	1.5	0.219	0.080	0.938	3.349	0.066
1.5	2	2	0.172	0.052	0.942	4.522	0.242
1.5	5	5	0.075	0.011	0.973	11.945	0.912
2	0.5	0.5	0.438	0.275	0.864	1.110	0.004
2	1.5	0.5	0.301	0.142	0.926	2.153	0.792
2	1.5	1.5	0.196	0.067	0.885	3.626	0.425
2	2	2	0.153	0.042	0.893	4.962	0.779
2	5	5	0.065	0.008	0.943	13.645	0.312
3	0.5	0.5	0.399	0.240	0.774	1.167	0.515
3	1.5	0.5	0.278	0.127	0.862	2.234	0.286
3	1.5	1.5	0.165	0.051	0.801	4.064	0.109
3	2	2	0.126	0.030	0.819	5.704	0.070
3	5	5	0.051	0.005	0.894	16.775	0.204
5	0.5	0.5	0.348	0.200	0.657	1.229	0.096
5	1.5	0.5	0.248	0.107	0.766	2.320	0.108
5	1.5	1.5	0.128	0.034	0.669	4.557	0.003
5	2	2	0.094	0.019	0.716	6.868	0.062
5	5	5	0.036	0.003	0.821	22.274	0.639

$$\times \frac{E\left(\frac{1}{1 \pm S} \,\middle|\, S \gtrless 0\right) - E\left(\frac{1}{(1 \pm S)^2} \,\middle|\, S \gtrless 0\right)}{E\left(\frac{1}{(1 \pm S)^2} \,\middle|\, S \gtrless 0\right) - E^2\left(\frac{1}{1 \pm S} \,\middle|\, S \gtrless 0\right)}$$

and

$$q = \left\{1 - E\left(\frac{1}{1 \pm S} \,\middle|\, S \gtrless 0\right)\right\} \times \frac{E\left(\frac{1}{1 \pm S} \,\middle|\, S \gtrless 0\right) - E\left(\frac{1}{(1 \pm S)^2} \,\middle|\, S \gtrless 0\right)}{E\left(\frac{1}{(1 \pm S)^2} \,\middle|\, S \gtrless 0\right) - E^2\left(\frac{1}{1 \pm S} \,\middle|\, S \gtrless 0\right)}. \tag{4.3}$$

Note that the expressions given by (4.2) and (4.3) satisfy $p > 0$ and $q > 0$. The two moments $E(1/(1 \pm S) \mid S \gtrless 0)$ and $E(1/(1 \pm S)^2 \mid S \gtrless 0)$ can be computed numerically by evaluating the integrals

$$I_1 = \begin{cases} \displaystyle\int_0^{\infty} \frac{f_{S|S>0}(s)}{1+s} ds, & \text{when} \quad S > 0, \\ \displaystyle\int_{-\infty}^{0} \frac{f_{S|S<0}(s)}{1-s} ds, & \text{when} \quad S < 0 \end{cases}$$

and

Table 2. Check on robustness of the approximation for $1/(1+S) \mid S>0$ when $\alpha=-1$ and $c=1$.

b	a	β	I_1	I_2	p	q	p-value
1.5	0.5	0.5	0.548	0.378	1.203	0.992	0.719
1.5	0.5	2	0.307	0.156	0.755	1.704	0.148
1.5	1.5	0.5	0.572	0.406	1.196	0.895	0.545
1.5	1.5	2	0.329	0.176	0.741	1.512	0.489
2	0.5	0.5	0.494	0.319	1.158	1.185	0.837
2	0.5	2	0.250	0.110	0.739	2.218	0.133
2	1.5	0.5	0.533	0.364	1.142	1.000	0.459
2	1.5	2	0.280	0.136	0.702	1.806	0.007
3	0.5	0.5	0.412	0.236	1.098	1.567	0.460
3	0.5	2	0.176	0.059	0.737	3.449	0.005
3	1.5	0.5	0.470	0.297	1.074	1.211	0.831
3	1.5	2	0.207	0.081	0.668	2.566	0.054
4	0.5	0.5	0.351	0.180	1.060	1.959	0.891
4	0.5	2	0.133	0.035	0.767	5.011	0.001
4	1.5	0.5	0.416	0.243	1.015	1.426	0.984
4	1.5	2	0.156	0.050	0.663	3.575	0.001
5	0.5	0.5	0.305	0.141	1.034	2.359	0.190
5	0.5	2	0.105	0.022	0.800	6.787	0.033
5	1.5	0.5	0.370	0.201	0.967	1.649	0.559
5	1.5	2	0.122	0.031	0.687	4.944	0.000

$$I_2 = \begin{cases} \displaystyle\int_0^{\infty} \frac{f_{S|S>0}(s)}{(1+s)^2} ds, & \text{when} \quad S>0, \\ \displaystyle\int_{-\infty}^{0} \frac{f_{S|S<0}(s)}{(1-s)^2} ds, & \text{when} \quad S<0 \end{cases}$$

for given values of the parameters a, b, c, α and β, where $f_{S|S>0}(s)$ and $f_{S|S<0}(s)$ are given by (2.1), (2.4) and (2.5). Evaluating these integrals clearly requires computation of the Gauss hypergeometric function. We used the function hypergeom ([·, ·], [·], ·) in MAPLE.

