

ESTIMATION OF BOUNDED LOCATION AND SCALE PARAMETERS

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This paper addresses the issue of deriving estimators improving on the best location equivariant (or Pitman) estimator under the squared error loss when a location parameter is restricted to a bounded interval. A class of improved estimators is constructed, and it is verified that the Bayes estimator for the uniform prior over the bounded interval and the truncated estimator belong to the class. This paper also obtains the sufficient conditions for the density under which the class includes the Bayes estimators with respect to the two-point boundary symmetric prior and general continuous prior distributions when a symmetric density is considered for the location family. It is demonstrated that the conditions on the symmetric density can be applied to logistic, double exponential and t -distributions as well as to a normal distribution. These conditions can be also applied to scale mixtures of normal distributions. Finally, some similar results are developed in the scale family.

Key words and phrases: Bayes estimator, bounded mean, decision theory, improved estimator, location family, minimaxity, monotone likelihood ratio, Pitman estimator, restricted parameter, scale family, symmetric density, uniform prior.

1. Introduction

The problem of estimating a parameter restricted to a bounded subset has received theoretical attentions in the field of statistical decision theory. In the estimation of mean μ of a normal distribution $\mathcal{N}(\mu, 1)$ under the restriction $A = \{\mu \mid |\mu| \leq m\}$ for $m > 0$, Casella and Strawderman (1981) established that the nonrestricted estimator X , having $\mathcal{N}(\mu, 1)$, is not minimax and showed that the Bayes estimator against the two-point symmetric prior distribution putting mass on the endpoints $\{-m, m\}$, given by

$$\hat{\mu}^{BU} = (me^X - me^{-X})/(e^X + e^{-X}) = m \tanh X,$$

is minimax under the squared error loss if the boundary m satisfies the condition $m \leq 1.0567$. This result was extended by Berry (1990) and Marchand and Perron (2002) to a multivariate normal distribution and by DasGupta (1985) to a general parametric model. Marchand and Perron (2001) demonstrated that $\hat{\mu}^{BU}$ dominates the maximum likelihood estimator $\hat{\mu}^{TR} = (X/|X|) \min(|X|, m)$ if $m \leq 1$, and Marchand and Perron (2005) recently extended this result to a multivariate t -distribution. Although the estimator $\hat{\mu}^{BU}$ is minimax, the condition on the boundary m for the minimaxity is restrictive. An alternative is the

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Bayes estimator against the fully uniform prior over A , given by

$$\hat{\mu}^{FU} = \int_{-m}^m \mu \exp\{-(X - \mu)^2/2\} d\mu \bigg/ \int_{-m}^m \exp\{-(X - \mu)^2/2\} d\mu.$$

Gatsonis *et al.* (1987) proved the dominance result of $\hat{\mu}^{FU}$ over X , and illustrated that it has a favorable risk behavior in comparison with $\hat{\mu}^{BU}$: the risk is slightly higher near zero, but quite a bit smaller near the boundary. Hartigan (2004) provided an interesting method for establishing the dominance based on the Stein identity.

Using the IERD method given by Kubokawa (1994a, 1994b, 1998, 1999), Marchand and Strawderman (2005) recently constructed a broad class of estimators improving on X in the general location family, and demonstrated that the Bayes estimator $\hat{\mu}^{FU}$ against the fully uniform prior over A belongs to the class. These results inspired me to develop further studies about the following queries:

(i) Does the Bayes estimator $\hat{\mu}^{BU}$ against the two-point prior belong to the class of improved estimators given by Marchand and Strawderman (2005)?

(ii) What types of prior distributions of μ produce the Bayes estimators belonging to the class?

(iii) What kinds of conditions on the density in the location family are required to establish the dominance properties of the Bayes estimators over X ? Do similar kinds of dominance properties hold in the scale family?

The objective of this paper is to investigate and answer the above queries. In Section 2, the class of estimators improving on the best location-equivariant estimator $\hat{\mu}_0$ based on a sample with size n in the general location family is given. This is an extension of the result of Marchand and Strawderman (2005) who dealt with the case of a single observation. It is shown that the class includes the Bayes estimator against the fully uniform prior over A and a truncated estimator which corresponds to the maximum likelihood estimator in the case of the single observation. A new and simple estimator shrinking $\hat{\mu}_0$ towards the center of the restricted interval is also derived in the general setup, and it is verified to be superior to $\hat{\mu}_0$.

When we focus on a simple setup, some further studies can be developed and several interesting dominance results can be obtained. In Section 3, we treat the estimation of the location parameter based on a single observation from a symmetric distribution whose density is described by $f(x - \mu)$. Related to the query (ii), we consider the prior distribution with the symmetric density

$$\pi_h^U(\mu) = h(\mu)I(|\mu| \leq m),$$

where $h(\cdot)$ is a nonnegative function defined on the real numbers and $I(|\mu| \leq m)$ is the indicator function, namely, $I(|\mu| \leq m) = 1$ for $|\mu| \leq m$ and $I(|\mu| \leq m) = 0$ for $|\mu| > m$. Then it is proved that the resulting Bayes estimator $\hat{\mu}_h^U$ belongs to the class of improved estimators under the following conditions on the prior $h(\mu)$ and the density $f(x - \mu)$: The conditions on $h(\mu)$ are given as

(a) $h(\mu)$ is nondecreasing in μ for $\mu > 0$ and

(b) $\log h(\mu)$ is symmetric and concave in μ ;

The conditions on $f(x - \mu)$ are described as

(A.1) $f(u)$ is nonincreasing in $u > 0$,

(A.2) $f'(x - \mu)/f(x - \mu) + f'(x + \mu)/f(x + \mu) \leq 0$ for nonnegative x and $\mu \in [0, m]$.

