A Solvable Model of "Shear Dispersion"

W. C. Thacker

Atlantic Oceanographic and Meteorological Laboratories, NOAA, Miami, Fla. 33149

(Manuscript received 14 April 1975, in revised form 22 July 1975)

ABSTRACT

Shear dispersion results from vertical shear of horizontal velocity and vertical mixing, features which cannot be included explicitly in one-layer, vertically integrated models. The parametric description of shear dispersion as effective horizontal diffusion in one-layer models is investigated by comparing analytic solutions of two-layer dispersion equations to the corresponding solutions of a one-layer diffusion equation. The diffusion description is found to be poor for times comparable with or shorter than the vertical mixing time but excellent for longer times.

1. Introduction

This discussion of "shear dispersion" stems from a proposed study of the environmental effects of mining for gravel in Massachusetts Bay. Such a study would be directed toward estimating the extent of the dispersion of the silt introduced into the water by the mining operation. It is helpful to think of dispersion as being due to two distinct processes, advection and diffusion. If a numerical model were used to assess the extent of the dispersion, a numerical model of the flow would be used to provide the advection, and the diffusion would be eddy diffusion resulting from motion on length scales too small to be resolved on a numerical grid. Although eddy diffusion is actually advection on a scale too fine to resolve, it can be treated as diffusion when dealing with effects on a larger scale. In such a calculation, simplicity and convenience might dictate the use of a one-layer or vertically integrated model, in which case there is no vertical resolution at all. Just as subgrid-scale effects result in eddy diffusion, ignored variations in the vertical are responsible for horizontal dispersion. The combination of vertical shear and vertical diffusion yields a net horizontal dispersion in addition to horizontal eddy diffusion. The calculations presented here are the result of an effort to understand in a simple way the nature of this shear dispersion, and its relationship to diffusion.

Bowden (1965) suggested that this shear effect is like an effective horizontal diffusion. He argued that, for concentrations of contaminant which are steady in time and which have the same horizontal variations at each vertical level, the horizontal transport of the contaminant due to the shear effect could be attributed to an effective diffusion process characterized by a coefficient that is inversely proportional to the coefficient of vertical eddy diffusion and directly proportional to the square of the velocity shear. Earlier studies by Elder (1959) of turbulent flow in an open channel and by Taylor (1954) of turbulent flow through a pipe led to similar conclusions.

These results suggest that one possibility for incorporating the shear effect into one-layer calculations is by enhancing the horizontal diffusion coefficient along the direction of the shear. This simple modification requires, as additional inputs, the shear field and its associated eddy viscosities as functions of time. Perhaps, with a few observations and some simple assumptions, they might be related to the mean flow. Then the amount of enhancement might be estimated from the flow conditions. But the question still remains to what extent shear dispersion acts like diffusion.

Okubo (1967) discusses the spreading of an initially localized concentration of contaminant by a linear velocity shear. For times large compared to the vertical mixing time, he finds that the width of the contaminated region increases as $t^{\frac{3}{2}}$, i.e., as the square root of the elapsed time. This is the same time-dependence as given by a diffusion process, and his diffusion coefficient is exactly that which would be given by Bowden's argument. Okubo also considers the case of infinite depth, which corresponds to the short time limit, before the pollutant has spread far enough vertically to be affected by the presence of the surface and bottom boundaries. In this case he finds that the width of the contaminated region increases as $t^{\frac{3}{2}}$, or slower than for a diffusion process.

A very simple argument by Okubo and Carter (1966) in which the vertical diffusion and shear act sequentially also supports diffusion-like dispersion for long times, but assumes a linear time dependence for the spreading for short times. Thus, it should be expected that enhanced diffusion should give a good

approximation to shear dispersion a long time after the pollutant is introduced, but it is not clear that the approximation is valid for the initial interval of a few vertical diffusion times. For highly stratified flows, the vertical diffusion time might be several days. Even for this extreme case, enhanced diffusion should be adequate for calculating the dispersion after several weeks. But, before using such a calculation to obtain the dispersion after a few days, it would be useful to know that the approximation is adequate or, if it is not, to have some other approximation that is better.

In order to get a better idea of the nature of shear dispersion, I have chosen to study a simple two-layer model similar to that used by Schönfeld (1960) to study diffusion in tidal rivers. The model flow consists of two equally thick layers of fluid moving relative to each other, and the vertical exchange proceeds at a rate proportional to the difference in the concentrations of the layers. Because of the simplicity of this model, it is possible to solve analytically for the vertical mean of the contaminant distribution as a function of time. The equation governing the dispersion is a damped-wave equation, not a diffusion equation, but it approaches a diffusion equation for relatively steady conditions. Solutions are obtained for several different cases corresponding to different choices for initial conditions and for sources of pollution. In each case the solution approaches the corresponding diffusion solution in the long time limit, as expected. Comparisons between the two-layer model solutions and the diffusion solutions reveal that they are quite different after only one vertical diffusion time measured from the first introduction of the contaminant but that they are quite similar after ten vertical diffusion times.

In Section 3, the two-layer model is extended to include the effects of horizontal eddy diffusion. Again analytical solutions are obtained. After many vertical diffusion times, these solutions also approach diffusion solutions with a coefficient that is the sum of the horizontal eddy diffusion coefficient and the shear diffusion coefficient which would be used if there were no horizontal eddy diffusion. This is in agreement with Okubo's (1967) results that the second moment of the contaminant distribution is the sum of a term representing eddy diffusion and another representing the shear effect.

2. The two-layer model

The two equations

$$\left.\begin{array}{c}\dfrac{\partial C_1}{\partial t}+u\dfrac{\partial C_1}{\partial x}=-\alpha(C_1-C_2)+S_1 \\[4mm] \dfrac{\partial C_2}{\partial t}-u\dfrac{\partial C_2}{\partial x}=-\alpha(C_2-C_1)+S_2\end{array}\right\} \quad (1)$$

contain the essence of the shear effect, vertical shear and vertical mixing. The shear flow is idealized as two equally thick layers flowing in opposite directions with velocities $\pm u$. Contaminant is mixed vertically at a rate proportional to the difference in the concentrations C_1 and C_2 of contaminant in each layer. The coefficient α governs the rate of vertical exchange. Continuous sources of contaminant are represented by S_1 and S_2.

