Symmetric Finite-Amplitude Rotational Water Waves

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ABSTRACT

Two forms of a two-dimensional streamfunction solution for symmetric periodic water waves on a fluid with a vertical distribution of vorticity are presented. The magnitude of the vorticity varies linearly with the magnitude of the streamfunction, while remaining constant on a particular streamline. The analysis utilizes a numerical perturbation technique, which converges rapidly to a wave of given height and period in water of a specified depth with a given vorticity distribution. Computed results show the influence of the vorticity on the wavelength and crest elevation of the wave.

1. Introduction

Due to the ubiquity of wind over bodies of water and the fact that the wind which creates the waves also creates a wind-driven current, waves rarely propagate on a quiescent fluid. Waves which have traveled from their generation region will still encounter wind, tidal or gravity-induced flows. Therefore, it is inadequate to describe waves mathematically without including the effects of quasi-steady currents, with their spatial inhomogeneity. For a general description of the state of knowledge with respect to waves on currents, the reader is referred to a survey by Peregrine (1976).

The theoretical study of two-dimensional waves on a vertical shear current has proceeded by the solution of the water wave problem on progressively more complicated current profiles (over the depth). The easiest choice was a velocity profile which was constant over depth. This current profile, for small amplitude waves, can be accommodated by the Airy (1845) and Stokes (1847) theories. For more nonlinear waves, the techniques of Chappelear (1961) for velocity potential and Dean (1965) for the streamfunction are amenable to constant currents. Thompson (1949), Biesel (1950) and Tsao (1959) have investigated waves on mean currents which varied linearly in magnitude over the depth. For this case of constant fluid vorticity, Dalrymple (1974a) has extended Dean's (1965) streamfunction approach for nonlinear waves, both with symmetric and irregular free surface profiles. A more complicated current velocity profile may be modeled by dividing the fluid into two regions of differing vorticity leading to the bilinear shear current. Dalrymple (1974b) investigated both the small-amplitude and finite-amplitude waves for this case.

A more realistic current profile can be obtained by allowing the fluid vorticity to change continually with depth in the fluid. Small-amplitude wave theories for waves on exponential or sinusoidally varying current profiles have been developed by Abdullah (1949), Wehausen (1965) and Eliasson and Engelund (1972). The dispersion relation for these cases, which assigns the appropriate wavenumber to the given wave period T has also been determined by Dalrymple (1973) using a WKB approach, and more generally by Yih (1972) and Fenton (1973). Finite-amplitude wave models for these currents are developed herein. The symmetric wave models are developed for currents flowing in the direction of the wave and opposed to the wave. Both models are developed for any order theory and will converge directly on wave height.

2. Theoretical development

The governing equation for the wave motion and its solution follows after a number of assumptions. First, the waves are assumed to be long crested, thereby allowing a two-dimensional treatment, using a coordinate axis which is in the plane of the water motion. Second, the fluid is taken to be frictionless, which implies that the given vorticity distribution was imparted by external means and it remains constant. This assumption, as well as fixing the bottom to be horizontal and impermeable, indicates that the wave will propagate without change in form. Further, by translating the coordinate axis at the celerity C of the wave, the wave motion becomes steady (see Fig. 1).

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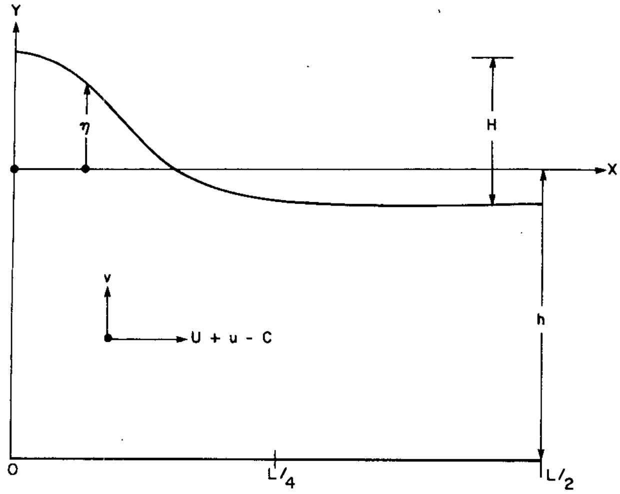

FIG. 1. Schematic of symmetric wave of height H in water of depth h. By translating the coordinate system with speed C (the wave celerity), the wave becomes stationary.