The assumption (A.1) means that the density function is unimodal, and (A.2) is guaranteed if the density has the monotone likelihood ratio property.

As an answer to the query (i), we shall show that the Bayes estimator $\hat{\mu}^{BU}$ against the two-point prior belongs to the class of improved estimators when we assume (A.1), (A.2) and the additional condition that

(A.3) $f'(x - m)/f(x - m) - f'(x + m)/f(x + m) \leq 2/m$ for $x > 0$.

As illustrated in some examples, the condition (A.3) seems to require the restrictive condition on m such that the boundary m is bounded above. Some distributional examples satisfying the assumptions (A.1), (A.2) and (A.3) are presented in Section 3, including logistic, double exponential and t -distributions as well as the normal distribution. We also derive conditions for general normal mixture distributions to satisfy the assumptions. Finally, Section 4 studies some similar dominance results in the scale family and provides an example of a gamma distribution. These answer the query (iii).

It is noted that the same notations are repeatedly used in the paper as long as they are not confusing. Throughout the paper, the notations $\hat{\mu}_0$, $\hat{\mu}^{FU}$ and $\hat{\mu}^{BU}$, respectively, denote the best location-equivariant estimator, the Bayes estimator against the fully uniform prior over the bounded interval, and the Bayes estimator against the boundary uniform prior putting mass on the endpoints.

Finally, we conclude this section with remarks on the dominance problem studied in this paper. Although the paper will derive the conditions for estimators to dominate the best location- or scale-equivariant estimator, it is more important to address the problem of finding Bayes estimators dominating the truncated or maximum likelihood estimator (MLE). This problem was investigated by Marchand and Perron (2001, 2005) for multivariate normal and t -distributions, and general conditions for the dominance over the MLE were derived. In the univariate normal distribution $\mathcal{N}(\mu, 1)$ under the restriction $|\mu| \leq m$, Table 1 of Marchand and Perron (2001) demonstrates that the dominance properties of the Bayes estimators over the MLE are guaranteed restrictively for small m . For example, $\hat{\mu}^{FU}$ has the dominance property for $m \leq 0.523$. Although the dominance of $\hat{\mu}^{FU}$ over the MLE is not guaranteed for $m > 0.523$, $\hat{\mu}^{FU}$ dominates $\hat{\mu}_0$ for any m and has a favorable risk behavior in large part of μ as illustrated in Gatsonis *et al.* (1987). On the other hand, it seems very hard to derive a Bayes estimator dominating the MLE for any m , and such a Bayes estimator has not been developed so far. Taking these facts into account, we consider it meaningful to begin with constructing classes of estimators improving on $\hat{\mu}_0$ for any m , which is the aim of this paper. Based on the results obtained in the paper, we can search for Bayes estimators having good risk behaviors within the classes of the improved estimators. We plan to consider the more difficult issue

of finding the Bayes estimators dominating the truncated ones in a future study.

2. Estimation in the location family

2.1. A class of improved estimators

We consider the estimation of the bounded location parameter in the general location family. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a set of random variables having the density function $f(\mathbf{x} - \mu)$ where $\mathbf{x} - \mu$ means $(x_1 - \mu, \dots, x_n - \mu)$ for scalar μ . Suppose that the location parameter μ is restricted to the bounded interval

$$A = \{\mu \mid a \leq \mu \leq b\} \quad \text{for known real } a \text{ and } b.$$

Estimator $\hat{\mu}$ of μ is evaluated by the risk function $R(\mu, \hat{\mu}) = E[L_\ell(\hat{\mu}, \mu)]$ relative to the squared error loss

$$L_\ell(\hat{\mu}, \mu) = (\hat{\mu} - \mu)^2.$$

The best location-equivariant estimator, called the Pitman estimator, of μ is given by

$$(2.1) \quad \hat{\mu}_0 = \hat{\mu}_0(\mathbf{X}) = \int_{-\infty}^{\infty} \xi f(\mathbf{X} - \xi) d\xi \Big/ \int_{-\infty}^{\infty} f(\mathbf{X} - \xi) d\xi,$$

which is the generalized Bayes estimator against the Lebesgue measure $d\xi$ over real line. To improve the best location-equivariant estimator $\hat{\mu}_0$ by using the restriction A , consider a class of the estimators

$$(2.2) \quad \hat{\mu}_\phi = \hat{\mu}_0(\mathbf{X}) - \phi(\hat{\mu}_0(\mathbf{X}), \mathbf{y}),$$

where $\mathbf{y} = \mathbf{X} - \hat{\mu}_0(\mathbf{X})$, and $\phi(w, \mathbf{y})$ is an absolutely continuous function. In this general location family, a class of estimators improving on $\hat{\mu}_0$ is constructed in the following theorem, which is an extension of the result of Marchand and Strawderman (2005) who addressed the case $n = 1$ and gave a class of estimators improving on $\hat{\mu}_0 = X_1$. The following theorem provides the result in the case of size n and the proof is instructively stated below.