There are three types of approximations that we would like to check the robustness of: approximation of $1/(1+S) \mid S>0$ by the beta pdf when $\alpha>0$ and $\beta>0$ (approximation 1); approximation of $1/(1+S) \mid S>0$ by the beta pdf when $\alpha<0$ and $\beta>0$ (approximation 2); and, approximation of $1/(1-S) \mid S<0$ by the beta pdf when $\alpha<0$ and $\beta>0$ (approximation 3). The robustness of these three approximations was checked for a range of values of (a,b,c,α,β). The following procedure based on simulation and the Kolmogorov-Smirnov test was used:

(i) for given (a,b,c,α,β), compute the integrals (I_1,I_2) with $f_{S|S>0}$ given by (2.1) for approximation 1, $f_{S|S>0}$ given by (2.4) for approximation 2, and $f_{S|S<0}$ given by (2.5) for approximation 3.

(ii) Obtain the corresponding estimates for (p,q) using (4.2) and (4.3).

(iii) Simulate 1000 random numbers of $S_i=\alpha X_i+\beta Y_i$ by simulating (X_i,Y_i)

Table 3. Check on robustness of the approximation for $1/(1-S) \mid S < 0$ when $\beta = 1$ and $c = 1$.

a	b	α	I_1	I_2	p	q	p-value
1.5	0.5	−0.5	0.548	0.378	1.200	0.989	0.244
1.5	0.5	−2	0.307	0.156	0.756	1.703	0.102
1.5	1.5	−0.5	0.572	0.406	1.187	0.889	0.777
1.5	1.5	−2	0.328	0.176	0.738	1.510	0.097
2	0.5	−0.5	0.494	0.319	1.161	1.189	0.929
2	0.5	−2	0.250	0.110	0.740	2.215	0.020
2	1.5	−0.5	0.534	0.364	1.145	0.999	0.355
2	1.5	−2	0.280	0.136	0.705	1.810	0.007
3	0.5	−0.5	0.412	0.236	1.099	1.568	0.093
3	0.5	−2	0.176	0.059	0.743	3.480	0.071
3	1.5	−0.5	0.470	0.297	1.078	1.214	0.295
3	1.5	−2	0.207	0.082	0.666	2.552	0.005
4	0.5	−0.5	0.351	0.180	1.058	1.955	0.873
4	0.5	−2	0.133	0.035	0.764	4.987	0.023
4	1.5	−0.5	0.416	0.243	1.014	1.425	0.995
4	1.5	−2	0.156	0.050	0.664	3.588	0.001
5	0.5	−0.5	0.304	0.141	1.036	2.368	0.504
5	0.5	−2	0.105	0.022	0.797	6.764	0.001
5	1.5	−0.5	0.370	0.201	0.970	1.655	0.474
5	1.5	−2	0.122	0.031	0.688	4.947	0.000

from (1.1) for which methods are widely available. Note that in practice it will be unusual to have a sample size greater than 1000.

(iv) Calculate the sample of values $\{1/(1+S_i)\}$ for approximation 1, $\{1/(1+S_i)$ with $S_i > 0\}$ for approximation 2 and $\{1/(1-S_i)$ with $S_i < 0\}$ for approximation 3.

(v) Perform the Kolmogorov-Smirnov test to see whether the sample in step 4 arises from the beta distribution with parameters p and q calculated in step 2. The p-values of this test for the three approximations are given in Tables 1–3.

Table 1 shows that approximation 1 is robust for most of the parameter choices. However, there is some evidence that the robustness is weak when both a and b are small (see lines 3, 6 and 18 of Table 1). Table 2 shows that approximation 2 is less robust when β is away from 0 (see lines 8, 10, 12, 14, 16, 18 and 20 of Table 2). Table 3 shows that approximation 3 is less robust when α is away to 0 (see lines 6, 8, 10, 12, 14, 16, 18 and 20 of Table 3). In general, it appears that approximations 2 and 3 are weaker than that provided by approximation 1.

We hope that the approximations presented will be useful—especially to the practitioners of the inverted Dirichlet distribution—since they avoid the use of the Gauss hypergeometric function and since the beta distribution is widely accessible in standard statistical packages.

5. Conclusions

We have derived the distribution of $S = \alpha X + \beta Y$ and the corresponding moment properties when (X, Y) has the inverted Dirichlet distribution. The formulas involve the Gauss hypergeometric function. We have proposed approximating the distribution of $1/(1 \pm S) \mid S \gtrless 0$ by the standard beta distribution. This monotonic transformation is chosen over others because of its popularity, simplicity and its association to the derived distribution. We have discussed evidence for the robustness of the approximation based on the powerful Kolmogorov-Smirnov test.

It would be of interest to extend the work of this paper for other less known bivariate beta distributions. These may include those proposed by Mihram and Hultquist (1967), Connor and Mosimann (1969), Lee (1981), Gupta and Wong (1985), and Olkin and Liu (2003). We hope to provide this treatment in a future paper.

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