Finite-Amplitude Baroclinic Disturbances in Downstream Varying Currents

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(Manuscript received 4 December 1975, in revised form 30 January 1976)

ABSTRACT

The finite-amplitude dynamics of baroclinic disturbances in currents whose cross-stream structure varies in the *downstream* direction is investigated.

It is first shown under what circumstances downstream variations of the current properties influence the local stability of the current. For flows near the neutral curve only the potential vorticity in *one* of the fluid layers is significant in determining the local stability.

For currents which are locally stable at some downstream locations and unstable at others, it is shown that the disturbance amplitude depends on the entire upstream structure of the current. In particular, simple examples illustrate the lack of a local relationship between "local" stability characteristics and the disturbance intensity.

The linear initial value problem for uniform (in the downstream direction) currents is also discussed to elucidate the relation between the temporal and spatial stability problems.

1. Introduction

One possible source of eddy motions in the ocean is the instability of swift currents in the ocean with respect to baroclinic disturbances. For currents which can be idealized as homogeneous in their properties in the direction down the current axis, stability criteria can be developed (e.g., Pedlosky, 1964) which depend solely on the cross-sectional distributions of velocity, temperature and potential vorticity. Yet most real currents vary significantly in their properties in the downstream direction. Such currents present the investigator with a continuous series of cross sections, each possessing (were they each the cross section of a downstream homogeneous flow) different stability characteristics. It is natural to attempt to associate with each cross section a "local" stability measure. Indeed, it is tempting to suppose that where the "local" cross-sectional profiles are "unstable," the current will locally produce disturbances and, where the current is locally "stable," disturbances should be absent or at least significantly less intense. In some hydrodynamic situations disturbances exist only by extracting energy from the mean to compensate for dissipation. In such cases such locality of relationship between local stability and fluctuation intensity may be expected. On the other hand, in many meteorological and oceanographic situations the advective time scale of the current is much less than the dissipation time. In such cases the disturbance may propagate into stable regions while still possessing, structurally, a certain memory of its history in a locally unstable region.

The purpose of this paper is to initiate a study of the nature of the downstream dependence of fluctuation amplitude in cases where the mean current varies sufficiently in the downstream direction so as to be locally stable at some downstream positions and unstable at others. In particular, attention is focused on the non-dissipative, finite-amplitude behavior of the disturbance field. Finite-amplitude effects are important to consider since sufficient time must be allowed the disturbance to transit substantial distances to observe variations of local stability while maintaining bounded amplitude.

Two results of interest are found in the following work. First, that only variations of some combination of local variables in the downstream direction are of significance and, second, that disturbances can extract energy from the mean and reach substantial amplitudes in stable regions if they have been formed initially in unstable regions.

2. The model and mathematical formulation

For the purposes of this study I consider a two-layer baroclinic model as in Pedlosky (1972). Two layers of fluid each with different constant density lie on a plane rotating with angular velocity Ω. The lighter fluid lies above the heavier fluid. The fluid is bounded above by a rigid horizontal plane. The lower boundary is nearly flat and its mean distance from the upper boundary is D. The deviation of the lower boundary from perfect flatness is given by the equation for the bottom

$$z_* = d\eta_B(x,y), \tag{2.1}$$

where d is an amplitude measure of the bottom variation and η_B yields the shape of the bottom. The interface between the two fluids is taken, in the absence of motion, to lie a distance D_1 from the upper boundary. The system is confined laterally to a channel infinite in length whose width is L. To include the effects of the earth's sphericity Ω is assumed to vary linearly in the northward (y_*) direction, e.g.,

$$2\Omega = f_0 + \beta_* y_*. \tag{2.2}$$

If scales $[L, D, L/U, U, (D/L)\,U]$ are chosen for the horizontal coordinates, the vertical coordinate, time and the horizontal and vertical velocities, respectively, the governing non-dimensional equations of motion for small Rossby number (U/f_0L) are the quasi-geostrophic potential vorticity equations

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial x}\right)[\nabla^2\psi_1+F_1(\psi_2-\psi_1)+\beta y] = T_1(x,y,t), \tag{2.3a}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi_2}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi_2}{\partial y}\frac{\partial}{\partial x}\right)[\nabla^2\psi_2+F_2(\psi_1-\psi_2)+\beta y + \delta B(x,y)] = T_2(x,y,t), \tag{2.3b}$$

and associated boundary conditions

$$\frac{\partial\psi_n}{\partial x}=0,\quad y=0,1,$$

where x and y are the longitude and latitude coordinates, and ψ_1 and ψ_2 are the non-dimensional geostrophic streamfunctions. The function

$$\delta B(x,y) = \frac{f_0L}{U}\frac{d}{D_2}\eta_B(x,y), \tag{2.4}$$

where $D_2 = D - D_1$, so that it is tacitly assumed that $\delta = (d/D_2)/(\text{Rossby number})$ is $O(1)$ or less. The other dimensionless parameters that appear are

$$\left.\begin{aligned} \beta &= \frac{\beta_* L^2}{U} \\ F_1 &= \frac{f_0^2L^2}{\dfrac{(\rho_2-\rho_1)}{\rho_1}gD_1} \\ F_2 &= \frac{f_0^2L^2}{\dfrac{(\rho_2-\rho_1)}{\rho_1}gD_2} \end{aligned}\right\}, \tag{2.5}$$

and the operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The functions $T_K(x,y,t)$ are some, as yet, unspecified sources of potential vorticity for each layer which will be responsible for altering the *mean* potential vorticity of the current along its path. They will be specified later. The plan of development is as follows. I consider the state of flow consisting of a mean shear, $U_1 - U_2$, which is independent of x, y and t and which is just marginally stable in the presence of the β-effect, but in the absence of bottom topography. I then consider the stability of the new mean state given by

$$\psi_n = -U_n y + \Delta\Psi_n(X,y,T), \quad \Delta \ll 1, \quad n = 1, 2, \tag{2.6}$$