THEOREM 2.1. *Assume that $\phi(w, \mathbf{y})$ is an absolutely continuous function satisfying the following conditions:*

- (a) *There exists a function $c(\mathbf{y})$ such that $\phi(c(\mathbf{y}), \mathbf{y}) = 0$,*
- (b) *$\phi(w, \mathbf{y})$ is nondecreasing in w ,*
- (c) *$\phi(w, \mathbf{y})$ is bounded as*

$$\phi(w, \mathbf{y}) \begin{cases} \leq \phi_{w-b, \infty}(w, \mathbf{y}) & \text{if } w \geq c(\mathbf{y}), \\ \geq \phi_{-\infty, w-a}(w, \mathbf{y}) & \text{if } w < c(\mathbf{y}), \end{cases}$$

where

$$\phi_{w-b, \infty}(w, \mathbf{y}) = \frac{\int_{w-b}^{\infty} u f(\mathbf{y} + u) du}{\int_{w-b}^{\infty} f(\mathbf{y} + u) du}, \quad \phi_{-\infty, w-a}(w, \mathbf{y}) = \frac{\int_{-\infty}^{w-a} u f(\mathbf{y} + u) du}{\int_{-\infty}^{w-a} f(\mathbf{y} + u) du}.$$

Then $\hat{\mu}_\phi$ given by (2.2) dominates the best location-equivariant estimator $\hat{\mu}_0$ relative to the L_ℓ -loss.

PROOF. The IERD method provided by Kubokawa (1994a, 1994b, 1998, 1999) is useful for the proof. The risk difference of the two estimators $\hat{\mu}_0$ and $\hat{\mu}_\phi$ is written by

$$\begin{aligned}\Delta &= R(\mu, \hat{\mu}_0) - R(\mu, \hat{\mu}_\phi) \\ &= E[\{\hat{\mu}_0 - \mu\}^2 - \{\hat{\mu}_0 - \phi(\hat{\mu}_0, \mathbf{y}) - \mu\}^2]\end{aligned}$$

which is, from the condition (a), expressed as

$$\begin{aligned}E[\{\hat{\mu}_0 - \phi(\hat{\mu}_0 + t, \mathbf{y}) - \mu\}^2 \Big|_{t=0}^{c(\mathbf{y}) - \hat{\mu}_0}] \\ &= E \left[\int_0^{c(\mathbf{y}) - \hat{\mu}_0} \frac{d}{dt} \{\hat{\mu}_0 - \phi(\hat{\mu}_0 + t, \mathbf{y}) - \mu\}^2 dt \right] \\ &= -2 \iint_0^{c(\mathbf{y}) - \hat{\mu}_0} \{\hat{\mu}_0 - \phi(\hat{\mu}_0 + t, \mathbf{y}) - \mu\} \phi'(\hat{\mu}_0 + t, \mathbf{y}) f(\mathbf{x} - \mu) dt d\mathbf{x},\end{aligned}$$

where $\phi'(t, \mathbf{y}) = (\partial/\partial t)\phi(t, \mathbf{y})$. By partitioning the space of \mathbf{x} into the two subsets $\{\mathbf{x} \mid \hat{\mu}_0(\mathbf{x}) > c(\mathbf{y})\}$ and $\{\mathbf{x} \mid \hat{\mu}_0(\mathbf{x}) \leq c(\mathbf{y})\}$, the risk difference Δ is written as

$$\begin{aligned}\Delta &= -2 \left\{ \int_{\hat{\mu}_0 > c(\mathbf{y})} + \int_{\hat{\mu}_0 \leq c(\mathbf{y})} \right\} \int_0^{c(\mathbf{y}) - \hat{\mu}_0} \{\hat{\mu}_0 - \phi(\hat{\mu}_0 + t, \mathbf{y}) - \mu\} \\ &\quad \times \phi'(\hat{\mu}_0 + t, \mathbf{y}) f(\mathbf{x} - \mu) dt d\mathbf{x} \\ &= 2 \int_{\hat{\mu}_0 > c(\mathbf{y})} \int_{c(\mathbf{y}) - \hat{\mu}_0}^0 \{\hat{\mu}_0 - \phi(\hat{\mu}_0 + t, \mathbf{y}) - \mu\} \phi'(\hat{\mu}_0 + t, \mathbf{y}) f(\mathbf{x} - \mu) dt d\mathbf{x} \\ &\quad - 2 \int_{\hat{\mu}_0 \leq c(\mathbf{y})} \int_0^{c(\mathbf{y}) - \hat{\mu}_0} \{\hat{\mu}_0 - \phi(\hat{\mu}_0 + t, \mathbf{y}) - \mu\} \phi'(\hat{\mu}_0 + t, \mathbf{y}) f(\mathbf{x} - \mu) dt d\mathbf{x} \\ &= \Delta_1 + \Delta_2. \quad (\text{say}).\end{aligned}$$