These equations have the virtue that they are simple enough to be solved analytically. It is clear that no real flow is quite so two-layered. The two layers should be thought of as giving the minimal amount of vertical resolution necessary to discuss vertical shear and vertical mixing. Additional simplicity stems from the fact that it is possible to ignore variations in the horizontal direction perpendicular to the direction of the shear, because the shear effect does not contribute to mixing in that direction. Another simplification is the result of ignoring a mean flow $\bar{u}$ in addition to the shear flow. This complication is not difficult to handle and is incorporated when discussing continuous sources of contaminant. The more difficult complication due to incorporating horizontal diffusion is discussed in Section 3. If the coefficients u and α are allowed to vary with x and t, Eqs. (1) would be much more difficult to solve than if they were constrained to be constant. Thus, u and α are constant both in space and time.

It is the vertical mean concentration $\bar{C}=\frac{1}{2}(C_1+C_2)$ that should be simulated in a one-layer model. Eqs. (1) can be added and subtracted, giving two equivalent equations in $\bar{C}$ and $\Delta C=\frac{1}{2}(C_1-C_2)$:

$$\left.\begin{array}{c}\dfrac{\partial \bar{C}}{\partial t}+u\dfrac{\partial \Delta C}{\partial x}=\bar{S} \\[4mm] \dfrac{\partial \Delta C}{\partial t}+u\dfrac{\partial \bar{C}}{\partial x}=-2\alpha\Delta C+\Delta S\end{array}\right\}, \quad (2)$$

where $\bar{S}=\frac{1}{2}(S_1+S_2)$ and $\Delta S=\frac{1}{2}(S_1-S_2)$. These equations can be solved for $\bar{C}(x,t)$ resulting from arbitrary initial distributions, $C_1(x, t=0)$ and $C_2(x, t=0)$, and from arbitrary distributions of contaminant sources, $S_1(x,t)$ and $S_2(x,t)$. Several different cases, corresponding to different choices for initial conditions and sources are discussed below and comparisons are made to corresponding solutions of a one-layer effective diffusion equation [see Eq. (4) and Figs. 1–8].

By eliminating ΔC from (2), a higher order equation for $\bar{C}$ can be obtained:

$$\dfrac{\partial^2 \bar{C}}{\partial t^2}+2\alpha\dfrac{\partial \bar{C}}{\partial t}-u^2\dfrac{\partial^2 \bar{C}}{\partial x^2}=\dfrac{\partial \bar{S}}{\partial t}+2\alpha\bar{S}-u\dfrac{\partial \Delta S}{\partial x}. \quad (3)$$

This is a damped-wave equation, not a diffusion equation. If there is no vertical mixing, $\alpha=0$, and

(3) becomes the usual wave equation. In that case, the solutions which are functions of $x \pm ut$ correspond to differential advection by the two layers. If the concentration $\bar{C}$ is sufficiently steady, so that $\partial^2 \bar{C}/\partial t^2 \ll 2\alpha(\partial \bar{C}/\partial t)$, then (3) becomes a diffusion equation

$$\left.\begin{array}{l} \dfrac{\partial \bar{C}}{\partial t} - K_H{}^* \dfrac{\partial^2 \bar{C}}{\partial x^2} = \bar{S} \\[2ex] K_H{}^* = \dfrac{u^2}{2\alpha} \end{array}\right\}, \qquad (4)$$

with the same coefficient of effective diffusion, $K_H{}^*$, as given by Bowden's (1965) method of analysis. Two initial conditions are needed to solve Eq. (3) $\{\bar{C}(x,t=0)$ and $\partial \bar{C}(x,t=0)/\partial t = -u[\partial \Delta C(x,t=0)/\partial x]\}$, but only one $[\bar{C}(x,t=0)]$ for (4). Since (4) cannot resolve any vertical variations, it cannot use the additional information about initial vertical asymetry. Likewise, for sufficiently steady conditions, no information concerning the vertical distribution of sources ΔS enters Eq. (4).

This two-layer model can be considered as an extrapolation back from asymptotically large times to intermediate times. It cannot describe the dispersion in its initial stage, before the contaminant has become relatively uniform throughout the water column. For this, greater vertical resolution is needed. By including more layers, a generalization to (3) can be constructed which contains higher time derivatives and more information about shorter times. The one-layer model should agree for asymptotically large times, but disagree for intermediate times since it contains less information about the initial stages of dispersion. Thus, by comparing solutions of (3), or equivalently (2), to solutions of (4), the two-layer model can be used to determine the limits of validity of the effective horizontal diffusion approximation.

It is convenient to introduce the matrix notation

$$\mathbf{C} = \begin{pmatrix} \bar{C} \\ \Delta C \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \bar{S} \\ \Delta S \end{pmatrix}, \quad \mathbf{G} = \begin{bmatrix} \dfrac{\partial}{\partial t} + 2\alpha & -u\dfrac{\partial}{\partial x} \\[2ex] -u\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial t} \end{bmatrix} G, \quad (5)$$

so that the solutions to (2) can be expressed compactly in terms of a Green's function G for arbitrary initial conditions and sources as

$$\mathbf{C}(x,t) = \int_{-\infty}^{+\infty} dx' \int_0^t dt' \mathbf{G}(x-x', t-t') \mathbf{S}(x',t')$$

$$+ \int_{-\infty}^{+\infty} dx' \mathbf{G}(x-x',t) \mathbf{C}(x',0). \quad (6)$$

The first term expresses the cumulated effects of the sources and the second, the effect of the initial distribution of contaminant. Both depend upon the matrix Green's function $\mathbf{G}$, which, in turn, depends upon the scalar Green's function G which is the solution to

$$\left(\frac{\partial^2}{\partial t^2} + 2\alpha \frac{\partial}{\partial t} - u^2 \frac{\partial^2}{\partial x^2} \right) G(x,t) = \delta(x)\delta(t), \qquad (7)$$

satisfying the casuality condition $G=0$ for $t<0$.