For a two-dimensional incompressible fluid, the following equation expresses the conservation of mass:

$$\frac{\partial(U+u-C)}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

where U is the mean (relative to the wave period) x-directed current, (u,v) are the wave-induced water motions in the (x,y) directions, and the wave celerity C appears as negative due to the translation of the coordinate axis. A streamfunction $\psi(x,y)$ may be introduced with the following definitions, which satisfy (1) exactly:

$$\left.\begin{aligned}(U+u-C)&=-\frac{\partial\psi}{\partial y}\\ v&=\frac{\partial\psi}{\partial x}\end{aligned}\right\}. \tag{2}$$

Substituting the streamfunction into the equation for the fluid vorticity yields (Lamb, 1945)

$$\frac{\partial v}{\partial x}-\frac{\partial(U+u-C)}{\partial y}=\nabla^2\psi=f(\psi), \tag{3}$$

where $\nabla^2(=\partial^2/\partial x^2+\partial^2/\partial y^2)$ is the two-dimensional Laplacian operator and $f(\psi)$ the vorticity. This equation states that the vorticity $\nabla^2\psi$ is a constant for a streamline and changes between streamlines according to $f(\psi)$. For the case of $f(\psi)=0$, a nonlinear high-order solution has been obtained by Dean (1965); for $f(\psi)$ equal to a constant, Dalrymple (1974a) developed a similar high-order solution. Here it will be taken that $f(\psi)=\pm\gamma^2\psi$, which will permit currents that vary as trigonometric and hyperbolic sine and cosine functions of the depth. The parameter γ governs the strength of the vorticity.

The boundary conditions to be satisfied by any solution to the above governing equations are located at the bottom, the water surface, and in the lateral direction. At the bottom, as mentioned previously, a no-flow condition is required in the vertical direction:

$$\frac{\partial\psi}{\partial x}=0 \quad \text{on} \quad y=-h. \tag{4}$$

Laterally, a periodic boundary condition is imposed in order to obtain a wave-like solution

$$\psi(x,y)=\psi(x+L,y), \tag{5}$$

where L is the length of the wave. On the free surface $y=\eta(x)$, the pressure is assumed to be a constant and, without loss of generality for free waves, is taken as zero. The Bernoulli equation, which is valid along a streamline such as the free surface, can be written as

$$\eta+\frac{\left[\frac{\partial\psi}{\partial x}\right]^2+\left[\frac{\partial\psi}{\partial y}\right]^2}{2g}=\bar{Q} \quad \text{on} \quad y=\eta(x), \tag{6}$$

where g is the acceleration of gravity and $\bar{Q}$ the free-surface streamline Bernoulli constant. This is referred to as the dynamic free surface boundary condition.

For wave problems expressed in terms of the velocity potential, it is also necessary to prescribe a further boundary condition at the free surface, which states that the free surface is a streamline. For this case, with the coordinate axis moving with the speed of the wave, this is true by definition, since the surface is one of constant ψ. However, in order that the free surface streamline yield a free surface displacement which has a zero mean about the mean water level, the further constraint

$$E_2\equiv\frac{2}{L}\int_0^{L/2}\eta(x)dx=0 \tag{7}$$

is needed on the solution. Since the wave form is taken to be symmetric, the above integral need only be taken over half the wave.

The boundary value problem for the waves is now fully prescribed and solutions may now be proposed. Two cases will be investigated: the first is for an aiding current, that is, one which flows in the direction of the waves. The mean current is assumed to have the following velocity profile $U(y)$ relative to a stationary observer:

$$U(y)-C=-(C-U_B)\cos\gamma(h+y), \tag{8}$$

where U_B is the mean bottom current. The governing equation for a wave on this current profile is

$$\nabla^2\psi=-\gamma^2\psi. \tag{9}$$

An assumed solution, which satisfies this governing equation, the bottom boundary condition (4), and the periodicity requirement (5) is

$$\psi(x,y)=\frac{(C-U_B)}{\gamma}\sin\gamma(h+y)+\sum_{n=1}^{N}X(n)\times\sinh[(k_n^2-\gamma^2)^{\frac{1}{2}}(h+y)]\cos k_n x, \quad (10)$$

where

$$k_n=2\pi n/L. \quad (11)$$

For an opposing current, a velocity profile may be taken as

$$U(y)-C=-(C-U_B)\cosh\gamma(h+y). \quad (12)$$

The governing equation now takes the form

$$\nabla^2\psi=\gamma^2\psi. \quad (13)$$

The assumed solution is now taken as

$$\psi(x,y)=\frac{(C-U_B)}{\gamma}\sinh\gamma(h+y)+\sum_{n=1}^{N}X(n)\times\sinh[(k_n^2+\gamma^2)^{\frac{1}{2}}(h+y)]\cos k_n x. \quad (14)$$