where $\Delta\Psi_n$ represents a small alteration of the mean flow in the x direction by an amount $O(\Delta)$. The variables X, T represent new slow space and time scales, i.e.,

$$X = \mu x, \quad T = \sigma t, \quad \mu \ll 1, \quad \sigma \ll 1, \tag{2.7}$$

so that (2.6) represents a *nearly* zonal shear flow whose intensity varies slowly in space and time by the small amount Δ. However, since the basic uniform flow is marginally stable this additional mean flow is sufficient to make the mean flow "locally" stable for some downstream positions, and "locally" unstable for others in the sense of the remarks in the Introduction. In addition, I choose

$$\left.\begin{aligned} \delta &= O(\Delta) \\ B &= B(X,y) \end{aligned}\right\}, \tag{2.8}$$

so that the topography consists of a small slope in the cross-stream direction, which also changes slowly in the downstream direction and, hence, also can affect the local stability properties differently at different downstream locations. The key question is then formulated as follows.

In view of the the downstream variation of the mean shear and the cross-stream topographic slope, how is the amplitude of a baroclinic disturbance affected by these variations?

Earlier analysis of the spatial dependence of baroclinic disturbances in a homogeneous flow (Pedlosky, 1972) showed that the intrinsic time and space scales for the packet were $\Delta^{-\frac{1}{2}}L/U$ and $\Delta^{-\frac{1}{2}}L$, respectively. The most natural choice for μ and σ is therefore

$$\mu = \sigma = \Delta^{\frac{1}{2}}, \tag{2.9}$$

where Δ is a measure of the supercriticality of the vertical shear, which, in the present problem, varies with X and T. Similarly, that earlier analysis suggests that the appropriate amplitude scaling for the disturbance streamfunction is $\Delta^{\frac{1}{2}}$. Thus the total streamfield is taken to be

$$\psi_n = -U_n y + \Delta^{\frac{1}{2}}\phi_n(x,y,t,X,T) + \Delta\Psi_n(X,y,T), \tag{2.10}$$

where the disturbance field ϕ_n has the asymptotic expansion

$$\phi_n = \phi_n^{(1)} + \Delta^{\frac{1}{2}}\phi_n^{(2)} + \Delta\phi_n^{(3)} + \cdots. \tag{2.11}$$

If (2.10) is inserted into (2.3) a sequence of linear problems emerges. First, however, I specify that $T_n = O(\Delta^{3/2})$ so that the secular change of the mean potential vorticity is $O(\Delta^{3/2})$. The details of the subsequent calculation are lengthy, straightforward, and similar to the derivation in Pedlosky (1972), so I will only quote the results.

For a current wherein $U_1 - U_2 > 0$[1], the minimum critical shear is β/F_2. At that value of the shear a marginally stable disturbance

$$\phi_n^{(1)}(x,y,t,X,T) = A_n(X,T)e^{ik(x-ct)}\sin m\pi y + * \quad (2.12)$$

is possible according to the $O(\Delta^{1/2})$ problem implied by (2.11) and (2.3) where

$$\left.\begin{aligned} a^2 &= k^2 + m^2\pi^2 = [F_2(F_1+F_2)]^{1/2} \\ c &= U_2 \\ \frac{A_2}{A_1} &\equiv \gamma = \frac{a^2 - F_2}{F_1} \end{aligned}\right\}. \quad (2.13)$$

It is also easy to demonstrate that although the two linear, marginal modes on the neutral curve have coalescing phase speeds equal to U_2, the modes have separated *group* speeds in the downstream direction, i.e.,

$$\left.\begin{aligned} c_1 &= U_2 + \frac{4\beta k^2}{F_2(a^2+F_1+F_2)} \\ c_2 &= U_2 \end{aligned}\right\}. \quad (2.14)$$

Both c_1 and c_2 will reappear as crucial physical parameters in the following work.

The order Δ problem yields a correction to the wave field's vertical phase and also a wave-induced correction to the $O(\Delta)$ mean flow; that is

$$\phi_1^{(2)} = \Phi_1^{(2)}(X,y,T), \quad (2.15a)$$

$$\phi_2^{(2)} = \Phi_2^{(2)}(X,y,T) - i\frac{(F_1+F_2)}{k\beta}\frac{F_2}{F_1} \times \left\{\frac{\partial A_1}{\partial T} + \left[U_2 + \frac{2\beta k^2}{F_2(F_1+F_2)}\right]\frac{\partial A_1}{\partial X}\right\} \times e^{[ik(x-ct)]}\sin m\pi y + *. \quad (2.15b)$$

The functions $\Phi_n^{(2)}(X,y,T)$ represent the as yet unknown alterations to the mean flow by the nonlinear self-advection of potential vorticity in the baroclinic wave field. The phase shift between ϕ_1 and ϕ_2 given by the second term in (2.15b) is produced by both slow temporal and spatial changes in the wave amplitude.

The order $\Delta^{3/2}$ problem is the crucial one. At this order the problem becomes cognizant of the alteration of the mean flow by $\Phi_n^{(2)}$ and Ψ_n. It is also at this order that the wave-induced mean flow $\Phi_n^{(2)}$ and the externally driven flow Ψ_n are determined. As in Pedlosky (1972), they are determined by a condition in the $O(\Delta^{3/2})$ problem which ensures the suppression of secular terms at that order. Thus one condition for the problem to remain uniformly valid in time is

$$\left[\frac{\partial}{\partial T} + U_2\frac{\partial}{\partial X}\right]\left[\frac{d^2S_2}{dy^2} - F_2(S_2-S_1) + \frac{\delta}{\Delta}B(X,y)\right] = -\gamma m\pi\sin 2m\pi y\left[\frac{F_2(F_1+F_2)(a^2+F_2)}{\beta F_1}\right]\left[\frac{\partial|A_1|^2}{\partial T} + c_1\frac{\partial|A_1|^2}{\partial X}\right] + \tau_2(X,y,T), \quad (2.16)$$

where

$$\left.\begin{aligned} \tau_2 &= T_2(X,y,T)/\Delta^{3/2} \\ S_n &= \Phi_n^{(2)}(y,X,T) + \Psi_n(y,X,T) \end{aligned}\right\}. \quad (2.17)$$