The scalar Green's function G can be found using Fourier and Laplace transform techniques. By writing

$$G(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} g(k,\omega) e^{i(kx-\omega t)}, \qquad (8)$$

and using an integral representation for the Dirac delta functions,

$$\delta(x)\delta(t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i(kx-\omega t)}, \qquad (9)$$

Eq. (7) can be transformed into

$$\left.\begin{array}{l} g(k,\omega) = -[(\omega+i\alpha-\Omega)(\omega+i\alpha+\Omega)]^{-1} \\[1ex] \Omega \quad = (u^2 k^2 - \alpha^2)^{\frac{1}{2}} \end{array}\right\}. \qquad (10)$$

For $t>0$, the ω contour can be closed in the lower half-plane and ω integration can be evaluated by residues, giving

$$G(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\sin \Omega t}{\Omega} e^{ikx} e^{-\alpha t}. \qquad (11)$$

With the aid of a table of Fourier transforms (Selby, 1971), the k integration can be evaluated in terms of the zeroth-order modified Bessel function, yielding

$$\left.\begin{array}{l} G(x,t) = \dfrac{1}{2u} e^{-\alpha t} I_0\{\alpha t[1-(x/ut)^2]^{\frac{1}{2}}\} \\[2ex] \qquad\qquad \times [\theta(x+ut) - \theta(x-ut)]\theta(t) \\[2ex] \theta(y) = \begin{cases} 1, & y>0 \\ 0, & y<0 \end{cases} \end{array}\right\}. \qquad (12)$$

The unit step function θ serves to indicate that there is no contamination outside of the region $|x| < ut$ or before $t=0$ and allows $\mathbf{G}$ to be evaluated from (5) with the aid of the identity $d\theta/dy = \delta(y)$.

Having the expressions (12) for $G(x,t)$ and (5) for $\mathbf{G}(x,t)$, $\mathbf{C}(x,t)$ can be calculated for arbitrary initial conditions and source terms according to (6). Although it is quite straightforward to derive (6), it is somewhat lengthy so the details are omitted here. A general discussion of the procedure to express the solution of partial differential equations in terms of a Green's function is given in Morse and Feshbach (1953). Using expression (6) along with (12) and (5), several
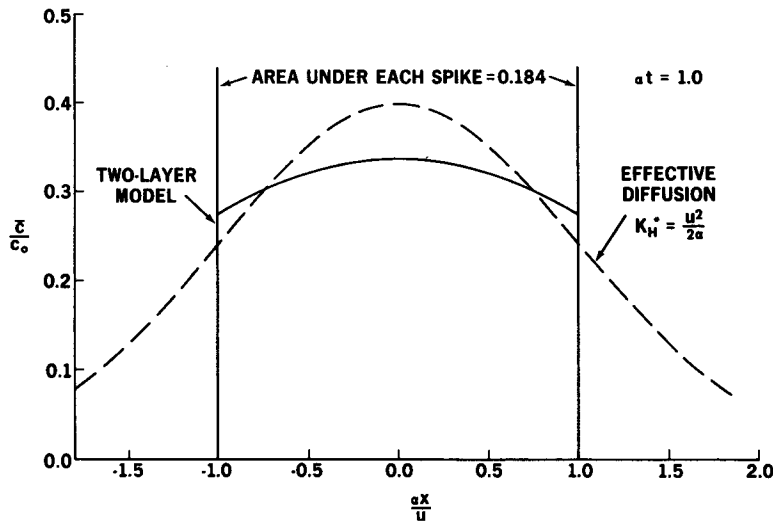
FIG. 1. *Case 1.* One-layer effective diffusion (dashed line) poorly approximates two-layer shear dispersion (solid line) after only one vertical exchange time. These distributions result from horizontally localized, vertically uniform initial distributions. The two-layer distribution is confined to the region $[\alpha x/u] \leqslant 1.0$, with spikes representing the delta function terms. The Gaussian diffusion distribution lacks these sharp features.

cases will be considered, each corresponding to a different choice for initial conditions and sources. Each will be compared to the corresponding solution of the diffusion equation (4) describing shear dispersion as effective diffusion. The solutions obtained from (6) will be referred to as the two-layer model solutions and those from (4) as the effective diffusion solutions.

a. Case 1

We first consider the case of an initially localized distribution with equal quantities of contaminant in each layer and no continuous sources of contaminant. There are two equivalent ways of describing this situation using Eq. (6): with initial conditions $\bar{C}(x,t=0) = C_0\delta(x)$ and $\Delta C(x,t=0) = 0$ and no sources $[\bar{S} = \Delta S = 0]$, or with instantaneous sources $\bar{S} = C_0\delta(x)\delta(t)$ and $\Delta S = 0$ and initially clean water $[\bar{C}(x,t=0) = \Delta C(x,t=0) = 0]$. Both ways yield the

same answer, as they must. Using either choice for $\bar{C}$, ΔC, $\bar{S}$ and ΔS in (6), along with (5) and (12), the vertically averaged distribution of contaminant is given by

$$
\begin{aligned}
\bar{C}(x,t) &= C_0\left(\frac{\partial}{\partial t}+2\alpha\right)G(x,t) \\[6pt]
&= C_0 e^{-\alpha t}\left\{\frac{\alpha}{2u}\left[I_0(z)+\frac{\alpha l}{z}I_1(z)\right]\right. \\[4pt]
&\qquad \times[\theta(x+ut)-\theta(x-ut)] \\[4pt]
&\qquad \left. +\tfrac{1}{2}[\delta(x+ut)+\delta(x-ut)]\right\}\theta(t) \\[6pt]
z &= \left[(\alpha t)^2-\left(\frac{\alpha x}{u}\right)^2\right]^{\frac{1}{2}}
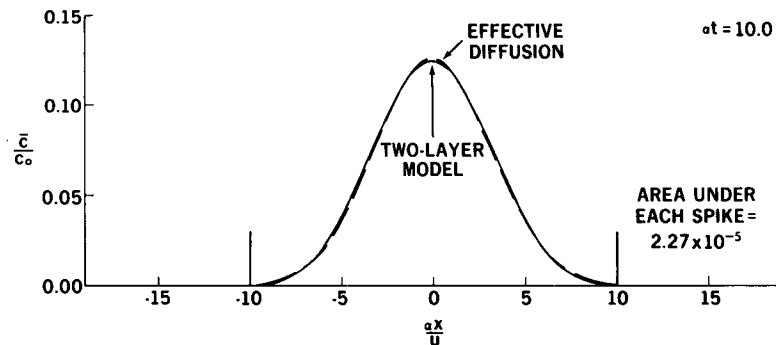\end{aligned}
\quad\right\} \quad (13)
$$

FIG. 2. *Case 1.* After ten exchange times, the one-layer diffusion distribution agrees quite well with the two-layer shear dispersion distribution. The spikes and fronts are negligible. Compare with Fig. 1.

