J. Japan Statist. Soc.
Vol. 35 No. 1 2005 61–71

A SKEWED TRUNCATED PEARSON TYPE VII DISTRIBUTION

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Skewed symmetric distributions have attracted a great deal of attention in the last few years. One of them, the *skewed Pearson type VII distribution* suffers from limited applicability because it is well known that the Pearson type VII distribution does not have finite moments of all orders. This note proposes an alternative referred to as *skewed truncated Pearson type VII distribution* and defined by the pdf $f(x) = 2g(x)G(\lambda x)$, where $g(\cdot)$ and $G(\cdot)$ are taken, respectively, to be the pdf and the cdf of a truncated Pearson type VII distribution. This distribution possesses finite moments of all orders and could therefore be a better model for certain practical situations. Two such situations are discussed. The note also derives various properties of the distribution, including its moments.

Key words and phrases: Gauss hypergeometric function, skewed truncated Pearson type VII distribution, truncated Pearson type VII distribution.

1. Introduction

Skewed symmetric distributions have attracted a great deal of attention in the last few years. Most notable is the work by Professor A. K. Gupta and his colleagues (see, for example, Gupta and Chen (2001), where goodness-of-fit tests for the skew normal distribution are proposed; Gupta and Huang (2002), where a characterization of the skew normal as well as relevant results on quadratic and linear forms are given; Gupta *et al.* (2002), where properties of the skew normal, skew uniform, skew t, skew Cauchy, skew Laplace, and the skew logistic distributions are explored; and, Gupta (2003), Gupta and Chang (2003), where some multivariate skew symmetric distributions are studied). One of the well known skew symmetric distributions is the *skewed Pearson type VII distribution* given by the probability density function (pdf):

$$f(x) = 2h(x)H(\gamma x) \tag{1.1}$$

for $-\infty < x < \infty$, where $-\infty < \gamma < \infty$, $m > 0$, $N > 1$, $h(\cdot)$ is the pdf of the Pearson type VII distribution given by:

$$h(x) = \frac{\Gamma(N-1/2)}{\sqrt{m\pi}\Gamma(N-1)} \left(1 + \frac{x^2}{m}\right)^{1/2-N}, \tag{1.2}$$

Received April 28, 2004. Revised July 12, 2004. Accepted October 9, 2004.

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and $H(\cdot)$ is the corresponding cumulative distribution function (cdf). This distribution, however, suffers from limited applicability because it is well known that the Pearson type VII distribution does not have finite moments of all orders (see, for example, Johnson *et al.* (1995)). In this note, we propose an alternative to (1.1) which overcomes this weakness. We refer to it as the *skewed truncated Pearson type VII distribution*. It is defined as follows: consider a truncated version of the standard Pearson type VII distribution with the pdf and cdf specified by

$$g(x) = \frac{h(x)}{D} \tag{1.3}$$

and

$$G(x) = \frac{1}{D}\{H(x) + H(A) - 1\}, \tag{1.4}$$

respectively, for $-A \leq x \leq A$ and $A > 0$, where $D = 2H(A) - 1$. The cdf H of the Pearson type VII distribution can be expressed by:

$$H(x) = \begin{cases} \frac{1}{2} I_{m/(m+x^2)}\left(N-1, \frac{1}{2}\right), & \text{if} \quad x \leq 0, \\ 1 - \frac{1}{2} I_{m/(m+x^2)}\left(N-1, \frac{1}{2}\right), & \text{if} \quad x > 0, \end{cases}$$

where $I_x(a, b)$ denotes the incomplete beta function ratio defined by

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x w^{a-1}(1-w)^{b-1} dw. \tag{1.5}$$

Thus, one can express the difference $D = 2H(A) - 1$ as

$$D = \begin{cases} I_{m/(m+A^2)}\left(N-1, \frac{1}{2}\right) - 1, & \text{if} \quad A \leq 0, \\ 1 - I_{m/(m+A^2)}\left(N-1, \frac{1}{2}\right), & \text{if} \quad A > 0. \end{cases}$$