Eq. (2.16) has a simple interpretation. The minimum critical shear for $U_1 - U_2 > 0$ is determined by the condition that the potential vorticity gradient in the lower layer just vanish, i.e., $U_1 - U_2 = \beta/F_2$. The variations of the lower layer's potential vorticity about this marginal state then becomes crucial. Eq. (2.16) states that the rate of change of the total potential vorticity of the lower layer, as seen by a fluid column moving with the $O(1)$ downstream speed of that layer, consists of two parts. The first is the production of lower layer potential vorticity by the nonlinearities associated with the temporal and spatial growth of the disturbance. This is given by the first term on the right-hand side of (2.16). The second part is that part driven directly by the external source of potential vorticity for the lower layer.

On the other hand, removal of secular terms of the form $\exp[ik(x-ct)]\sin m\pi y$ yields the second and final condition:

$$\left(\frac{\partial}{\partial T} + c_1\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T} + c_2\frac{\partial}{\partial X}\right)A_1 = \frac{2m\pi k^2\gamma\beta F_1 A_1}{F_2(a^2+F_2)(F_1+F_2)} \times \int_0^1 dy\,\sin 2m\pi y\left[\frac{d^2S_2}{dy^2} - F_2(S_2-S_1) + \frac{\delta}{\Delta}B\right], \quad (2.18)$$

where c_1 and c_2 are the two group speeds given by (2.14). It follows that the evolution of the amplitude field A depends only on the potential vorticity distribution of the lower[2] layer and, in fact, the projection of its cross-stream gradients on the square of the basic waves cross-stream eigenfunction. Furthermore, it is only the total potential vorticity of the lower layer that

[1] The case $U_1 - U_2 < 0$ follows trivially with the interchange of subscripts 1 and 2 in the following results.

[2] If $U_1 - U_2 < 0$, it would be the upper layer's potential vorticity that would be determining.

affects the wave growth—both externally determined and self-induced.

Let

$$\Pi=\int_0^1 \sin 2m\pi y\left[\frac{d^2S_2}{dy^2}-F_2(S_2-S_1)+\frac{\delta}{\Delta}B(X,y)\right]dy$$

$$=-p\Pi_A+q\Pi_\theta, \quad (2.19)$$

where

$$\left.\begin{aligned} p&=\frac{(a^2+F_2)(F_1+F_2)F_2}{2m\pi k^2\gamma\beta F_1}\\ q&=\frac{F_2(F_1+F_2)(a^2+F_2)}{2\gamma m\pi k^2(a^2+F_2)}\end{aligned}\right\}. \quad (2.20)$$

It then follows that the amplitude of the baroclinic disturbance in this downstream varying current is governed by the system of equations

$$\left.\begin{aligned} &\left(\frac{\partial}{\partial T}+c_1\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T}+c_2\frac{\partial}{\partial X}\right)A=A\Pi_\theta-A\Pi_A\\ &\left(\frac{\partial}{\partial T}+c_2\frac{\partial}{\partial X}\right)\Pi_A=\left(\frac{\partial}{\partial T}+c_1\frac{\partial}{\partial X}\right)|A|^2\\ &\left(\frac{\partial}{\partial T}+c_2\frac{\partial}{\partial X}\right)\Pi_\theta=\theta(X,T)\end{aligned}\right\}, \quad (2.21\text{a,b,c})$$

where

$$\theta(X,T)=\frac{1}{q}\int_0^1\frac{T_2(X,y,T)}{\Delta^{\frac{1}{2}}}\sin 2m\pi y dy, \quad (2.22\text{a})$$

$$A=m\pi\gamma k A_1(X,T). \quad (2.22\text{b})$$

Note that the evolution of A, the amplitude of the disturbance, is altered as it propagates by its interaction with the externally imposed alteration of the potential vorticity of the lower layer (Π_θ) as well as its interaction with its own self-induced alteration of the potential vorticity represented by Π_A.

3. Steady forcing

The system (2.21) is rather general and can describe a variety of interesting situations in which the wave interacts with both temporally and spatially varying ambient potential vorticity. For the remainder of this paper I will consider only the case where T_2 is a function of X and y but not of T. Since that implies that θ is then a function of X alone, a particular solution to (2.21c) is

$$\Pi_\theta(X)=\int_0^X\frac{\theta(X')}{c_2}dX'+\Pi_\theta(0), \quad (3.1)$$

as long as $c_2=U_2$ is not identically zero. To the solution given by (3.1) any solution of the form

$$\Pi_\theta=\Pi_\theta(X-c_2T)$$

may be added to satisfy initial conditions. After this transient has passed through the system, Π_θ will be given by (3.1). Henceforth, I assume that the *externally imposed* potential vorticity is steady, and given by (3.1). It therefore follows that (2.21a) may be rewritten in this case as

$$\left(\frac{\partial}{\partial T}+c_1\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T}+c_2\frac{\partial}{\partial X}\right)A=\sigma^2(X)A-A\Pi_A, \quad (3.2\text{a})$$

where

$$\sigma^2(X)=\Pi_\theta(0)+\int_0^X\frac{\theta(X')}{c_2}dX'. \quad (3.2\text{b})$$

The quantity $\sigma(X)$ is the local growth rate, i.e., the growth rate deduced solely from consideration of the properties of the mean flow at each cross section independently. It depends only on the potential vorticity in the lower layer of the externally imposed flow. If the potential vorticity source $\theta(X)$ vanishes, the problem for A is homogeneous in X due to the conservation of potential vorticity for the mean flow itself, even though Ψ_2 and B (the mean flow and the topography) are functions of X. In that case, (3.2a) would reduce to the problem examined in Pedlosky (1972) which is homogeneous in X. *Thus, only the downstream variation of potential vorticity and not its component elements enters the stability problem and sources of potential vorticity are required to produce a varying* σ.