We first show that $\Delta_1 \geq 0$. Since the conditions that $c(\mathbf{y}) - \hat{\mu}_0(\mathbf{x}) < t < 0$ and $c(\mathbf{y}) - \hat{\mu}_0(\mathbf{x}) < 0$ are equivalent to the conditions that $-\infty < t < 0$ and $\hat{\mu}_0(\mathbf{x}) - c(\mathbf{y}) > -t$, the quantity Δ_1 is expressed by

$$\Delta_1 = 2 \int_{-\infty}^0 \int_{\hat{\mu}_0 - c(\mathbf{y}) > -t} \{\hat{\mu}_0 - \phi(\hat{\mu}_0 + t, \mathbf{y}) - \mu\} \phi'(\hat{\mu}_0 + t, \mathbf{y}) f(\mathbf{x} - \mu) d\mathbf{x} dt.$$

Note that the estimator $\hat{\mu}_0(\mathbf{x})$ is location-equivariant, that is, $\hat{\mu}_0(\mathbf{x}) + t = \hat{\mu}_0(\mathbf{x} + t)$. Making the transformations $\mathbf{v} = \mathbf{x} + t$ and $u = \hat{\mu}_0(\mathbf{v}) - t - \mu$ in turn with $d\mathbf{v} = d\mathbf{x}$ and $du = -dt$, we can rewrite Δ_1 as

$$\begin{aligned}(2.3) \quad \Delta_1 &= 2 \int_{-\infty}^0 \int_{\hat{\mu}_0(\mathbf{x}+t) - c(\mathbf{y}) > 0} \{\hat{\mu}_0(\mathbf{x}) - \phi(\hat{\mu}_0(\mathbf{x} + t), \mathbf{y}) - \mu\} \\ &\quad \times \phi'(\hat{\mu}_0(\mathbf{x} + t), \mathbf{y}) f(\mathbf{x} - \mu) d\mathbf{x} dt \\ &= 2 \int_{-\infty}^0 \int_{\hat{\mu}_0(\mathbf{v}) - c(\mathbf{y}) > 0} \{\hat{\mu}_0(\mathbf{v}) - t - \phi(\hat{\mu}_0(\mathbf{v}), \mathbf{y}) - \mu\} \\ &\quad \times \phi'(\hat{\mu}_0(\mathbf{v}), \mathbf{y}) f(\mathbf{v} - t - \mu) d\mathbf{v} dt\end{aligned}$$

$$\begin{aligned}
&= 2 \int_{\widehat{\mu}_0(\mathbf{v})-c(\mathbf{y})>0} \int_{-\infty}^0 \{\widehat{\mu}_0(\mathbf{v}) - t - \phi(\widehat{\mu}_0(\mathbf{v}), \mathbf{y}) - \mu\} \\
&\quad \times \phi'(\widehat{\mu}_0(\mathbf{v}), \mathbf{y}) f(\mathbf{v} - t - \mu) dt d\mathbf{v} \\
&= 2 \int_{\widehat{\mu}_0(\mathbf{v})-c(\mathbf{y})>0} \int_{\widehat{\mu}_0(\mathbf{v})-\mu}^{\infty} \{u - \phi(\widehat{\mu}_0(\mathbf{v}), \mathbf{y})\} \\
&\quad \times \phi'(\widehat{\mu}_0(\mathbf{v}), \mathbf{y}) f(\mathbf{v} - \widehat{\mu}_0(\mathbf{v}) + u) du d\mathbf{v},
\end{aligned}$$

where $\mathbf{y} = \mathbf{x} - \widehat{\mu}_0(\mathbf{x}) = \mathbf{v} - \widehat{\mu}_0(\mathbf{v})$ since $\widehat{\mu}_0$ is equivariant. From the condition (b), $\phi'(w, \mathbf{y}) \geq 0$, so that Δ_1 is nonnegative if the function $\phi(w, \mathbf{y})$ is bounded above by

$$\phi(w, \mathbf{y}) \leq \int_{w-\mu}^{\infty} u f(\mathbf{y} + u) du \Big/ \int_{w-\mu}^{\infty} f(\mathbf{y} + u) du \quad \text{for any } a \leq \mu \leq b,$$

which is equivalent to

$$\phi(w^* + \mu, \mathbf{y}) \leq \int_{w^*}^{\infty} u f(\mathbf{y} + u) du \Big/ \int_{w^*}^{\infty} f(\mathbf{y} + u) du \quad \text{for any } a \leq \mu \leq b,$$

for $w^* = w - \mu$. Since $\phi(w, \mathbf{y})$ is nondecreasing in w , it follows that $\phi(w^* + \mu, \mathbf{y}) \leq \phi(w^* + b, \mathbf{y})$ for $a \leq \mu \leq b$, so that we get the sufficient condition that

$$\phi(w^* + b, \mathbf{y}) \leq \int_{w^*}^{\infty} u f(\mathbf{y} + u) du \Big/ \int_{w^*}^{\infty} f(\mathbf{y} + u) du,$$

which is rewritten by

$$\phi(w, \mathbf{y}) \leq \int_{w-b}^{\infty} u f(\mathbf{y} + u) du \Big/ \int_{w-b}^{\infty} f(\mathbf{y} + u) du.$$

This condition is guaranteed by the condition (c), and the requirement that $\Delta_1 \geq 0$ is proved.