For small t, this distribution reduces to $\bar{C}(x,t)$ $=\frac{1}{2}C_0[\delta(x+ut)+\delta(x-ut)]$, which corresponds to the contaminant remaining localized in each layer, but moving to the right in one and to the left in the other. No significant amount of contaminant is mixed between the two layers for $\alpha t \ll 1$.

As these localized quantities of contaminant move, they decay exponentially in time by transferring some of the contamination into the other layer where it is fed back toward the origin. This process can be visualized by imagining two thin strings of dye, one in each layer, being advected away from each other, dissolving as they go and leaving a trail in the other layer. The trails leave trails of their own, with the result that between the two decaying dye strings there is a diffuse dye distribution. This diffuse distribution is given by the term involving the modified Bessel functions I_0 and I_1 in (13). The factor formed from the difference of the step functions limits the diffuse region to $|x| < ut$.

The corresponding solution to the diffusion equation (4),

$$\bar{C}(x,t) = \frac{C_0}{[4\pi(u^2/2\alpha)t]^{\frac{1}{2}}} \exp\left[-\frac{x^2}{4(u^2/2\alpha)t}\right], \quad (14)$$

does not have sharp fronts nor localized quantities of contaminant. Nevertheless, for large t, the two-layer model distribution (13) does approach this effective horizontal diffusion solution. [For limits of I_0 and I_1, see Abramowitz and Stegun (1970).] Figs. 1 and 2 present a comparison between the two-layer model distribution (13) and the one-layer effective diffusion distribution (14) for $\alpha t = 1$ and $\alpha t = 10$. Although diffusion poorly approximates the two-layer shear dispersion after only one vertical exchange time (see Fig. 1), the approximation is quite good after ten exchange times (Fig. 2). For $\alpha t = 10$, the amount of contaminant that is still localized, the discontinuity across the fronts, and the Gaussian tails are all negligible.

By comparing (3) and (4), it can be seen that the corresponding solutions (13) and (14) should agree if the second time derivative in (3) is unimportant. In other words, (14) is a non-stationary time average of (13) with frequencies higher than $O(\alpha)$ filtered out. Equivalently, (14) should be obtainable from (13) by a corresponding spatial average. The connection between the two methods of averaging can be found in the relationship between diffusion length and time scales. The spatial average filters out spatial variations on length scales smaller than $O(L)$, where $L^2 = 2K_H^* \alpha^{-1} = u^2/\alpha^2$. Thus, the lack of sharp fronts in (14) can be thought of as resulting from a spatial averaging of (13) which smooths over these sharp features.

b. Case 2

We now consider an initially localized distribution with all of the contaminant in one layer and, as before,

no continuous sources. Again, this could be described either with an instantaneous source adding contaminant into initially clean water at $t = 0$ using $\tilde{S}(x,t) = \Delta S(x,t) = C_0\delta(x)\delta(t)$ and $\bar{C}(x,t=0) = \Delta C(x,t=0) = 0$, or with the initial contamination in the initial conditions using $\bar{C}(x,t=0) = \Delta C(x,t=0) = C_0\delta(x)$ and $\tilde{S}(x,t) = \Delta S(x,t) = 0$. For definiteness, suppose that the contaminant is initially in the upper layer and the upper layer is moving to the right. If it were in the lower, left-moving layer, the sign of ΔC or ΔS would change. For this case, the vertically averaged distribution given by (6) using (5) and (12) is

$$
\left.
\begin{aligned}
\bar{C}(x,t) &= C_0\left(\frac{\partial}{\partial t} + 2\alpha - u\frac{\partial}{\partial x}\right)G(x,t) \\
&= C_0 e^{-\alpha t}\left\{\frac{\alpha}{2u}\left[I_0(z) + \frac{\alpha t + (\alpha x/u)}{z}I_1(z)\right]\right. \\
&\quad \times [\theta(x+ut) - \theta(x-ut)] + \delta(x-ut)\Big\}\theta(t) \\
z &= \left[(\alpha t)^2 - \left(\frac{\alpha x}{u}\right)^2\right]^{\frac{1}{2}}
\end{aligned}
\right\}. \quad (15)
$$

Since the contaminant was initially in only one layer, the solution is no longer symmetric about $x = 0$. There is only one delta function term corresponding to the localized residue of the initial distribution moving to the right in the upper layer. The diffuse distribution is also not symmetric due to the initial motion to the right, but, as in case 1, it is limited to the region, $|x| < ut$.