4. Linear initial value problem

It is useful, before examining the nonlinear, inhomogeneous $[\sigma^2=\sigma^2(X)]$ problem, to consider the linear initial value problem in the homogeneous case. This corresponds to setting $\theta=0$ and neglecting the Π_A term in (3.2a). The linear problem which results is

$$\left(\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial X}\right)A=\sigma^2(0)A. \quad (4.1)$$

Consider the following initial and boundary value problem for $A(X,T)$. At $X=0$

$$\left.\begin{aligned} &A(0,T)=A_0\\ &\frac{\partial A}{\partial X}(0,T)=0\end{aligned}\right\}, \quad (4.2\text{a,b})$$

while at $T=0$

$$\left.\begin{aligned} &A(X,0)=0\\ &\frac{\partial A}{\partial T}(X,0)=0\end{aligned}\right\}, \quad (4.3\text{a,b})$$

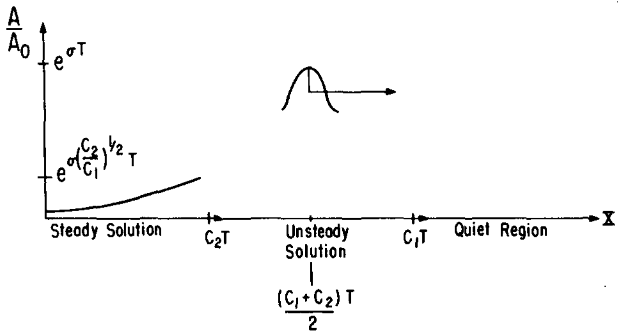

FIG. 1. A schematic diagram illustrating the various regions of response for the linear problem discussed in Section 4. The figure shows the region of the spatial growth of the steady solution $0\leqslant X\leqslant C_2T$, of the unsteady solution (which assumes a packet-pulse form) $c_2T\leqslant Xc_1T$, and the quiet region $c_1T<X$.

for the region $X\geqslant 0$, $T\geqslant 0$. This corresponds to introducing a perturbation at $X=0$ whose amplitude[3] is constant in time, while at the initial instant the region $X>0$ is free of any disturbance. The condition (4.2b) is chosen for symmetry.

It is important to recall that $c_1>c_2(\beta\neq 0)$. For definiteness I will assume that the product $c_1c_2>0$. It then follows that the solution to the problem posed above, which may be found by Laplace transform methods, is

$$\frac{A}{A_0}=[H(X)-H(X-c_2T)]\cosh\mu X+[H(X-c_1T)-H(X-c_2T)]\left\{\frac{1}{2\pi}\int_{-\kappa}^{\kappa}d\xi\frac{\cos\xi X\cosh c_S(\kappa^2-\xi^2)^{\frac{1}{2}}T}{(\kappa^2-\xi^2)^{\frac{1}{2}}}\times\left(\frac{\xi^2c_m/c_S}{\xi^2+\mu^2}-2\frac{c_m}{c_S}\right)+\frac{1}{2\pi}\int_{-\kappa}^{\kappa}d\xi\sin\xi X\sinh c_S\times(\kappa^2-\xi^2)^{\frac{1}{2}}T\left(\frac{\xi}{\xi^2+\mu^2}\right)\right\},\qquad(4.4)$$

where

$$\left.\begin{aligned}c_m&=\frac{c_1+c_2}{2},\quad \kappa^2=\sigma^2(0)/c_S^2\\ c_S&=\frac{c_1-c_2}{2},\quad \mu^2=\sigma^2(0)/c_1c_2\end{aligned}\right\},\qquad(4.5)$$

and $H(r)$ is the Heaviside function, *viz*:

$$\left.\begin{aligned}H(r)&=1,\quad r>0\\ H(r)&=0,\quad r<0\end{aligned}\right\}.\qquad(4.6)$$

The solution naturally separates into three regions (see Fig. 1). For $X>c_1T$ there is no disturbance. This is natural since c_1 is the faster group speed. For

$$c_2T<X<c_1T$$

only the second term in (4.4) remains. This is the region of the traveling, time-dependent pulse. For $X<c_2T$ only the first term in (4.4) remains, the steady solution

$$\frac{A}{A_0}=\cosh\mu X=\cosh\left[\frac{\sigma_0X}{(c_1c_2)^{\frac{1}{2}}}\right]\qquad(4.7)$$

which satisfies the boundary but not the initial conditions. Thus there will be a region of steady amplitude envelope only if $c_1c_2>0$. If $c_1c_2<0$, the boundary between steady and unsteady regions will occur for $X<0$ and *the steady solution has no relevance for the region $X>0$.* This resolves an apparent conundrum associated with (4.1). Apparently, (4.1) implies that if $c_1c_2<0$, temporal instability ($\sigma^2>0$) corresponds to spatial stability ($\sigma^2/c_1c_2<0$) for A a function of X alone. The present calculation shows that the steady solutions alone have relevance only when spatial and temporal instability coincide, i.e., when $c_1c_2>0$. I will make use of this in the discussion of the nonlinear problem.