Because (1.3) is defined over a finite interval, the truncated Pearson type VII distribution has all its moments. Following the usual definition of skew symmetric distributions (see, for example, Gupta *et al.* (2002)), we define a random variable X to have the skewed truncated Pearson type VII distribution if its pdf is given by

$$f(x) = 2g(x)G(\gamma x), \tag{1.6}$$

where $-A \leq x \leq A$. We assume without loss of generality that $\gamma \geq 0$ in (1.6) since the corresponding properties for $\gamma < 0$ can be obtained using the fact $G(\gamma x) = 1 - G(-\gamma x)$. It follows from (1.3), (1.4) and (1.6) that the pdf of X is

$$f(x) = \frac{2}{D^2} \{H(\gamma x) + H(A) - 1\} h(x) \tag{1.7}$$

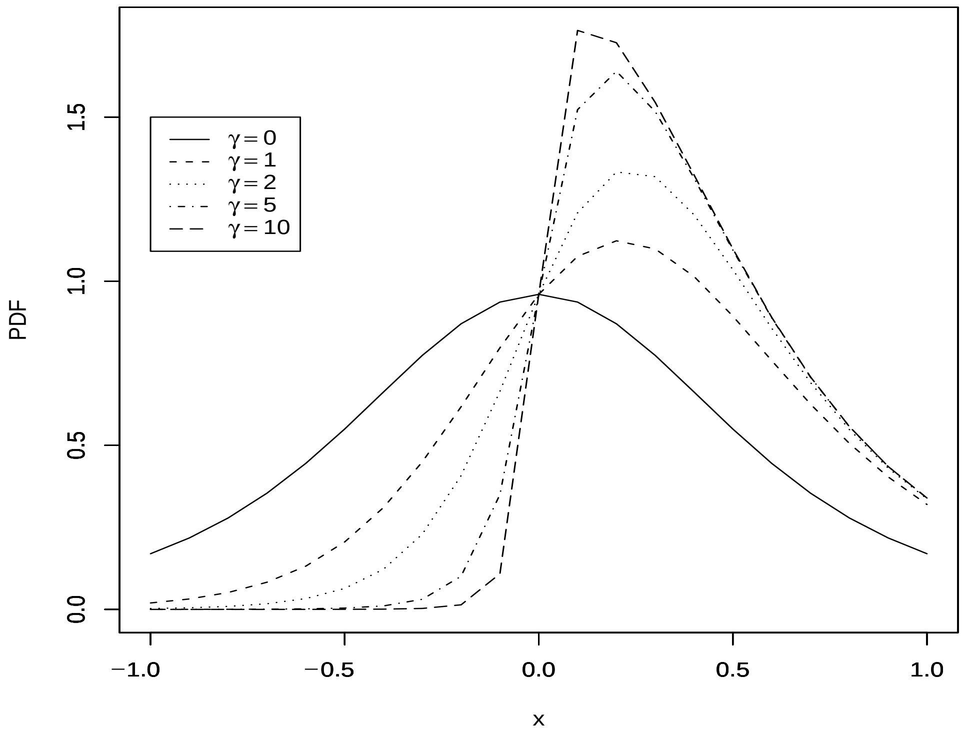

Figure 1. The skewed truncated Pearson type VII pdf (1.7) for $\gamma = 0, 1, 2, 5, 10$, $A = 1$, $m = 1$ and $N = 3$.

for $-A \leq x \leq A$. When $\gamma = 0$, (1.7) reduces to the truncated Pearson type VII pdf (1.3). When $\nu = 1$, (1.7) reduces to a skewed truncated Cauchy pdf (1.3). When $N = 1 + \nu/2$ and $m = \nu$, (1.7) reduces to a skewed truncated t distribution with degrees of freedom ν. Figure 1 below illustrates the shape of the pdf (1.7) for a range of values of γ.