The diffusion equation (4) cannot incorporate the vertical structure contained in the initial conditions for this case. Thus, the corresponding solution to (4) for this case should be the same as that corresponding to case 1, i.e., that given by Eq. (14). The limit of the expression given in (15) for $\alpha t \gg 1$ is exactly that given in (14), just as expected. Comparisons between the two-layer distribution and the one-layer effective diffusion distribution for this case are shown, for $\alpha t = 1$ and $\alpha t = 10$, in Figs. 3 and 4. As for case 1, the agreement is not good for $\alpha t = 1$. For $\alpha t = 10$, the agreement is better, but not as good as for case 1, since the centers of the two distributions are not in the same place. The position of the center of the two-layer distribution is given by the first moment, $\bar{x} = (u/\alpha)e^{-\alpha t}\sinh\alpha t$. Thus, the center of the distribution moves ultimately as far as $u/(2\alpha)$, which is essentially the location of the center for $\alpha t = 10$ in Fig. 4.

c. Case 3

Here we consider the case of a steady continuous source discharging the quantity C_0 of contaminant in one vertical mixing time into previously clean
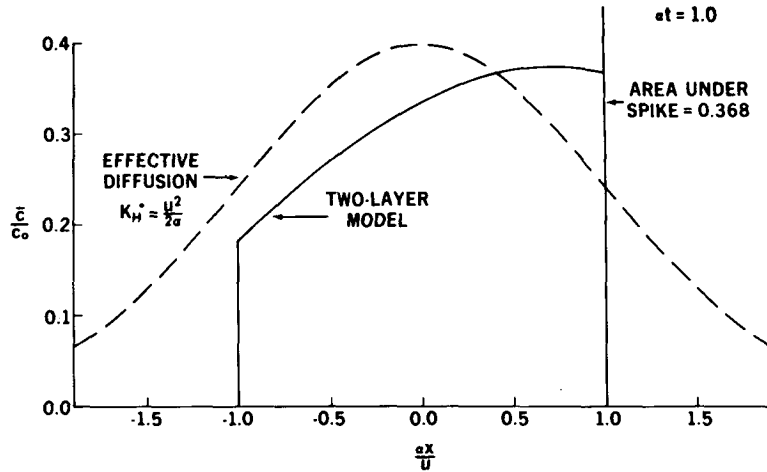
FIG. 3. *Case 2.* An initial distribution with all of the contaminant in the upper, right-moving layer leads to short time horizontal asymmetry. The one-layer diffusion approximation is the same as in Fig. 1, since the vertical structure must be ignored.

water, i.e., $\bar{S}(x,t)=\alpha C_0\delta(x)\theta(t)$, $\Delta S(x,t)=\bar{C}(x,t=0)$ $=\Delta C(x,t=0)=0$. As in case 1, the contaminant is added equally to both layers, so the resulting distribution should be symmetric about $x=0$. From (6), this distribution can be expressed as the superposition of distributions from a sequence of instantaneous sources, each discharging the quantity $\alpha C_0 dt'$ of contaminant in a time interval dt'. Thus, the solution for this case can be constructed from the solution for case 1, i.e.,

$$\bar{C}(x,t)=\int_0^t \alpha \bar{C}_I(x',t')dt', \qquad (16)$$

where $\bar{C}_I$ is given by the right-hand side of (13). With continuous sources it is necessary to be careful if there is a mean flow $\bar{u}$ relative to the source. For this reason the integrand of (16) depends upon x' $=x-\bar{u}t$ and not upon x. For this case, however, let $\bar{u}=0$, and consider $\bar{u}\neq0$ in case 4.

The delta function parts of (13) contribute spatially decaying exponentials to (16) which dominate for short times. There are no localized parts to the dis-

tribution unless $\bar{u}=\pm u$, in which case one of the layers is not moving and contaminant can accumulate at $x=0$ in the layer. However, there are sharp fronts as in cases 1 and 2 with the contaminant restricted to the region $\min[0, (\bar{u}-u)t]<x<\max[0, (\bar{u}+u)t]$ as expected.

If $\bar{C}_I$ in (16) is given by (14), then (16) represents the corresponding solution of the diffusion equation with a continuous source. Again, the two-layer distribution and the one-layer diffusion distribution have the same limit for $\alpha t\gg1$. These two distributions are obtained numerically from (16) using the right-hand sides of (13) and (14) for $\bar{C}_I$. They are compared for $\alpha t=1$ and $\alpha t=10$ in Figs. 5 and 6. For $\alpha t=1$, the sharp fronts are evident, as are the spatially decaying exponentials. For $\alpha t=10$, the fronts are insignificant, and the agreement is quite good, being worst near the source at $x=0$.

d. Case 4

This case is exactly the same as case 3 except that $\bar{u}=\frac{3}{2}u$ rather than $\bar{u}=0$. Again the contaminant is added equally to both layers, but in this case that
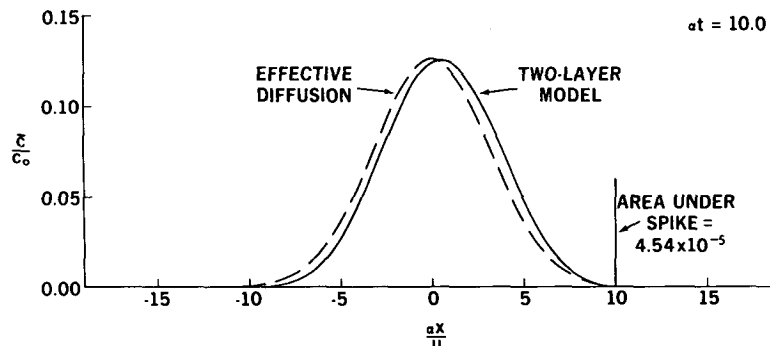

FIG. 4. *Case 2.* After ten vertical exchange times, the horizontal asymmetry is small and the diffusion approximation is good. Compare with Fig. 3 and with Fig. 2.
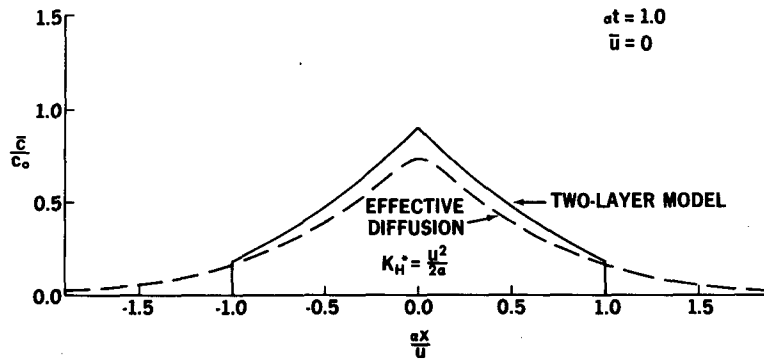
FIG. 5. *Case 3.* For horizontally localized, vertically uniform sources discharging contaminant uniformly for one exchange time, the agreement is poor. Compare with Fig. 1. The two-layer distribution (solid line) is again limited, $[\alpha x/u] < 1.0$. In this case there are no spikes, but exponential decay away from the origin.

does not insure a symmetrical distribution about $x = 0$. The mean flow washes the contaminant downstream to the right. Since in this case $\bar{u} > u$, none of the contaminant can enter the region to the left of the origin. The one-layer diffusion equation does allow upstream diffusion. As in the three previous cases, this difference between the two-layer distribution and the diffusion distribution becomes negligible for $\alpha t \gg 1$.