It is interesting to examine the nature of the solution in the region of the transient pulse. This, of course, requires approximate calculation since the integrals in (4.4) are quite difficult. The method of steepest descent, valid for large X and T, and

$$-c_ST<X-c_mT<c_ST,$$

yields for the transient pulse

$$\frac{A}{A_0}\sim\alpha(X,T)\exp\left\{\kappa c_ST\left[1-\frac{(X-c_mT)^2}{c_S^2T^2}\right]\right\},\qquad(4.8)$$

where α is the algebraic function

$$\alpha(X,T)=\frac{2^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}[c_S^2T^2-(X-c_mT)^2]^{\frac{1}{4}}}\left[\frac{2c_mT^{\frac{1}{2}}}{T(c_S\kappa)^{\frac{1}{2}}}-\frac{\kappa(X-c_mT)T^{\frac{1}{2}}\kappa^{\frac{1}{2}}}{\mu^2c_S^{\frac{1}{2}}T\left(|X-c_mT|\frac{c_m}{c_S}+c_ST\right)}\right].$$

Note that the amplitude exponentially grows as it propagates. The maximum growth, and hence the peak of the disturbance, will occur at $X=\frac{1}{2}(c_1+c_2)T$. The exponential factor there is $\exp(\sigma_0T)$, i.e., the growth factor corresponding to purely temporal growth. Note that the maximum amplitude of the steady solution is obtained at $X=c_2T$ where its exponential factor is $\exp[\sigma_0(c_2/c_1)^{\frac{1}{2}}T]$. Since $c_2/c_1<1$ it follows, according to linear theory, that the maximum amplitude occurs at the traveling pulse, although a significant and exponentially growing steady spatial disturbance is left behind.

[3] Although the amplitude is constant this corresponds to a fluctuating disturbance at $X=x=0$ with a frequency kc_2.

Naturally, when the solution in either the transient or steady region becomes large, nonlinear effects will become significant. Further, in situations where σ is a function of X, the disturbance amplitude will begin to sense the loss of downstream homogeneity. The addition of these two important features returns us to the problem posed by (3.2a,b). It is almost hopeless in the more general situation to extract useful analytical information from the general initial value problem. Henceforth, guided by the results of the linear problem, I will limit myself to examining those situations where $c_1c_2>0$. This usually implies that the deep flow be in the same direction as the surface flow for the O(1) component. In these cases the steady problem would appear to be physically relevant and illuminating. Note, however, that this will restrict only the envelope A to be steady. The wave disturbances will still thread through the envelope with the speed $c=U_2=c_2$ as implied by (2.13).

5. Steady solutions

I consider in this section what I believe to be the prototypical situation of interest. Imagine a current whose local properties are such that in the region $0\leqslant X\leqslant l$ the mean flow appears unstable, i.e.,

$$\sigma^2>0,\quad 0\leqslant X\leqslant l,$$

while further downstream, $X>l$, the flow is stable, i.e.,

$$\sigma^2<0,\quad l\leqslant X.$$

To simplify the nonlinear analysis, I consider the case where

$$\frac{\sigma^2}{c_1c_2}=\left\{\begin{array}{ll}\mu^2>0, & 0\leqslant x<l\\ -\nu^2<0, & l<x\end{array}\right\}. \tag{5.1}$$

This idealization corresponds to a change from uniform local instability to uniform stability on spatial scales short compared to the envelope scale, but still long compared to the wavelength of the carrier wave; that is, in the transition zone about $x=l$, for purposes of analytic simplicity, $\sigma^2(X)$ is considered as the limit of

$$\sigma^2(X)=\frac{\mu^2-\nu^2}{2}-\frac{(\mu^2+\nu^2)}{2}\tanh(X/\epsilon)$$

as $\epsilon\rightarrow 0$. Since $X=\Delta^{\frac{1}{2}}x$, this is consistent with the slowly varying hypothesis of σ^2 as long as

$$\Delta^{\frac{1}{2}}\ll\epsilon\ll 1,$$

so that the discontinuity of σ occurs as a limit of rapid change on the long spatial scale. For purposes of comparison I take the same boundary conditions as in the linear problem of Section 4, namely

$$\left.\begin{array}{l}A=0,\quad \text{at}\quad X=0\\ \dfrac{dA}{dX}=0\end{array}\right\}. \tag{5.2}$$

The problem for A is then

$$\left.\begin{array}{l}c_1c_2\dfrac{d^2A}{dX^2}=\sigma^2(X)A-A\Pi_A\\ \dfrac{d\Pi_A}{dX}=\dfrac{c_1}{c_2}\dfrac{d|A|^2}{dX}\end{array}\right\}. \tag{5.3a,b}$$

The solution to (5.3b) is

$$\Pi_A=\frac{c_1}{c_2}|A|^2+\text{constant}. \tag{5.4}$$

The constant in (5.4) is determined by the condition at $X=0$ that the potential vorticity distribution in the current corresponds to that of the mean flow, i.e., $\Pi_A=0$ at $X=0$, so that

$$\Pi_A=\frac{c_1}{c_2}(|A|^2-|A(0)|^2). \tag{5.5}$$

Furthermore, the boundary conditions (5.2) imply that if A_0 is real, A will be real so that in the region $0\leqslant X<l$, the equation for A is

$$\frac{d^2A}{dX^2}=\mu^2A-\frac{A}{c_2^2}(A^2-A_0^2). \tag{5.6}$$

The solution of (5.6) is easily obtained in terms of elliptic functions. The solution is

$$\frac{A}{A_0}=\left[dn\left(\frac{A_{\max}X}{\sqrt{2}c_2},\alpha\right)\right]^{-1} \tag{5.7}$$

and corresponds to a spatial oscillation between the minimum amplitude $A_0=A(0)$ and the maximum amplitude

$$A_{\max}=(2c_2^2\mu^2+A_0^2)^{\frac{1}{2}}. \tag{5.8}$$

The modulus of the elliptic function α is given by

$$\alpha^2=\frac{(2c_2^2\mu^2)}{(A_0^2+2c_2^2\mu^2)}. \tag{5.9}$$

The spatial period of the oscillation is given by

$$\left.\begin{array}{l}X_p=\dfrac{\sqrt{2}c_2 2K(\alpha)}{A_{\max}}=\dfrac{2\alpha K(\alpha)}{\mu},\\ K(\alpha)=\displaystyle\int_0^{\pi/2}\frac{d\theta}{(1-\alpha^2\sin^2\theta)^{\frac{1}{2}}}.\end{array}\right. \tag{5.10}$$