We next show that $\Delta_2 \geq 0$. By the same arguments as in (2.3), we observe that

$$\begin{aligned}
\Delta_2 &= -2 \int_{\widehat{\mu}_0 \leq c(\mathbf{y})} \int_0^{c(\mathbf{y})-\widehat{\mu}_0} \{\widehat{\mu}_0 - \phi(\widehat{\mu}_0 + t, \mathbf{y}) - \mu\} \phi'(\widehat{\mu}_0 + t, \mathbf{y}) f(\mathbf{x} - \mu) dt d\mathbf{x} \\
&= -2 \int_0^{\infty} \int_{\widehat{\mu}_0(\mathbf{x})-c(\mathbf{y}) \leq -t} \{\widehat{\mu}_0(\mathbf{x}) - \phi(\widehat{\mu}_0(\mathbf{x}) + t, \mathbf{y}) - \mu\} \\
&\quad \times \phi'(\widehat{\mu}_0(\mathbf{x}) + t, \mathbf{y}) f(\mathbf{x} - \mu) d\mathbf{x} dt \\
&= -2 \int_0^{\infty} \int_{\widehat{\mu}_0(\mathbf{v})-c(\mathbf{y}) \leq 0} \{\widehat{\mu}_0(\mathbf{v}) - t - \mu - \phi(\widehat{\mu}_0(\mathbf{v}), \mathbf{y})\} \\
&\quad \times \phi'(\widehat{\mu}_0(\mathbf{v}), \mathbf{y}) f(\mathbf{v} - t - \mu) d\mathbf{v} dt \\
&= -2 \int_{\widehat{\mu}_0(\mathbf{v})-c(\mathbf{y}) \leq 0} \int_{-\infty}^{\widehat{\mu}_0(\mathbf{v})-\mu} \{u - \phi(\widehat{\mu}_0(\mathbf{v}), \mathbf{y})\} \phi'(\widehat{\mu}_0(\mathbf{v}), \mathbf{y}) f(\mathbf{y} + u) du d\mathbf{v}.
\end{aligned}$$

Since $\phi'(w, \mathbf{y}) \geq 0$, it is sufficient to show that

$$\phi(w, \mathbf{y}) \geq \int_{-\infty}^{w-\mu} u f(\mathbf{y} + u) du \Big/ \int_{-\infty}^{w-\mu} f(\mathbf{y} + u) du,$$

which is guaranteed by the condition (c) as verified by the same way as in the case that $\Delta_1 \geq 0$. Therefore, the proof of Theorem 2.1 is complete. \square

The following theorem is useful for showing the dominance property of the typical Bayes estimators introduced in Section 1.

THEOREM 2.2. *Assume that $\phi(w, \mathbf{y})$ is an absolutely continuous function satisfying the following conditions:*

- (a) *There exists a function $c(\mathbf{y})$ such that $\phi(c(\mathbf{y}), \mathbf{y}) = 0$,*
- (b) *$\phi(w, \mathbf{y})$ is nondecreasing in w ,*
- (c) *$\phi(w, \mathbf{y})$ is bounded as*

$$\phi(w, \mathbf{y}) \begin{cases} \leq \phi^U(w, \mathbf{y}) & \text{if } w \geq c(\mathbf{y}), \\ \geq \phi^U(w, \mathbf{y}) & \text{if } w < c(\mathbf{y}), \end{cases}$$

where

$$(2.4) \quad \phi^U(w, \mathbf{y}) = \int_{w-b}^{w-a} uf(\mathbf{y} + u)du \Big/ \int_{w-b}^{w-a} f(\mathbf{y} + u)du.$$

Then $\hat{\mu}_\phi$ given by (2.2) dominates the best location-equivariant estimator $\hat{\mu}_0$ relative to the L_ℓ -loss.

PROOF. From Theorem 2.1, we need to show that

$$(2.5) \quad \phi^U(w, \mathbf{y}) \leq \frac{\int_{w-b}^{\infty} uf(\mathbf{y} + u)du}{\int_{w-b}^{\infty} f(\mathbf{y} + u)du} \quad \text{for } w \geq c(\mathbf{y}),$$

and

$$(2.6) \quad \phi^U(w, \mathbf{y}) \geq \frac{\int_{-\infty}^{w-a} uf(\mathbf{y} + u)du}{\int_{-\infty}^{w-a} f(\mathbf{y} + u)du} \quad \text{for } w < c(\mathbf{y}).$$

To check the inequality (2.5), it is sufficient to show that the function

$$h(s) = \int_{w-b}^s uf(\mathbf{y} + u)du \Big/ \int_{w-b}^s f(\mathbf{y} + u)du$$

is increasing in s . The derivative of $h(s)$ with respect to s is proportional to the quantity that

$$\begin{aligned} sf(\mathbf{y} + s) \int_{w-b}^s f(\mathbf{y} + u)du - \int_{w-b}^s uf(\mathbf{y} + u)du f(\mathbf{y} + s) \\ = f(\mathbf{y} + s) \int_{w-b}^s (s - u)f(\mathbf{y} + u)du, \end{aligned}$$

which is nonnegative, so that $h(s)$ is increasing. Hence, $h(w - a) \leq h(\infty)$, which shows the inequality (2.5). Similarly, we can show the inequality (2.6). \square

2.2. Derivation of improved estimators

Now we derive some estimators improving the best location-equivariant estimator $\hat{\mu}_0$.