The pdf (1.7) has all its moments and could therefore be a better model for practical situations than one based on just the skewed Pearson type VII distribution. Two such situations are discussed in Section 3. Section 2 provides various representations for the moments of (1.7). The calculations use the following important lemma:

Lemma 1. (equation (3.194.1), Gradshteyn and Ryzhik, 2000) *For* $\mu > 0$,

$$\int_0^u \frac{x^{\mu-1}}{(1+\beta x)^\nu} dx = \frac{u^\mu}{\mu} {}_2F_1(\nu, \mu; 1+\mu; -\beta u),$$

where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$$

denotes the Gauss hypergeometric function and $(z)_k = z(z+1)\cdots(z+k-1)$ *denotes the ascending factorial.*

The properties of the Gauss hypergeometric function can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2. Moments

Theorem 1 provides an expression for $E(X^n)$ for n even. The expression involves the Gauss hypergeometric function.

THEOREM 1. *If X has the pdf* (1.3) *with then*

$$\begin{aligned} E(X^n) &= \frac{2\Gamma(N-1/2)}{(n+1)\sqrt{m\pi}\Gamma(N-1)D} A^{n+1} \\ &\quad \times {}_2F_1\left(N-\frac{1}{2}, \frac{n+1}{2}; \frac{n+3}{2}; -\frac{A^2}{m}\right) \end{aligned} \tag{2.1}$$

for even integers $n \geq 2$.

PROOF. By Lemma 2 in Gupta *et al.* (2002), the nth even order moment of X is the same as the nth moment of (1.3). The latter can be rewritten as:

$$\begin{aligned} E(X^n) &= \frac{\Gamma(N-1/2)}{\sqrt{m\pi}\Gamma(N-1)D} \int_{-A}^{A} x^n \left(1+\frac{x^2}{m}\right)^{1/2-N} dx \\ &= \frac{2\Gamma(N-1/2)}{\sqrt{m\pi}\Gamma(N-1)D} \int_{0}^{A} x^n \left(1+\frac{x^2}{m}\right)^{1/2-N} dx \\ &= \frac{\Gamma(N-1/2)}{\sqrt{m\pi}\Gamma(N-1)D} \int_{0}^{A^2} y^{(n-1)/2} \left(1+\frac{y}{m}\right)^{1/2-N} dy. \end{aligned} \tag{2.2}$$

By application of Lemma 1, the integral in (2.2) can be calculated as

$$\begin{aligned} &\int_{0}^{A^2} y^{(n-1)/2} \left(1+\frac{y}{m}\right)^{1/2-N} dy \\ &\quad = \frac{2A^{n+1}}{n+1} {}_2F_1\left(N-\frac{1}{2}, \frac{n+1}{2}; \frac{n+3}{2}; -\frac{A^2}{m}\right). \end{aligned} \tag{2.3}$$

The result in (2.1) follows by combining (2.2) and (2.3).

Corollaries 1 to 6 provide simpler and explicit expressions for the first five even order moments.

COROLLARY 1. *If X has the pdf* (1.7) *with $N = 2$ then its first five even order moments are*:

$$\begin{aligned} E(X^2) &= m\{\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m}) - A\}/(\sqrt{m+A^2}D), \\ E(X^4) &= m\{A^3 + 3mA - 3m\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\}/(2\sqrt{m+A^2}D), \\ E(X^6) &= m\{15m^2\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m}) \\ &\qquad +2A^5 - 5mA^3 - 15m^2A\}/(8\sqrt{m+A^2}D), \\ E(X^8) &= m\{8A^7 - 14mA^5 + 35m^2A^3 + 105m^3A \end{aligned}$$

$$-105m^3\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})\}/(48\sqrt{m+A^2}D),$$

$$E(X^{10}) = m\{315m^4\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m}) + 16A^9$$
$$-24mA^7 + 42m^2A^5 - 105m^3A^3 - 315m^4A\}/(128\sqrt{m+A^2}D).$$

Proof. Set $N = 2$ and $n = 2, 4, \ldots, 10$ into (2.1) and use properties of the Gauss hypergeometric function.