Note that for strongly unstable situations where $A_{\max}\gg A_0$, i.e., where $c_2^2\mu^2\gg A_0^2$, α tends to unity and

$$X_p\sim\frac{2\ln 4(1+2\sigma^2c_2/c_1)^{\frac{1}{2}}}{\sigma/(c_1c_2)^{\frac{1}{2}}}, \tag{5.11}$$

so that the period of oscillation decreases as the degree of instability increases and is related to but not identical to the e-folding scale of the linear problem. The amplification factor

$$\frac{A_{\max}}{A_0}=\left(1+\frac{2c_2^2\mu^2}{A_0^2}\right)^{\frac{1}{2}} \tag{5.12}$$

increases strongly with the degree of local instability. Fig. 2a shows the amplitude as a function of downstream position in the unstable region for the case where $A_{\max}/A_0$ is 5.759. Note the strong departure of the linear solution from the nonlinear solution within half a period of the latter. By the time the disturbance has moved a distance $\mu X=3.1$ from the origin, the energy has increased by a factor $(A_{\max}/A_0)^2=33.166$. This is a considerable amplification but is important to note that this is the maximum enhancement of the disturbance and is substantially less than that predicted by linear theory (93.586).

In the stable region $x\geqslant l$, the amplitude equation is

$$\frac{d^2A}{dX^2}+A\left[\nu^2-\frac{A^2(0)}{c_2^2}\right]+\frac{A^3}{c_2^2}=0, \tag{5.13}$$

the first integral of which is

$$\left(\frac{dA}{dX}\right)^2+A^2\left[\nu^2-\frac{A^2(0)}{c_2^2}\right]+\frac{A^4}{2c_2^2}=E. \tag{5.14}$$

To evaluate the constant E it is necessary to examine the jump condition for the equation for A at a discontinuity in σ^2. As long as σ^2 is finite, even though discontinuous, it is a simple matter to show that at $x=l$ both A and dA/dX must be continuous. Since in the region $x\leqslant l$

$$\left(\frac{dA}{dX}\right)^2-\frac{1}{c_2^2}[A^2(0)+c_2^2\mu^2]A^2$$
$$+\frac{A^2(0)}{2c_2^2}[A^2(0)+2c_2^2\mu^2]+\frac{A^4}{2c_2^2}=0, \tag{5.15}$$

it follows that for $x>l$

$$E=A_0^2\left\{(\nu^2+\mu^2)\left[\frac{A^2(l)}{A_0^2}-1\right]+\left(\nu^2-\frac{A_0^2}{2c_2^2}\right)\right\}. \tag{5.16}$$

In principle, E can be positive or negative. Unless l is chosen to be *exactly* an integral multiple of periods downstream from the origin, the sign of E will be determined by the first term in (5.16) and hence positive. Even in that special case E will still be positive if $\nu^2>A_0^2/2c_2^2$, i.e., if the local stability is sufficiently large. I therefore restrict my attention to the more typical case where $E>0$. In that case the solution for $x>l$, which is continuous with the solution for $x<l$, is

$$A=\alpha_{\max}cn\left[\nu\theta(X-l)-cn^{-1}\left(\frac{A(l)}{\alpha_{\max}},r\right),r\right], \tag{5.17}$$

where

$$\left.\begin{aligned}\theta&=\left\{1+2\frac{(\nu^2+\mu^2)}{\nu^2}\frac{A_0^2}{c_2^2\nu^2}\left[\frac{A^2(l)}{A_0^2}-1\right]\right\}^{\frac{1}{4}}\\ r^2&=\frac{1}{2}\left[1-\frac{(1-A_0^2/c_2^2\nu^2)}{\theta^2}\right]\end{aligned}\right\}, \tag{5.18a,b}$$

while the maximum amplitude of the oscillation in the locally stable region is

$$\left(\frac{\alpha_{\max}}{A_0}\right)^2=1-\frac{c_2^2\nu^2}{A_0^2}+\frac{c_2^2\nu^2}{A_0^2}\left\{1+2\frac{(\nu^2+\mu^2)}{\nu^2}\times\left[\frac{A^2(l)-A^2(0)}{c_2^2\nu^2}\right]\right\}^{\frac{1}{2}}. \tag{5.19}$$

The maximum amplitude achieved in the stable region is a function of the amplitude of the disturbance in the unstable region at the transition point $X=l$, and this information is remembered in the further downstream oscillation. There are some limiting cases of interest. If $A(l)=A_0$, then $\alpha_{\max}=A_0$, so that the subsequent oscillation remains bounded by the small initial amplitude A_0. On the other hand, if $A(l)$ is $A_{\max}$, then $\alpha_{\max}$, ***the maximum amplitude in the locally stable region, is as large as the amplitude that can be achieved in the locally unstable region.*** It is easy to prove though that

$$\alpha_{\max}^2\leqslant A_{\max}^2, \tag{5.20}$$

so that the amplitude in the stable region is never greater than the maximum *possible* amplitude in the stable region. For regions of very high local stability,

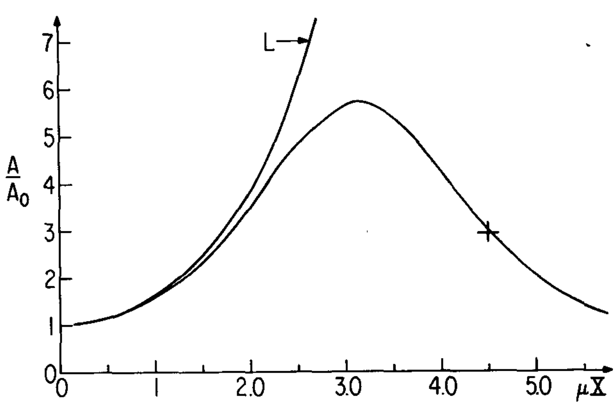

FIG. 2a. The amplitude behavior in the locally unstable region. The linear solution (L) is also shown to emphasize the limited region in X of its validity. The $+$ symbol at $\mu X=4.5$ marks the spot where the current becomes locally stable. Beyond this point the amplitude is given in Fig. 2b.