Figs. 7 and 8 compare the two-layer distribution to the diffusion distribution for this case for $\alpha t = 1$ and $\alpha t = 10$. For $\alpha t = 1$, the fronts and spatially decaying exponentials are again evident. For $\alpha t = 10$, the agreement is again good except near the source. Again, it is easy to see that the diffusion distribution can be considered to be a spatial average of the two-layer distribution which filters out variations smaller than $O(u/\alpha)$.

In a similar manner, it is possible to use (6) to calculate many more cases. However, these four cases should be sufficient to illustrate the type of results to be expected from the two-layer model and how they should compare with the one-layer effective diffusion results. At this point it is interesting to note that several of the features of shear dispersion illustrated in these examples were anticipated qualitatively by Stommel (1950).

3. Two-layer model with horizontal eddy diffusion

The two-layer model presented in Section 2 describes horizontal dispersion due only to the combined effects of shear and vertical diffusion and ignores the possibility of horizontal eddy diffusion in each layer. This eddy diffusion is a parameterization of the dispersion due to variations in the flow not specifically resolved by the two-layer model. One possible mechanism contributing to this eddy diffusion process is the combination of shear and vertical diffusion within each layer. Another is the turbulent mixing due to seemingly random fluctuations which occur on smaller spatial and temporal scales than the two-layer model can describe. For the purpose of this study, both can be lumped into one, called horizontal eddy diffusion, and characterized by one parameter K_H. This parameter is not to be confused with the parameter $K_H^* = u^2/2\alpha$ of Eq. (4) which represents the dispersion due to the shear effect; K_H represents the dispersion which would occur in the absence of shear between the two layers.

It is not difficult to modify the two-layer model of Section 2 to incorporate horizontal eddy diffusion. This is done simply by replacing the operator $\partial/\partial t$
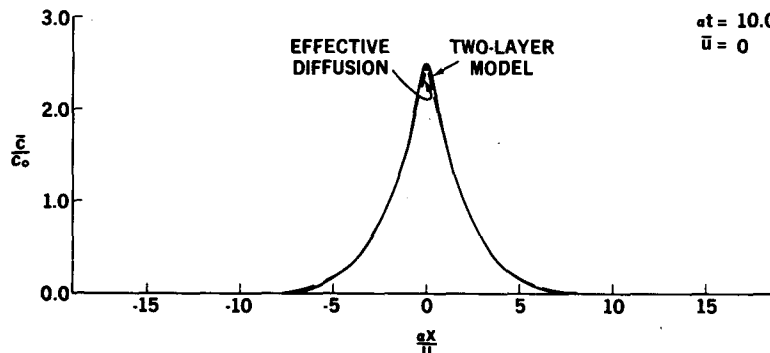

FIG. 6. *Case 3.* After ten exchange times the one-layer diffusion distribution for continuous sources is a good approximation to the two-layer distribution.
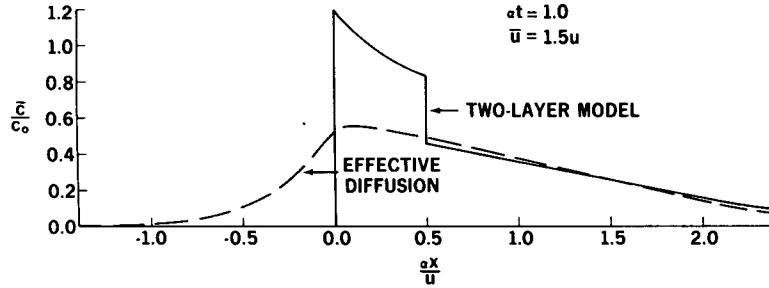
FIG. 7. *Case 4.* When there is a mean flow relative to the source, a horizontal asymmetry results. Compare with Fig. 5. Again, after one exchange time, the agreement is poor. Since $\bar{u}=1.5\ u>u$, two-layer shear dispersion can have no contaminant spreading upstream, whereas upstream diffusion is possible.

with $\partial/\partial t - K_H(\partial^2/\partial x^2)$. Then Eqs. (1) become

$$\left.\begin{array}{l}\left(\dfrac{\partial}{\partial t}-K_H\dfrac{\partial^2}{\partial x^2}\right)C_1+u\dfrac{\partial C_1}{\partial x}=-\alpha(C_1-C_2)+S_1\\[4mm]\left(\dfrac{\partial}{\partial t}-K_H\dfrac{\partial^2}{\partial x^2}\right)C_2-u\dfrac{\partial C_2}{\partial x}=-\alpha(C_2-C_1)+S_2\end{array}\right\}. \quad (17)$$

The procedure for solution is the same as before. In this case, there is a new Green's function, G', which satisfies

$$\left[\left(\dfrac{\partial}{\partial t}-K_H\dfrac{\partial^2}{\partial x^2}\right)^2+2\alpha\left(\dfrac{\partial}{\partial t}-K_H\dfrac{\partial^2}{\partial x^2}\right)-u^2\dfrac{\partial^2}{\partial x^2}\right]G'(x,t)$$

$$=\delta(x)\delta(t), \quad (18)$$

which can be obtained by Fourier and Laplace transform techniques. If

$$G'(x,t)=\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}g'(k,\omega)e^{i(kx-\omega t)}, \quad (19)$$

then

$$g'(k,\omega)=-\left[(\omega+iK_Hk^2+i\alpha-\Omega)\right.$$
$$\left.\times(\omega+iK_Hk^2+i\alpha-\Omega)\right]^{-1}, \quad (20)$$

where, as before, $\Omega=(u^2k^2-\alpha^2)^{\frac{1}{2}}$. Thus, for $t>0$,

$$G'(x,t)=\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\exp(-k^2K_Ht)\frac{\sin\Omega t}{\Omega}e^{ikx}e^{-\alpha t}. \quad (21)$$