Corollary 2. *If X has the pdf* (1.7) *with $N = 3$ then its first five even order moments are*:

$$E(X^2) = mA^3/\{2(m+A^2)^{3/2}D\},$$
$$E(X^4) = m^2\{3m\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$+3\sqrt{m+A^2}A^2\text{arcsinh}(A/\sqrt{m}) - 4A^3 - 3mA\}/\{2(m+A^2)^{3/2}D\},$$
$$E(X^6) = m^2\{3A^5 + 20mA^3 + 15m^2A$$
$$-15m^2\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$-15m\sqrt{m+A^2}A^2\text{arcsinh}(A/\sqrt{m})\}/\{4(m+A^2)^{3/2}D\},$$
$$E(X^8) = m^2\{105m^3\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$+105m^2\sqrt{m+A^2}A^2\text{arcsinh}(A/\sqrt{m}) + 6A^7 - 21mA^5$$
$$-140m^2A^3 - 105m^3A\}/\{16(m+A^2)^{3/2}D\},$$
$$E(X^{10}) = m^2\{8A^9 - 18mA^7 + 63m^2A^5 + 420m^3A^3 + 315m^4A$$
$$-315m^4\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$-315m^3\sqrt{m+A^2}A^2\text{arcsinh}(A/\sqrt{m})\}/\{32(m+A^2)^{3/2}D\}.$$

Proof. Set $N = 3$ and $n = 2, 4, \ldots, 10$ into (2.1) and use properties of the Gauss hypergeometric function.

Corollary 3. *If X has the pdf* (1.7) *with $N = 4$ then its first five even order moments are*:

$$E(X^2) = mA^3\{5m + 2A^2\}/\{8(m+A^2)^{5/2}D\},$$
$$E(X^4) = 3m^2A^5/\{8(m+A^2)^{5/2}D\},$$
$$E(X^6) = m^3\{15m^2\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$+30mA^2\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$+15A^4\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$-23A^5 - 35mA^3 - 15m^2A\}/\{8(m+A^2)^{5/2}D\},$$
$$E(X^8) = m^3\{15A^7 + 161mA^5 + 245m^2A^3 + 105m^3A$$
$$-105m^3\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$-210m^2A^2\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})$$
$$-105mA^4\sqrt{m+A^2}\text{arcsinh}(A/\sqrt{m})\}/\{16(m+A^2)^{5/2}D\},$$

$$
\begin{aligned}
E(X^{10}) = 3m^3\{&315m^4\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&+630m^3A^2\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&+315m^2A^4\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})+10A^9-45mA^7\\
&-483m^2A^5-735m^3A^3-315m^4A\}/\{64(m+A^2)^{5/2}D\}.
\end{aligned}
$$

PROOF. Set $N=4$ and $n=2,4,\ldots,10$ into (2.1) and use properties of the Gauss hypergeometric function.

COROLLARY 4. *If X has the pdf* (1.7) *with $N=5$ then its first five even order moments are*:

$$
\begin{aligned}
E(X^2) &= mA^3\{35m^2+28mA^2+8A^4\}/\{48(m+A^2)^{7/2}D\},\\
E(X^4) &= m^2A^5\{7m+2A^2\}/\{16(m+A^2)^{7/2}D\},\\
E(X^6) &= 5m^3A^7/\{16(m+A^2)^{7/2}D\},\\
E(X^8) &= m^4\{105m^3\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad+315m^2A^2\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad+315mA^4\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad+105A^6\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad-176A^7-406mA^5-350m^2A^3-105m^3A\}/\{48(m+A^2)^{7/2}D\},\\
E(X^{10}) &= m^4\{35A^9+528mA^7+1218m^2A^5+1050m^3A^3+315m^4A\\
&\quad-315m^4\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad-945m^3A^2\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad-945m^2A^4\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad-315mA^6\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\}/\{32(m+A^2)^{7/2}D\}.
\end{aligned}
$$

PROOF. Set $N=5$ and $n=2,4,\ldots,10$ into (2.1) and use properties of the Gauss hypergeometric function.