[1] **Fully uniform prior Bayes estimator.** Consider the fully uniform prior distribution over the bounded interval, described by

$$\pi^U(\mu) = (b - a)^{-1}d\mu I(a \leq \mu \leq b),$$

where $I(\cdot)$ denotes the indicator function. The resulting Bayes estimator is given by

$$\begin{aligned} (2.7) \quad \hat{\mu}^{FU} &= \int_a^b \xi f(\mathbf{x} - \xi) d\xi \Big/ \int_a^b f(\mathbf{x} - \xi) d\xi \\ &= \hat{\mu}_0 - \int_a^b (\hat{\mu}_0 - \xi) f(\mathbf{x} - \xi) d\xi \Big/ \int_a^b f(\mathbf{x} - \xi) d\xi \\ &= \hat{\mu}_0 - \phi^U(\hat{\mu}_0, \mathbf{y}), \end{aligned}$$

which we here call *the fully uniform prior Bayes estimator*, where $\phi^U(w, \mathbf{y})$ is defined by (2.4). We shall show that the Bayes estimator $\hat{\mu}^{FU}$ belongs to the class provided in Theorem 2.2. The condition (c) is trivially satisfied. The derivative of $\phi^U(w, \mathbf{y})$ with respect to w is proportional to the quantity

$$\begin{aligned} (2.8) \quad &\{(w - a)f(\mathbf{y} + w - a) - (w - b)f(\mathbf{y} + w - b)\} \int_{w-b}^{w-a} f(\mathbf{y} + u) du \\ &- \int_{w-b}^{w-a} u f(\mathbf{y} + u) du \{f(\mathbf{y} + w - a) - f(\mathbf{y} + w - b)\} \\ &= \int_{w-b}^{w-a} f(\mathbf{y} + u) [\{(w - a) - u\} f(\mathbf{y} + w - a) \\ &\quad + \{u - (w - b)\} f(\mathbf{y} + w - b)] du, \end{aligned}$$

which is positive. Thus, $\phi^U(w, \mathbf{y})$ satisfies the condition (b). Noting that $\phi^U(w, \mathbf{y})$ has one sign change from negative to positive, we see that there exists a function $c(\mathbf{y})$ such that $\phi^U(c(\mathbf{y}), \mathbf{y}) = 0$.

PROPOSITION 2.1. *The fully uniform prior Bayes estimator $\hat{\mu}^{FU}$ given by (2.7) dominates the best location-equivariant estimator $\hat{\mu}_0$ relative to the L_ℓ -loss.*

[2] **Truncated estimator.** Every estimator taking values outside the parameter space $a \leq \mu \leq b$ can be improved on by truncating it at the boundary points a and b . Thus, the estimator $\hat{\mu}_0$ is dominated by the truncated estimator

$$\begin{aligned} (2.9) \quad \hat{\mu}^{TR} &= \max\{a, \min\{\hat{\mu}_0, b\}\} \\ &= \hat{\mu}_0 - \phi^{TR}(\hat{\mu}_0, \mathbf{y}), \end{aligned}$$

where

$$\phi^{TR}(w, \mathbf{y}) = \min\{w - a, \max\{w - b, 0\}\}.$$

Noting that $c(\mathbf{y})$ is between a and b , we can see that $\phi^{TR}(w, \mathbf{y})$ satisfies all the conditions in Theorem 2.1, that is, $\hat{\mu}^{TR}$ belongs to the class given in Theorem 2.1.

[3] **Shrinkage estimator.** Let $R_0 = E[(\hat{\mu}_0 - \mu)^2]$ and $B_0 = E[\hat{\mu}_0 - \mu]$, both of which are independent of μ since $\hat{\mu}_0$ is equivariant. Based on R_0 and B_0 , consider a shrinkage estimator of the form

$$(2.10) \quad \hat{\mu}^S = \hat{\mu}_0 - A_0 \left(\hat{\mu}_0 - \frac{a+b}{2} \right),$$

where

$$A_0 = \frac{R_0 - |B_0(a-b)|/2}{R_0 - |B_0(a-b)| + (a-b)^2/4}.$$

The shrinkage constant A_0 satisfies the condition $0 \leq A_0 \leq 1$ when $|B_0| \leq \min(|a-b|/2, 2R_0/|a-b|)$. Then $\hat{\mu}^S$ shrinks $\hat{\mu}_0$ towards the center $(a+b)/2$ of the interval. Although it is not sure that $\hat{\mu}^S$ belongs to the class given in Theorem 2.1, the dominance of $\hat{\mu}^S$ over X can be directly verified.