The integrand can be recognized as a product of the Fourier transform of $G(x,t)$, the Green's function given in (11) and (12), and the transform of

$$G_{K_H}(x,t)=\frac{1}{(4\pi K_Ht)^{\frac{1}{2}}}\exp[-x^2/(4K_Ht)]\theta(t), \quad (22)$$

the Green's function for horizontal diffusion alone; thus G' must be the convolution of the Green's functions for the separate processes,

$$G'(x,t)=\int_{-\infty}^{+\infty}dx'G_{K_H}(x-x',t)G(x',t). \quad (23)$$

With the substitution of $\partial/\partial t - K_H(\partial^2/\partial x^2)$ for $\partial/\partial t$ in the matrix Green's function of Eq. (5), the matrix Green's function with diffusion can be reduced to a convolution of the matrix Green's function without diffusion and the Green's function for diffusion alone:

$$\mathbf{G}_{K_H}=\int_{-\infty}^{+\infty}dx'G_{K_H}(x-x',t)\mathbf{G}(x',t). \quad (24)$$

This follows directly from Eq. (23). From this follows the simple expression

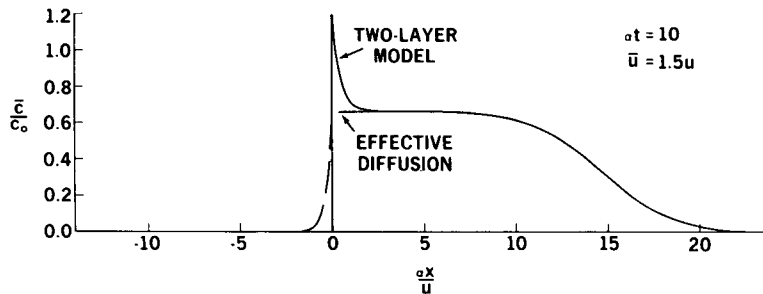$$\bar{C}(x,t)=\int_{-\infty}^{+\infty}dx'G_{K_H}(x-x',t)\bar{C}_{K_H=0}(x',t) \quad (25)$$

FIG. 8. *Case 4.* After ten exchange times, the agreement is excellent, except close to the source. Again, upstream diffusion is possible, but upstream shear dispersion is forbidden. Compare with Fig. 7 and with Fig. 6.
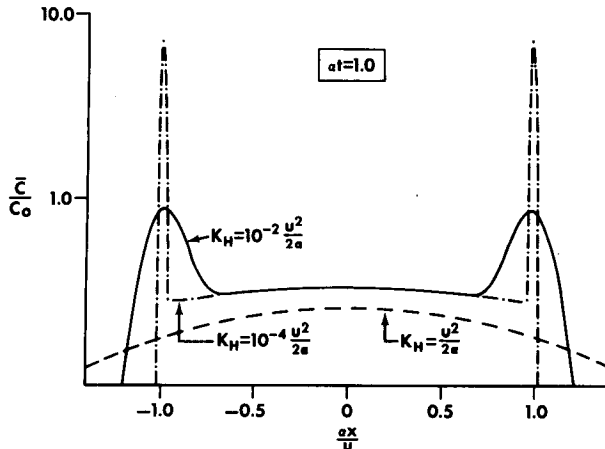
FIG. 9. Distributions after one exchange time are shown for horizontal eddy diffusion within the layers characterized by a coefficient K_H. For small values, $K_H = 10^{-4} \, u^2/2\alpha$ or $K_H = 10^{-2} \, u^2/2\alpha$, the effect of this diffusion is to spread out the spikes and fronts. Compare with Fig. 1, noting that a logarithmic scale is used here. If $K_H = u^2/2\alpha = K_H^*$, there is no indication of the spikes or fronts since the diffusion within the layers is equal in strength to the dispersion due to shear and exchange between the layers.

relating the solutions $\bar{C}(x,t)$ to Eqs. (17) to the corresponding solutions $\bar{C}_{K_H=0}$, to Eqs. (2) which are given by (6). Thus, the solutions to the two-layer model with horizontal eddy diffusion are simply a convolution of the eddy diffusion Green's function given in (22) with the corresponding solutions without eddy diffusion.

It would not be hard to evaluate the integral in (25) for each of the cases discussed in Section 2, each for several values of K_H, and to compare the dispersion due to the shear effect alone with that due to the combination of the shear effect and horizontal diffusion. However, a less detailed discussion is sufficient to show how the two mechanisms work together. From Section 2, we know that $\bar{C}_{K_H=0}$ is essentially Gaussian for $\alpha t \gg 1$. The Green's function G_{K_H} is also a Gaussian distribution. Using the fact that the convolution of two Gaussian distributions is another Gaussian distribution with a width which is the sum of the two component widths, it must follow that $\bar{C}$ of (25) must approach a solution of a diffusion equation with a coefficient of effective diffusion given by $K_H^* = K_H + u^2/2\alpha$. To combine the two mechanisms, we simply add their diffusion coefficients, but, as before, this approximation holds only after many vertical exchange times.

For short times, when the localized parts of the distribution and the fronts would be important if $K_H = 0$, the horizontal eddy diffusion serves to spread out these sharp features. This is illustrated in Fig. 9 which shows a numerical integration of (25) for the same situation as case 1 in Section 2, a localized initial distribution of contaminant uniformly distributed in the vertical, for $\alpha t = 1$. In order to show the height

of the spikes conveniently, the vertical axis is logarithmic. Three values of K_H are considered in Fig. 9, and a fourth, $K_H = 0$, in Fig. 1. In Fig. 1 the spikes corresponding to a localized distribution are perfectly localized with zero width and infinite height. For $K_H = 10^{-4} \, u^2/2\alpha$ the spikes shown in Fig. 9 are still quite pronounced, as are the fronts. For $K_H = 10^{-2} \, u^2/2\alpha$ these sharp features are still evident but less pronounced, and for $K_H = u^2/2\alpha$ the sharp features disappear altogether. For $K_H = u^2/2\alpha$ the shear effect and horizontal diffusion can be considered to be equal in strength. If $K_H \gg u^2/2\alpha$, the shear effect becomes negligible and horizontal diffusion dominates. These curves are for $\alpha t = 1$, for which a one-layer description of the dispersion is not very good because the vertical mixing is not complete. The corresponding curves for $\alpha t = 10$ were not calculated because their features can be anticipated as discussed in the preceding paragraph.