COROLLARY 5. *If X has the pdf* (1.7) *with $N=6$ then its first five even order moments are*:

$$
\begin{aligned}
E(X^2) &= mA^3\{105m^3+126m^2A^2+72mA^4+16A^6\}/\{128(m+A^2)^{9/2}D\},\\
E(X^4) &= m^2A^5\{63m^2+36mA^2+8A^4\}/\{128(m+A^2)^{9/2}D\},\\
E(X^6) &= 5m^3A^7\{9m+2A^2\}/\{128(m+A^2)^{9/2}D\},\\
E(X^8) &= 35m^4A^9/\{128(m+A^2)^{9/2}D\},\\
E(X^{10}) &= m^5\{315m^4\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad+1260m^3A^2\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad+1890m^2A^4\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad+1260mA^6\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})\\
&\quad+315A^8\sqrt{m+A^2}\mathrm{arcsinh}(A/\sqrt{m})-563A^9-1746mA^7\\
&\quad-2268m^2A^5-1365m^3A^3-315m^4A\}/\{128(m+A^2)^{9/2}D\}.
\end{aligned}
$$

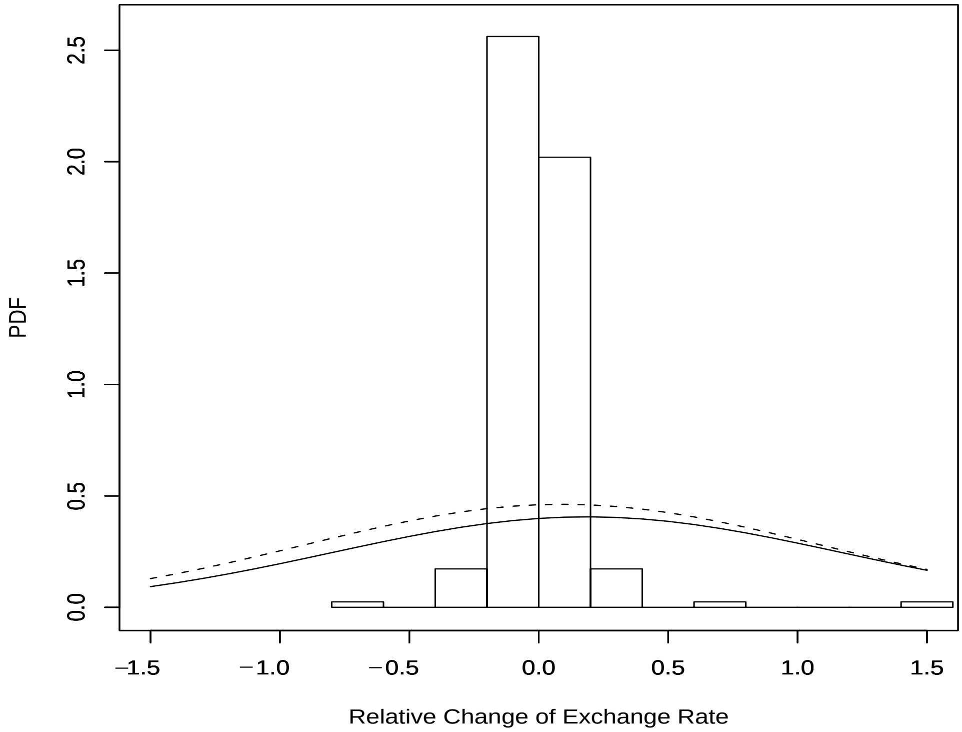

Figure 2. Fits of the skewed t distribution (solid line) and the truncated skewed t distribution (broken line) for the United Kingdom exchange rate data (with $A = 1.5$).

Proof. Set $N = 6$ and $n = 2, 4, \ldots, 10$ into (2.1) and use properties of the Gauss hypergeometric function.