In both assumed solutions a series of periodic terms is included, each of which incorporates unknown coefficients $X(n)$, which will be different for each term and for each case. The only remaining boundary condition to be satisfied is the dynamic free surface boundary condition (6), subject to the mean sea level constraint (7). Therefore, the $X(n)$ must be chosen so as to satisfy these requirements on the free surface.

It first becomes necessary to evaluate the free surface function $\eta(x)$. This is accomplished by substituting $y=\eta(x)$ into the definitions of $\psi(x,y)$, namely, (10) and (14), and solving the resulting transcendental equation for $\eta(x)$ for each x. An accurate solution technique, which was used in this study, was developed by Traub (1960). This solution for η, however, requires a prior knowledge of the value of the free surface streamline $\psi(x,\eta)$, which is unknown. Therefore, $\psi(x,\eta)$ is defined as another unknown, $X(N+2)$, and will be allowed to vary in the now iterative solution technique so as to best satisfy the free surface conditions.

The nonlinear dynamic free surface boundary condition is expressed in a least squares form E_1, which should equal zero for an exact solution:

$$E_1=\frac{2}{L}\int_0^{L/2}[Q(x)-\bar{Q}]^2dx, \quad (15)$$

where

$$\bar{Q}=\frac{2}{L}\int_0^{L/2}Q(x)dx \quad \text{on} \quad y=\eta(x) \quad (16)$$

and $Q(x)$ is the Bernoulli constant at different positions along the free surface.

3. Solution technique

The mean-squared dynamic free surface boundary condition is to be made small subject to the aforementioned free surface constraint and also to a wave height constraint E_3:

$$E_3=\eta(0)-\eta(L/2)-H. \quad (17)$$

If E_3 is equal to zero, then the solution will result in a wave of height H. The problem then becomes, using a Lagrange multiplier technique (e.g., Hildebrand, 1965), a minimization of the function F defined as

$$F=E_1+\lambda_1E_2+\lambda_2E_3. \quad (18)$$

For an exact solution F will be zero. The solution technique involves minimizing F with respect to the unknowns, the $X(n)$ and λ_1 and λ_2. There are $N+2$ unknown $X(n)$ consisting of the series coefficients $[X(n), n=1, N]$, but also $X(N+1)$ and $X(N+2)$, which are the unknown wavelength and the value of the free surface streamline, respectively. Due to the nonlinearity of F, a quasi-linearization as developed by Dean (1965) is used along with an iterative technique in order that a matrix solution can be used. Therefore, for iteration $j+1$ it is assumed that F^{j+1} can be expressed as a first-order Taylor's series in the $X^j(n)$:

$$F^{j+1}=F^j+\sum_{n=1}^{N+2}\frac{\partial F^j}{\partial X(n)}X'(n), \quad (19)$$

where the $X'(n)$ are assumed to be small corrections to the $X(n)^j$, and F^{j+1} is now linear in the $X'(n)$. Minimizing F^{j+1} with respect to the $X'(n)$, λ_1 and λ_2, retaining only first derivatives, will yield a set of equations for the $X'(n)$ which may be solved by usual matrix methods. The $X'(n)$ are then added to the $X^j(n)$ as a correction:

$$X(n)^{j+1}=X(n)^j+\alpha X'(n), \quad (20)$$

where α is a stability parameter ranging from 0.3 for steep waves to 1.0 for small waves (see Chappelear, 1961; Dean, 1965). This procedure is then repeated until F^{j+1} is acceptably small; this usually takes about 15–20 iterations to achieve values of $F/H\sim0.00002$.