4. Conclusions

The objective of this study has been to determine under what conditions shear dispersion can be parameterized as diffusion in one-layer circulation models. The conclusion is that, if the contaminant is well mixed through the water column, then this parameterization is appropriate.

The converse is also true. If the contaminant is poorly mixed vertically, then shear dispersion is like differential advection and not like diffusion. This situation is expected to arise near the edge of a patch of contaminant or near a continuous source of contaminant. The extent of these regions is of the order of the shear velocity multiplied by the vertical mixing time. If the time scale of the circulation calculation is much greater than the vertical mixing time and the horizontal scale much larger than this coherence length, then the deviation from diffusion-like behavior should be insignificant.

Because of the feedback provided by the shear and cross-shear mixing, the contaminant tends in time to become well mixed through the water column. This feedback is like an averaging process. It is useful to think of the one-layer model with the shear effect parameterized as diffusion as being the result of averaging the vertical average of the contaminant distribution over time and space, filtering out variations shorter than the vertical mixing time and the horizontal coherence length. The diffusion parameterization holds so long as these small-scale variations, which reflect the vertical inhomogeneity of the contaminant distribution, are unimportant.

Because this model is so idealized, it is important to understand its relationship to more sophisticated models capable of a more realistic description of the vertical profiles of contaminant, shear and mixing. The next most simple model would have three layers to describe these vertical variations. Such a model

would have an equation for the vertical mean of the contaminate concentration involving a third time derivative. This term would dominate for short times, corresponding to differential advection within each layer. After a sufficient time for the contaminant to be well mixed through the water column, the diffusion parameterization should be appropriate. For intermediate times, some but not all of the vertical structure will be mixed out and the result should compare with the two-layer model. More layers give higher time derivatives and thus better short-time resolution due to better vertical resolution, but the diffusion parameterization should still be appropriate when the mixing is complete throughout all of the layers. Thus, the two-layer model is sufficient to determine the domain of validity of the diffusion parameterization, but a more sophisticated description of vertical variation should yield a better estimate of the diffusivity.

To parameterize shear dispersion as diffusion, the size of the diffusivity in the direction of the shear should be enhanced. This enchancement is additive in the sense that a term dependent only upon the shear and vertical mixing should be added to the horizontal diffusivity which is appropriate when there is no shear. This two-layer model was chosen for simplicity of analysis and does not give the best possible prescription for this term. Bowden's (1965) method for evaluating this term is more sophisticated. In any case, in order to evaluate this enhancement, statistical information describing the vertical shear and the vertical mixing must be used. In practice, especially for situations where one-layer circulation models are used, these data are poor or unavailable, so the expression $u^2/2\alpha$ from the two-layer model discussed here is good enough.

It is possible to push this two-layer model further. In most situations the vertical shear and the vertical mixing change with time. This can be studied by allowing u and α to vary with time. It can be shown that, if they do not vary too fast, then the magnitude of the enhancement can be evaluated using instantaneous values of the shear and mixing rate. A similar result should also be obtainable for spatial variations, where u and α depend upon x, but that analysis has not been carried out. Also, it is not difficult to extend the model to layers of unequal thickness or to incorporate mixing rates that are different in the two layers. The result is intuitive that, if the contaminant spends more time in one layer than the other, then it is transported in one direction more often than in the other. This can be accounted for through an effective mean flow. These results are not difficult to work out, but their inclusion would make this paper unreasonably long.

These results suggest that the shear effect can be properly included in numerical studies of pollution dispersion by simply enhancing the diffusion coefficient in the direction of the shear. However, if results are needed showing spatial variations of smaller scale than the horizontal coherence length or temporal variations that take less than the vertical mixing time, then more vertical resolution is needed. The problem of estimating the shear and vertical exchange rate in order to obtain the effective diffusion coefficient K_H^* is essentially the same problem as that of estimating eddy diffusion coefficients for turbulent mixing. It requires some understanding of the nature of the flows which must come from observational data, not from this model.

REFERENCES

Abramowitz, M., and I. A. Stegan, 1970: *Handbook of Mathematical Functions.* National Bureau of Standards, Gov. Printing Office, Washington, D. C., p. 377.
Bowden, K. F., 1965: Horizontal mixing in the sea due to a shearing current. *J. Fluid Mech.*, 21, 83–95.
Elder, J. W., 1959: The dispersion of marked fluid in turbulent shear flow. *J. Fluid Mech.*, 5, 544–560.
Morse, P. M., and H. Feshbach, 1953: *Methods of Theoretical Physics.* McGraw-Hill, Chap. 7.
Okubo, A., 1967: The effect of shear in an oscillatory current on horizontal diffusion from an instantaneous source. *Intern. J. Oceanol. Limnol.*, 1, 194–204.
——, and H. H. Carter, 1966: An extremely simplified model of the "shear effect" on horizontal mixing in a bounded sea. *J. Geophys. Res.*, 71, 5267–5270.
Schönfeld, J. C., 1960: The mechanism of longitudinal diffusion in a tidal river. Presented at the Symposium on Tidal Rivers, Helsinki.
Selby, S. M., 1971: *Standard Mathematical Tables.* Chemical Rubber Co., Cleveland, Ohio, p. 510.
Stommel, H., 1950: *Proc. Colloquium on the Flushing of Estuaries.* Ref. No. 50-37, Woods Hole Oceanographic Institution, p. 206.
Taylor, G. I., 1954: The dispersion of matter in turbulent flow through a pipe. *Proc. Roy. Soc. London*, A223, 446–468.