Corollary 6. *If X has the pdf* (1.7) *with $N = 7$ then its first five even order moments are*:

$$
\begin{aligned}
E(X^2) &= mA^3\{1155m^4 + 1848m^3A^2 + 1584m^2A^4 \\
&\qquad\qquad +704mA^6 + 128A^8\}/\{1280(m + A^2)^{11/2}D\}, \\
E(X^4) &= 3m^2A^5\{231m^3 + 198m^2A^2 \\
&\qquad\qquad +88mA^4 + 16A^6\}/\{1280(m + A^2)^{11/2}D\}, \\
E(X^6) &= m^3A^7\{99m^2 + 44mA^2 + 8A^4\}/\{256(m + A^2)^{11/2}D\}, \\
E(X^8) &= 7m^4A^9\{11m + 2A^2\}/\{256(m + A^2)^{11/2}D\}, \\
E(X^{10}) &= 63m^5A^{11}/\{256(m + A^2)^{11/2}D\}.
\end{aligned}
$$

Proof. Set $N = 7$ and $n = 2, 4, \ldots, 10$ into (2.1) and use properties of the Gauss hypergeometric function.

Table 1. Exchange rate data for the United Kingdom Pound.

Year	ER	Year	ER	Year	ER	Year	ER
1800	4.4623	1826	4.9652	1852	4.89	1878	4.875
1801	4.363	1827	4.9432	1853	4.8497	1879	4.845
1802	4.4743	1828	4.8662	1854	4.8123	1880	4.845
1803	4.662	1829	4.8614	1855	4.817	1881	4.85
1804	4.529	1830	4.7281	1856	4.845	1882	4.85
1805	4.3956	1831	4.8757	1857	4.8473	1883	4.855
1806	4.3956	1832	4.8286	1858	4.8709	1884	4.85
1807	4.4623	1833	4.662	1859	4.8638	1885	4.895
1808	4.8828	1834	4.7125	1860	4.6168	1886	4.85
1809	4.4405	1835	4.8403	1861	4.8444	1887	4.87
1810	4.1068	1836	4.7893	1862	6.5025	1888	4.895
1811	3.73	1837	5.0736	1863	7.3914	1889	4.84
1812	3.7736	1838	4.89	1864	11.0905	1890	4.845
1813	3.8956	1839	4.8497	1865	7.0621	1891	4.855
1814	4.4843	1840	4.8239	1866	6.5208	1892	4.885
1815	4.7506	1841	4.845	1867	6.5086	1893	4.885
1816	4.5725	1842	4.7237	1868	6.5701	1894	4.895
1817	4.529	1843	4.8239	1869	5.8583	1895	4.91
1818	4.3403	1844	4.8828	1870	5.3958	1896	4.88
1819	4.529	1845	4.7962	1871	5.3105	1897	4.86
1820	4.6232	1846	4.717	1872	5.4563	1898	4.855
1821	4.9285	1847	4.89	1873	5.3773	1899	4.885
1822	4.985	1848	4.8054	1874	5.4686	1900	4.86
1823	4.7893	1849	4.8239	1875	5.5051	1901	4.875
1824	4.8614	1850	4.8614	1876	5.2132	1902	4.875
1825	4.845	1851	4.9068	1877	5.0062	1903	4.855

3. Applications

In this section, we illustrate two possible applications of the skewed truncated Pearson type VII distribution given by the pdf (1.7).

The Student's t distribution (particular case of (1.2) for $N = 1+\nu/2$ and $m = \nu$) has been applied in the past as models for depth map data, prices of speculative assets such as stock returns, and the phase derivative (random frequency of a narrow band mobile channel) of air components in an urban environment. For data of this kind, there is no reason to believe that empirical moments of any order should be infinite. Thus, the choice of the t distribution or the skewed t distribution as a model is unrealistic since its moments are not finite for all order $n \geq \nu$. The alternative given by (1.7) will be a more appropriate model for the kind of data mentioned. For example, consider the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003. The data—obtained from the web-site http://www.globalfindata.com/—are displayed in the table above.

Following common practice for exchange rate data, we transformed the val-

Table 1. (continued).