In order to start the solution procedure, trial values of the $X(n)^1(j=1)$ are necessary. These can be chosen simply on the basis of small-amplitude theory in the absence of current. The wavelength $L^1[=X^1(N+1)]$ and $C^1(=L^1/T)$ can be found using the small-amplitude Airy wave theory dispersion relationship

$$L^1=\frac{g}{2\pi}T^2\tanh\left(\frac{2\pi}{L^1}h\right). \quad (21)$$

The $X(N+2)[=\psi(x,\eta)]$ can be approximated as

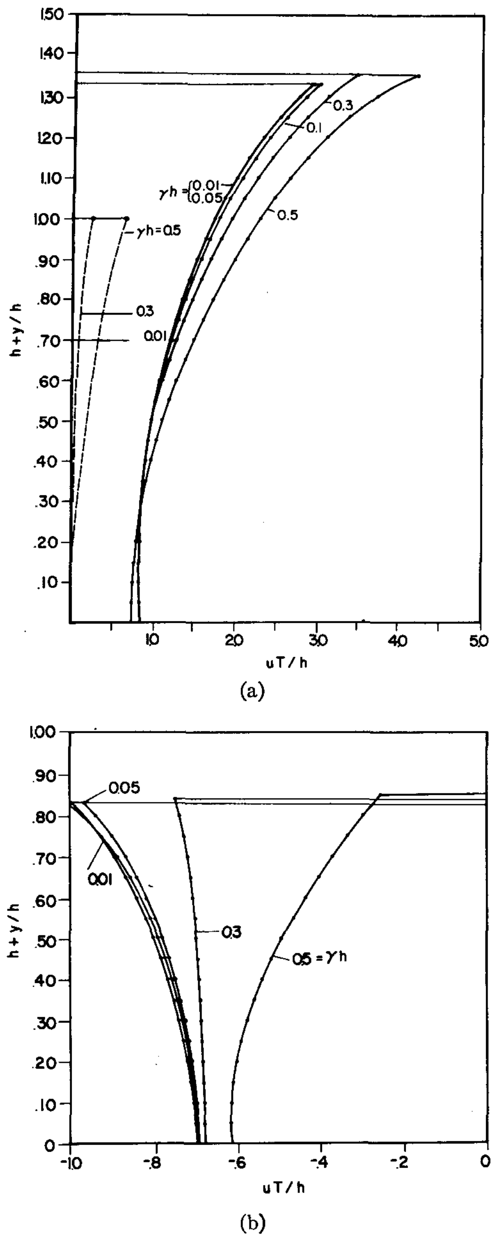

FIG. 2. Dimensionless horizontal velocity profiles (solid lines) for the (a) crest and (b) trough positions for selected values of γh for the deep water wave example, as seen by a stationary observer. The current is flowing in the same direction as the wave and is denoted by dashed lines in (a).

$\psi(x,0)$ using (10) or (14) neglecting the series term and the $X^1(1)$ from first order Stokes theory:

$$X(1) = -\frac{HC^1}{2\sinh\left(\frac{2\pi}{L^1}h\right)}. \tag{22}$$

The remaining coefficients $[X^1(n),\ 2 \leqslant n \leqslant N]$ are set equal to zero.

4. Results

The presence of a mean current affects the wavelength, the free surface profile and, of course, the water particle kinematics within the wave. Note that a wave propagating from a quiescent fluid onto a current will undergo wave height changes, as first explored by Unna (1942) and in detail by Longuet-Higgins and Stewart (1961) among others, but in the present case the wave height is fixed on the current and the wave height transformation not considered. Two example waves have been chosen to illustrate