PROPOSITION 2.2. *If the bias B_0 of the estimator $\hat{\mu}_0$ satisfies the condition*

$$|B_0| \leq \min(|a-b|/2, 2R_0/|a-b|),$$

then the shrinkage estimator $\hat{\mu}^S$ given by (2.10) dominates $\hat{\mu}_0$ relative to the L_ℓ -loss. If $R_0 > |B_0(a-b)|/2$, then the estimator $\hat{\mu}_0$ is not minimax.

PROOF. The risk function of the estimator $\hat{\mu}^S$ is written by

$$\begin{aligned} R(\mu, \hat{\mu}^S) &= E[(\hat{\mu}^S - \mu)^2] \\ &= (1 - A_0)^2 R_0 + A_0^2 \left(\mu - \frac{a+b}{2} \right)^2 - 2A_0(1 - A_0)B_0 \left(\mu - \frac{a+b}{2} \right). \end{aligned}$$

Noting that $-(b-a)/2 \leq \mu - (a+b)/2 \leq (b-a)/2$, we see that $\{\mu - (a+b)/2\}^2 \leq (b-a)^2/4$ and

$$-2B_0 \left(\mu - \frac{a+b}{2} \right) \leq |B_0(a-b)|,$$

which implies that

$$\begin{aligned} R(\mu, \hat{\mu}^S) &\leq (1 - A_0)^2 R_0 + A_0^2 (a-b)^2/4 + A_0(1 - A_0)|B_0(a-b)| \\ &= R_0 - \frac{(R_0 - |B_0(a-b)|/2)^2}{R_0 - |B_0(a-b)| + (a-b)^2/4}. \end{aligned}$$

This shows that the risk of the estimator $\hat{\mu}^S$ is bounded by the constant strictly smaller than R_0 if $R_0 > |B_0(a-b)|/2$, that is, the unrestricted estimator $\hat{\mu}_0$ is not minimax. \square

[4] **Two-point boundary prior Bayes estimator.** Consider the discrete prior distribution putting mass on the endpoints $\{a, b\}$, described by

$$\pi^B(\mu) = pP[\mu = a] + (1 - p)P[\mu = b],$$

where p is a known constant in the interval $[0, 1]$. The resulting Bayes estimator is given by

$$(2.11) \quad \hat{\mu}^B = \frac{pa f(\mathbf{X} - a) + (1 - p)bf(\mathbf{X} - b)}{pf(\mathbf{X} - a) + (1 - p)f(\mathbf{X} - b)} = \hat{\mu}_0 - \phi^B(\hat{\mu}_0, \mathbf{y}),$$

where for $\mathbf{y} = \mathbf{X} - \hat{\mu}_0$,

$$\phi^B(w, \mathbf{y}) = \frac{p(w - a)f(\mathbf{y} + w - a) + (1 - p)(w - b)f(\mathbf{y} + w - b)}{pf(\mathbf{y} + w - a) + (1 - p)f(\mathbf{y} + w - b)}.$$

It is too hard to get conditions under which $\hat{\mu}^B$ belongs to the class of improved estimators provided by Theorems 2.1 or 2.2. For some specific symmetric distributions, we can obtain such conditions as stated in the next section.

3. Examples in symmetric distributions

The improved procedures and the dominance results given in Section 2.2 can be applied to the estimation of the bounded location based on random samples of size n from various distributions, which include non-symmetric distributions such as an exponential distribution. Instead of stating such examples in detail, we here focus our attention on symmetric distributions and develop some interesting dominance properties. In particular, characterizations with respect to the underlying symmetric distributions and prior distributions will be presented to guarantee the dominance of the Bayes estimators in Section 2.

3.1. Symmetric distributions

Let X be a single random variable having the symmetric density $f(x - \mu)$, namely, $f(u) = f(-u)$ for any $u \in \mathbf{R}$, where the location parameter μ is restricted to the bounded interval $A = \{\mu \mid |\mu| \leq m\}$ for a positive constant m . When the estimation of μ is treated under the squared error loss, the best location-equivariant, the truncated and the shrinkage estimators corresponding to (2.1), (2.9) and (2.10) are, respectively, given by $\hat{\mu}_0 = X$, $\hat{\mu}^{TR} = (X/|X|) \min(|X|, m)$ and $\hat{\mu}^S = \{m^2/(R_0 + m^2)\}X$ for $R_0 = \int_{-\infty}^{\infty} u^2 f(u)du$. Also the fully uniform prior Bayes estimator (2.7) is written by

$$\hat{\mu}^{FU} = \int_{-m}^m \mu f(X - \mu) d\mu \Big/ \int_{-m}^m f(X - \mu) d\mu = X - \phi^U(X),$$

where

$$(3.1) \quad \begin{aligned} \phi^U(x) &= \int_{-m}^m (x - \mu) f(x - \mu) d\mu \Big/ \int_{-m}^m f(x - \mu) d\mu \\ &= \frac{\int_0^m \{(x - \mu) f(x - \mu) + (x + \mu) f(x + \mu)\} d\mu}{\int_0^m \{f(x - \mu) + f(x + \mu)\} d\mu}. \end{aligned}$$

From the results in Section 2, the estimators $\hat{\mu}^{TR}$, $\hat{\mu}^S$ and $\hat{\mu}^{FU}$ dominate $\hat{\mu}_0 = X$.