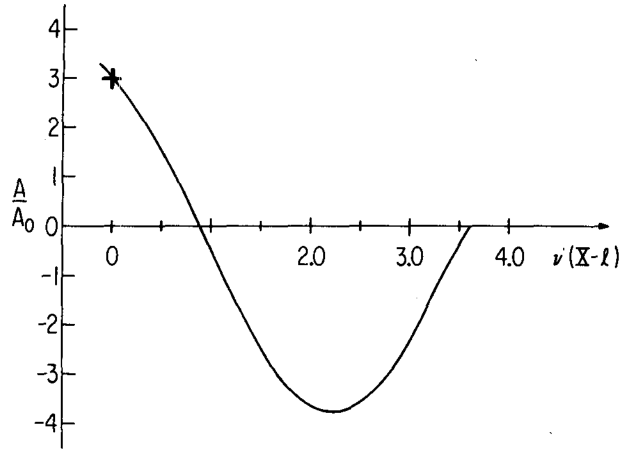

FIG. 2b. The amplitude behavior in the locally stable region downstream of an unstable region. Note the similarity in amplitude intensity with Fig. 2a and the zero crossings of A. The amplitude will continue this oscillation downstream.

i.e., where

$$\left.\begin{array}{l} c_2^2\nu^2 \ll A_0^2 \\ \theta \rightarrow 1 \\ r \rightarrow 0 \end{array}\right\}, \tag{5.21}$$

the amplitude can then be approximated by the linear solution

$$A \sim A(l) \cos\nu(X-l), \quad X>l. \tag{5.22}$$

It is also important to note that although the amplitude of the disturbance in the locally stable region is typically commensurate with that of the locally unstable region, the downstream structure of the disturbance differs. For $0 \leqslant X \leqslant l$, A is always positive, i.e., the amplitude envelope never diminishes to zero. For $X \geqslant l$, the disturbance given by (5.17) oscillates around zero. Fig 2b shows the amplitude in the region $X>l$ for the case $\nu^2=\mu^2$, $l=4.5\,\mu^{-1}$. The period of the oscillation in the stable region is given by $X_p=5.62\,\mu^{-1}$ which is less than that of the unstable region where for this case $X_p=6.21\,\mu^{-1}$.

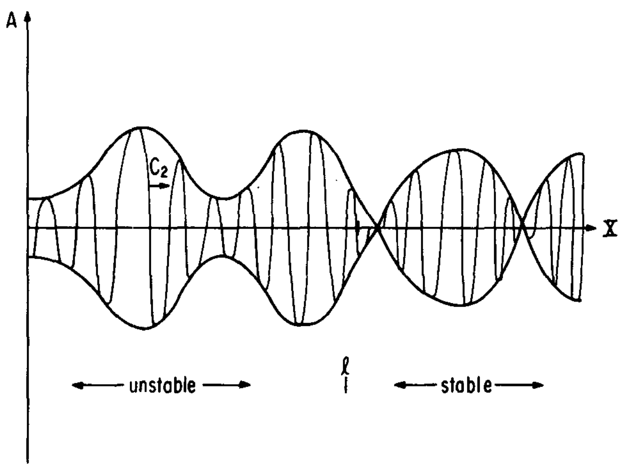

FIG. 3. A schematic rendering of the entire solution joining a stable to an upstream unstable flow. The carrier wave threading through the envelope is also shown.

Fig. 3 is a schematic representation of the total steady solution. It is clear from this diagram and from the analysis above, that *once the disturbance has grown in an unstable region to its maximum allowable amplitude the stabilization of the current downstream will not substantially reduce the amplitude of the disturbance* (unless of course strong dissipative mechanisms are added). Thus, for very weakly dissipative systems, there does not seem to be a local connection between local linear stability and the amplitude of the disturbance, at least in the steady state.

6. The reverse problem

For completeness I consider the opposite situation, namely, where the current is stable in the region $0 \leqslant X \leqslant l$ and becomes unstable for $X>l$. That is

$$\sigma^2/c_1c_2 = \begin{cases} -\nu^2, & 0 \leqslant X \leqslant l \\ \mu^2, & X \geqslant l \end{cases}$$

Then, with the same boundary conditions at $X=0$, it follows that the solution for A in the stable region is

$$\left.\begin{array}{l} A = A_0 cn[\nu X, \alpha] \\ \alpha = \dfrac{A_0}{\sqrt{2}c_2\nu} \end{array}\right\} \tag{6.1}$$

if $A_0^2 \ll 2c_2^2\nu^2$. In this case the disturbance undergoes a simple spatial oscillation whose maximum amplitude is its starting amplitude A_0, i.e., no amplification takes place. For $X>l$, the amplitude equation has a first integral which can be written

$$\left(\frac{dA}{dX}\right)^2 - \left(\mu^2 + \frac{A_0^2}{c_2^2}\right)A^2 + \frac{A^4}{2c_2^2} = E. \tag{6.2}$$

If use is again made of the continuity of A and dA/dX at $X=l$, it follows that

$$E = \nu^2[A_0^2 - A^2(l)] - \mu^2 A^2(l). \tag{6.3}$$

Hence if

$$\mu^2 > \nu^2 \frac{[A_0^2 - A^2(l)]}{A^2(l)},$$

i.e., if the downstream region is sufficiently unstable and $A(l)$ is not near zero, the downstream oscillation will be of the form given by (5.7). The maximum amplitude of the oscillation is given by

$$A_{\max}^2 = A_0^2 + \mu^2 c_2^2 + \{\mu^4 c_2^4 + 2(\mu^2+\nu^2)c_2^2[A_0^2 - A^2(l)]\}^{\frac{1}{2}}. \tag{6.4}$$