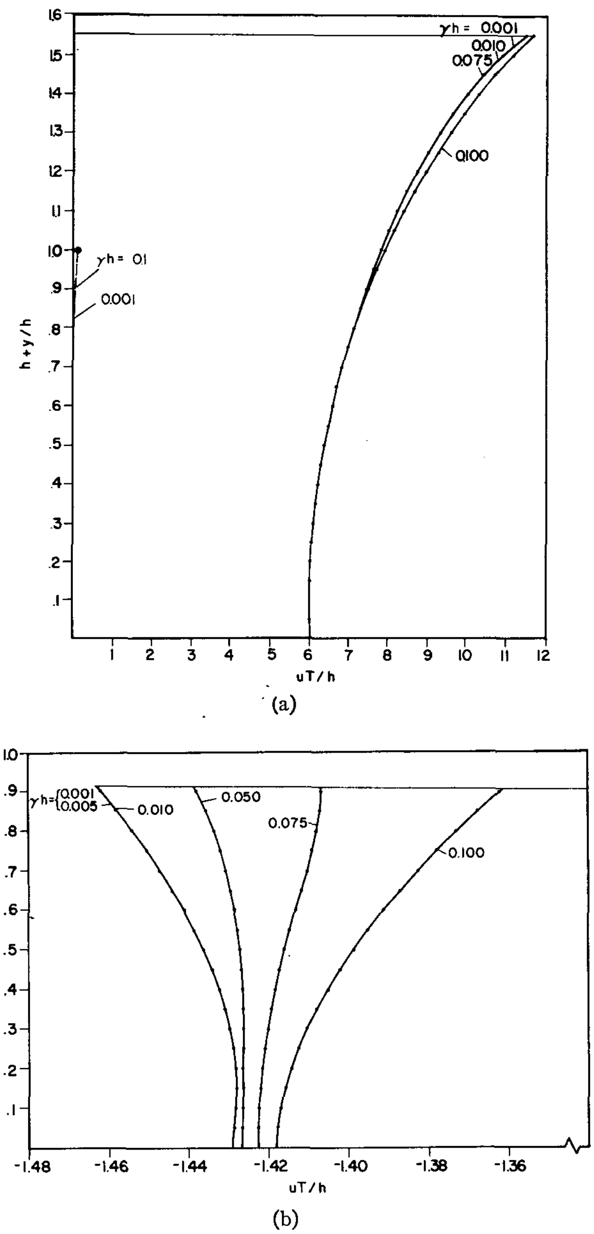

FIG. 3. As in Fig. 2 except for the shallow water wave example. Note change in scale of abscissa for (b).

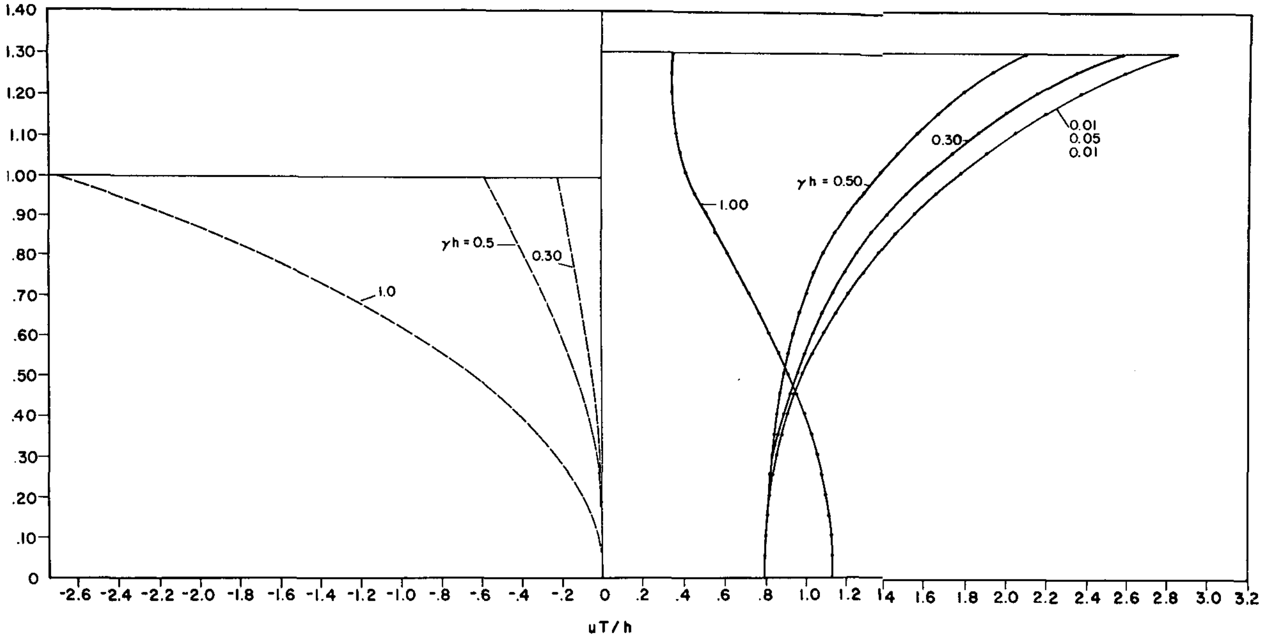

FIG. 4. As in Fig. 2 except for the deep-water wave example with the steady current (dashed lines) flowing in the direction opposite to the wave.

the effects of the currents. These are a deep-water wave with a height of 15.2 m in 30.5 m of water and a shallow-water wave with a height of 1.92 m in 3.05 m of water; both waves have a period of 10 s and seven terms were included in the series solution.