Year	ER	Year	ER	Year	ER	Year	ER
1904	4.88	1929	4.8744	1954	2.7856	1979	2.2145
1905	4.856	1930	4.8516	1955	2.8025	1980	2.395
1906	4.84	1931	3.3775	1956	2.7856	1981	1.917
1907	4.8425	1932	3.3275	1957	2.8093	1982	1.618
1908	4.871	1933	5.12	1958	2.8021	1983	1.4515
1909	4.8675	1934	4.9363	1959	2.7995	1984	1.158
1910	4.865	1935	4.93	1960	2.8038	1985	1.439
1911	4.87	1936	4.9088	1961	2.8081	1986	1.4819
1912	4.853	1937	4.9969	1962	2.8025	1987	1.8867
1913	4.855	1938	4.6363	1963	2.7969	1988	1.8089
1914	4.8525	1939	3.9537	1964	2.79	1989	1.611
1915	4.7362	1940	4.035	1965	2.8025	1990	1.932
1916	4.7556	1941	4.0325	1966	2.79	1991	1.865
1917	4.7512	1942	4.0325	1967	2.4067	1992	1.51
1918	4.7581	1943	4.02	1968	2.3849	1993	1.4765
1919	3.75	1944	4.02	1969	2.3989	1994	1.566
1920	3.525	1945	4.025	1970	2.3938	1995	1.55
1921	4.2063	1946	4.025	1971	2.552	1996	1.712
1922	4.6325	1947	4.0331	1972	2.348	1997	1.647
1923	4.3187	1948	4.0319	1973	2.3225	1998	1.6539
1924	4.7225	1949	2.8006	1974	2.347	1999	1.6176
1925	4.8481	1950	2.8013	1975	2.0242	2000	1.4957
1926	4.8481	1951	2.7814	1976	1.7025	2001	1.4541
1927	4.8762	1952	2.8096	1977	1.92	2002	1.611
1928	4.8488	1953	2.8106	1978	2.0435	2003	1.785

ues in the table by first taking logarithms and then computed the relative changes from one year to the next. We then fitted both the skewed t distribution and the skewed truncated t distribution to the transformed data by the method of maximum likelihood. The truncated limit A was chosen as $A = 1.5$. A quasi-Newton algorithm nlm in the R software package (Dennis and Schnabel (1983); Schnabel *et al.* (1985); Ihaka and Gentleman (1996)) was used to solve the likelihood equations. The following estimates were obtained:

$$\hat{\nu} = 124195.3, \qquad \hat{\lambda} = 0.243 \qquad \text{with} \qquad -\log L = 329.4$$

and

$$\hat{\nu} = 124195.3, \qquad \hat{\lambda} = 0.0 \qquad \text{with} \qquad -\log L = 300.3$$

for the two models ($-\log L$ denotes the negative logarithm of the maximized likelihood). Thus, it follows by the standard likelihood ratio test that the skewed truncated t distribution is a much better model for the exchange rate data. The fitted densities for the two models are shown in Figure 2. Similar observations were noted when this exercise was repeated for exchange rate data for the Japanese Yen, Euro, Canadian Dollar, Australian Dollar and the Swiss Franc.