This amplitude is somewhat larger than that given by

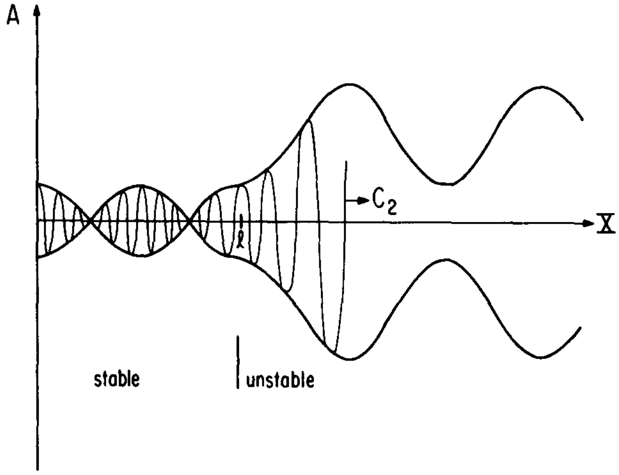

FIG. 4. A schematic rendering of the solution of the situation where an unstable region follows a stable region.

(5.7) if $dA/dx(l)$ is not zero due to the added "kick" given to the oscillation at $X=l$ by the non-zero slope at the joining point. A schematic picture of the amplitude behavior is shown in Fig. 4. After entering the unstable region the amplitude of the disturbance increases dramatically. Thus local stability acts to preserve the initial amplitude and regions of local instability to increase the disturbance amplitude. But regions of stability are not generally capable of suppressing the intensity of the disturbance already formed in a region of local instability. It therefore follows that the intensity of the disturbance depends, in a non-dissipative (or nearly so) fluid, on the entire structure of the current upstream of the station of interest and not merely on the local stability properties.

7. Energy

If Eqs. (2.3a,b) are multiplied by ψ_1 and ψ_2, respectively, the equation for the total energy of the flow can be obtained in the form (if ϕ_n is the *perturbation* streamfunction)

$$\frac{\partial}{\partial t}\left\{\sum_n \frac{(\nabla\phi_n)^2}{2F_n}+\frac{(\phi_1-\phi_2)^2}{2}\right.$$
$$\left.+\sum_n\left[U_n\frac{\partial}{\partial x}\frac{(\nabla\phi_n)^2}{2}+\frac{U_n}{2}\frac{\partial}{\partial x}\frac{(\phi_1-\phi_2)^2}{2}\right]\right\}+\nabla\cdot\mathbf{S}$$
$$=(U_1-U_2)(\phi_1+\phi_2)_x(\phi_1-\phi_2)+\sum T_n\phi_n, \quad (7.1a)$$

where

$$\mathbf{S}=\sum_n\frac{\phi_n}{F_n}\nabla\left(\frac{\partial\phi_n}{\partial t}+U_n\frac{\partial\phi_n}{\partial x}\right)+\mathbf{i}\beta\frac{\phi_n^2}{2}+\delta\frac{\phi_2^2}{2}\left(\mathbf{i}\frac{\partial B}{\partial y}-\mathbf{j}\frac{\partial B}{\partial x}\right)$$
$$+\sum_n\phi_n\Pi_n(\mathbf{k}\times\nabla\phi_n), \quad (7.1b)$$

and

$$\left.\begin{aligned}\Pi_1&=\nabla^2\phi_1-F_1(\phi_1-\phi_2)\\ \Pi_2&=\nabla^2\phi_2-F_2(\phi_2-\phi_1)\end{aligned}\right\}.$$

The divergence term $\mathbf{S}$ represents the mechanical flux of energy in the system. The baroclinic source term is the first term on the right-hand side of (7.1). It is merely the cross-stream advection of temperature (thickness, $\psi_1-\psi_2$) by the mean eddy velocity $(\psi_1+\psi_2)_x$ in the presence of the basic temperature (or thickness) gradient (U_1-U_2). In the cases studied in this paper the energy balances are in a steady state and the downstream divergence of energy, given by the second and third terms in (7.1), is balanced by the baroclinic production term,

$$W\equiv(U_1-U_2)(\phi_1+\phi_2)_x(\phi_1-\phi_2).$$

It is easy to show that

$$W\sim\frac{d}{dX}|A|^2 \quad (7.2)$$

aside from a constant, positive, multiplication factor. Thus there is baroclinic extraction of energy into the disturbance field whenever $|A|^2$ is increasing. It is important to recall that this happens, in the solutions presented earlier, in both the locally stable and unstable regions of flow. Furthermore, it follows from (2.15b) that the phase shift of the wave between the two layers will be proportional to $(1/A)dA/dX$. The phase shift of the wave with height will have all the characteristics of baroclinically unstable wave (upper wave lagging lower wave) whenever $(1/A)dA/dX>0$, and this will be observed in both locally unstable and stable regions of flow. Thus, even in terms of energy transformations it is not possible to restrict the usual characterizations of unstable waves to regions of flow which are unstable by a local analysis.

8. Conclusions

The examples presented here illustrate that disturbance energy will dramatically increase as the wave enters a locally unstable region of the current but will not be substantially diminished thereafter if the current subsequently becomes locally stable downstream. Indeed, it would be difficult to judge whether the current is locally unstable by examining only the local energy intensity of the disturbance. The amplitude and energy of the disturbance are bounded from above by nonlinear effects in the unstable region and are bounded from below in the stable region by the restoring mechanisms of both the linear and nonlinear dynamics. This feature would seem to hold whenever the dissipative time characteristic of the fluid is long compared to the advective time scale of the current.

It is interesting to note that the downstream variations in the envelope of the disturbance $A(X)$ are, in this model, determined primarily by the finite-amplitude dynamics and not by variations of stability, although both features do affect $A(X)$. Hence, the variations of eddy intensity downstream in a current are intrinsic to the dynamics and do not necessarily reflect variations in the local stability or the potential for energy exchange with the mean.

Acknowledgment. This research was supported in part by the Atmospheric Sciences Section, National Science Foundation.

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