In Figs. 2 and 3 the horizontal velocity profile, consisting of $U+u$, is shown for the crest and trough portions for both the deep- and shallow-water waves for aiding currents of different vorticities. For increasing values of the vorticity parameter γ, the effects of the current become more pronounced. In all of these cases U_B was taken as zero; therefore, the mean current is expressed as

$$U(y)=C[1-\cos\gamma(h+y)]. \tag{23}$$

For very small values of γ, $U(y)$ is nearly zero; that is, the solution and solution technique reduce to the streamfunction theory of Dean (1965). As γ increases, so does the current and, therefore, the surface current $U(0)$ is

$$U(0)=C(1-\cos\gamma h). \tag{24}$$

It is unfortunate that $U(0)$ is dependent on C as the wave celerity is *a priori* unknown, and, therefore, only after the solution is obtained is the exact current profile known.

For the opposing current, Fig. 4 shows the horizontal velocity profile obtained under the wave crest for the deep-water wave. For the case of $\gamma h=1.0$, there is a very strong opposing current with dimensionless magnitude of $U(0)/C=(1-\cosh\gamma h)=-0.54$. The effect of this strong flow is nearly to cancel out the crest velocities in the direction of the wave near the surface, while near the bottom, where the current is smaller in magnitude, the wave-induced motion is greater.

In addition to the change in horizontal velocity profiles, total accelerations within the fluid are changed

TABLE 1. Wavelengths and crest elevations for example waves.

		Aiding current			Opposing current		
	γh	L/H	η_C/H	$U(0)/C$	L/H	η_C/H	$U(0)/C$
Deep-water wave	0.010	10.05	0.66	5×10^{-5}	10.08	0.66	5×10^{-5}
	0.050	10.09	0.67	0.001	10.07	0.66	−0.001
	0.010	10.12	0.67	0.005	10.04	0.66	−0.005
	0.030	10.44	0.68	0.045	9.77	0.65	−0.045
	0.050	11.11	0.71	0.122	9.35	0.64	−0.128
Shallow-water wave	0.001	32.73	0.8626	5×10^{-7}	32.73	0.8626	-5×10^{-7}
	0.010	32.73	0.8626	5×10^{-5}	32.73	0.8626	-5×10^{-5}
	0.075	32.77	0.8630	0.003	32.68	0.8622	−0.003
	0.100	32.80	0.8633	0.005	32.66	0.8617	−0.005

by the current, as well as the wavelength, the crest elevation and all other wave parameters not prescribed as given. Table 1 illustrates the changes in wavelength and crest elevation for both aiding and opposing currents for the example waves. The wave crest elevation η_C is defined as

$$\eta_C = \eta(0)$$

and is of importance in the design of freeboard. Note that only for small-amplitude waves is $\eta_C = H/2$. For the deep-water wave, over the range of $\gamma h = 0.010$ to 0.50 for the aiding current, there is just a 7% change in η_C, while there is a 10% increase in wavelength.

Although not illustrated in this paper, the governing vorticity equation for the fluid is valid even if there is a uniform transverse flow (in the direction of the wave crest) as noted by Benny (1966). Therefore, with suitable modification of the dynamic free surface boundary condition (see Dalrymple, 1973) a cross flow can be accommodated for these wave models. The resulting horizontal velocity profiles, for any transverse flow, will show a changing direction as well as magnitude over the water depth.

5. Conclusions

The water wave models, including opposing and aiding currents, are analytically valid, satisfying all boundary conditions exactly with the exception of the dynamic free surface boundary condition, which is satisfied in a least-squares sense. The root-mean-square error in satisfying this remaining boundary condition is typically of the order of 0.0001 H for these examples.

The effects of the mean current is to change all the properties of the waves, except wave height and period which are fixed. Aiding currents have been shown to increase the horizontal velocities under the wave crest and to increase wavelength and the crest elevation, while an opposing current has the opposite effect.

The importance of these changes occurs in the design of offshore structures where small percentage changes in horizontal velocities result in twice as large changes in drag forces on structures. Further, the deck elevation of structures is dependent on knowledge of crest elevation.

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