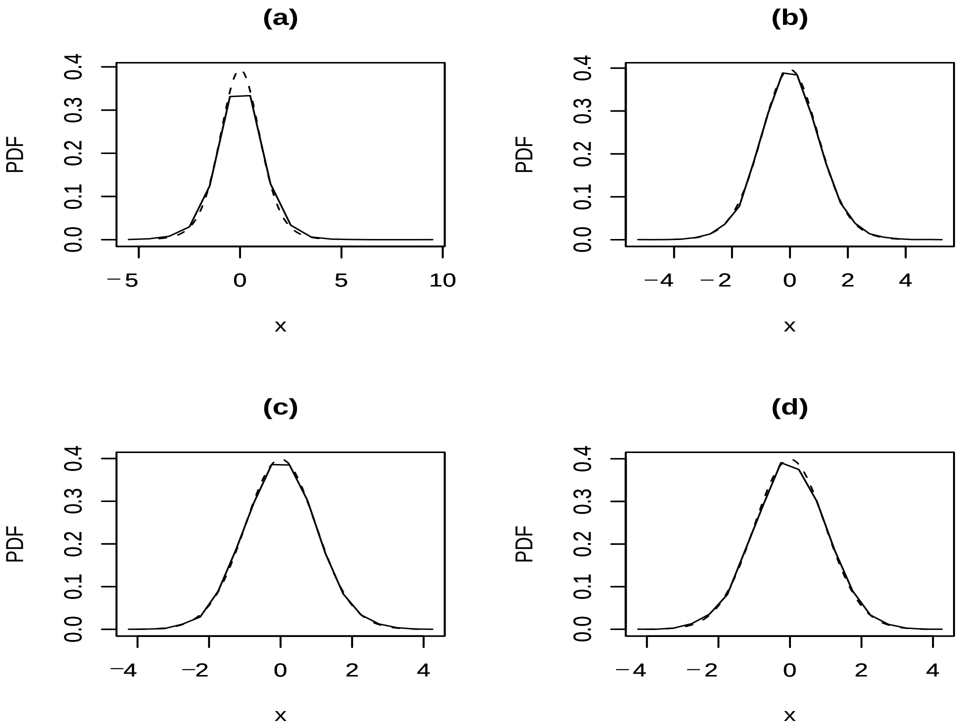

Figure 3. Exact and approximated pdfs (solid and broken lines) of (3.1) for (a): $n = 10$; (b): $n = 20$; (c): $n = 50$; and (d): $n = 100$ when $\bar{x}$ is the sample mean of a random sample of size n from a Beta $(-4, 4)$ distribution.

The second application concerns construction of confidence intervals. Statistical inference about construction of tests and confidence intervals for finite range data has not been well established in the literature. It can be based on the beta distribution; but, the tables and programs for this distribution are not widely available. An alternative and a more pragmatic approach would be to use (1.3). Suppose $x_1, \ldots, x_n$ is a random sample taking values in the range $[-A, A]$ with the population mean of μ. Instead of assuming

$$\frac{\bar{x} - \mu}{s/\sqrt{n}} \tag{3.1}$$

has the Student's t distribution (where $\bar{x}$ is the sample mean and s is the sample standard deviation), assume it follows the truncated version given by (1.3) for $N = 1 + \nu/2$ and $m = \nu$. Then, it follows by usual arguments (see, for example, Rohatgi (1984)) that a $100(1 - \alpha)\%$ confidence interval for μ can be written as

$$\left(\bar{x} - t_{n-1, H(-A)+D\alpha/2} \frac{s}{\sqrt{n}}, \bar{x} + t_{n-1, H(-A)+D(1-\alpha/2)} \frac{s}{\sqrt{n}} \right), \tag{3.2}$$

where $t_{\nu,a}$ denotes the usual $100(1-a)\%$ percentile of the Student's t distribution. Note that this approach requires no additional tables.

The new confidence interval given by (3.2) will be quite robust for large A and large n. This is intuitive because for large A the confidence interval given by (3.2) approximates to the one based on the usual Student's t statistic:

$$\left(\bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + t_{n-1,1-\alpha/2}\frac{s}{\sqrt{n}}\right),$$

which is known to be robust for all large n (see again Rohatgi (1984)). To investigate this, in practice, we performed a simulation study. We considered the distribution of (3.1) when $x_1, \ldots, x_n$ is a random sample from a Beta $(-4, 4)$ distribution. We derived the *exact* distribution of (3.1) by computing its value over 10,000 random samples of size n. We also approximated the distribution of (3.1) by a truncated t distribution with $A = 4$. The exact and the approximated pdfs are compared in Figure 3 for $n = 10, 20, 50, 100$. The fit is not very good when $n = 10$ but it is clear that the approximation is excellent for all the other values of n.

Acknowledgements

The authors would like to thank the referees and the editor for carefully reading the paper and for their great help in improving the paper.

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