The dominance property of the Bayes estimator $\hat{\mu}^{FU}$ can be extended to more general priors described by

$$\pi_h^U(\mu) = h(\mu)I(|\mu| \leq m),$$

where $h(\mu)$ is nonnegative and symmetric about zero, that is, $h(\mu) = h(-\mu)$ for $\mu > 0$. The resulting Bayes estimator is

$$\hat{\mu}_h^U = \int_{-m}^m \xi h(\mu) f(X - \mu) d\mu \bigg/ \int_{-m}^m h(\mu) f(X - \mu) d\mu = X - \phi_h^U(X),$$

where

$$(3.2) \quad \begin{aligned} \phi_h^U(x) &= \int_{-m}^m (x - \mu) h(\mu) f(x - \mu) d\mu \bigg/ \int_{-m}^m h(\mu) f(x - \mu) d\mu \\ &= \frac{\int_0^m \{(x - \mu) h(\mu) f(x - \mu) + (x + \mu) h(\mu) f(x + \mu)\} d\mu}{\int_0^m \{h(\mu) f(x - \mu) + h(\mu) f(x + \mu)\} d\mu}. \end{aligned}$$

To establish the dominance result, we need the following assumptions for the density $f(x - \mu)$.

(A.1) $f(u)$ is nonincreasing in $u > 0$.

(A.2) For the derivative $f'(u) = (d/du)f(u)$ and nonnegative x and $\mu \in [0, m]$,

$$(3.3) \quad f'(x - \mu)/f(x - \mu) + f'(x + \mu)/f(x + \mu) \leq 0.$$

The assumption (A.1) means that the density is unimodal. The assumption (A.2) is guaranteed under the assumption (A.1) if

(A.2') $(d/du) \log f(u) = f'(u)/f(u)$ is nonincreasing in u for $0 < u \leq 2m$, which is satisfied if the density $f(x - \mu)$ has the monotone likelihood ratio property. In fact, in the case that $x - \mu \geq 0$ for $x > 0$ and $\mu \in [0, m]$, the inequality (3.3) follows from the assumption (A.1). In the case that $x - \mu < 0$ or $x < \mu$, on the other hand, note that $0 < \mu - x < x + \mu < 2\mu \leq 2m$ since $0 < x < \mu \leq m$. Also note that

$$f'(x - \mu)/f(x - \mu) = -f'(\mu - x)/f(\mu - x),$$

since $f'(u)/f(u)$ is an odd function. Then from the monotonicity in (A.2'), it is seen that $-f'(\mu - x)/f(\mu - x) + f'(x + \mu)/f(x + \mu)$ is not positive.

Under these assumptions, we shall prove the monotonicity of the function

$$(3.4) \quad G(t, x) = \frac{(x - t)f(x - t) + (x + t)f(x + t)}{f(x - t) + f(x + t)},$$

which will be very useful for developing dominance results.

LEMMA 3.1. *Assume that the symmetric density $f(x - \mu)$ satisfies the conditions (A.1) and (A.2). Then $G(t, x)$ is nonincreasing in t for $0 < t < m$ and $x > 0$.*

PROOF. The derivative $(d/dt)G(t, x)$ is proportional to the quantity

$$\begin{aligned} & \{-f(x-t) + f(x+t) - (x-t)f'(x-t) + (x+t)f'(x+t)\} \\ & \quad \times \{f(x-t) + f(x+t)\} \\ & \quad - \{(x-t)f(x-t) + (x+t)f(x+t)\} \times \{-f'(x-t) + f'(x+t)\} \\ & = \{f(x+t) - f(x-t)\} \times \{f(x+t) + f(x-t)\} \\ & \quad + 2t\{f'(x-t)f(x+t) + f'(x+t)f(x-t)\}, \end{aligned}$$

which is not positive from the assumptions (A.1) and (A.2). Hence $G(t, x)$ is nonincreasing in t for $0 < t < m$. \square

PROPOSITION 3.1. *Assume that the symmetric density $f(x - \mu)$ satisfies the assumptions (A.1) and (A.2). If the nonnegative function $h(t)$ satisfies the following conditions:*

(a) *$h(t)$ is nondecreasing in t for $0 < t < m$,*

(b) *$\log h(t)$ is symmetric and concave,*

then the Bayes estimator $\hat{\mu}_h^U$ dominates X under the restriction A relative to the squared error loss.

PROOF. All the conditions in Theorem 2.2 will be checked. The condition (a) is clearly satisfied since $\phi_h^U(0) = 0$. Since $\phi_h^U(-x) = -\phi_h^U(x)$ and $\phi^U(-x) = -\phi^U(x)$ for $x > 0$, for the condition (c), it is sufficient to show that for $x > 0$,

$$\phi_h^U(x) \leq \phi^U(x),$$

which is, from (3.2), written by

$$(3.5) \quad \frac{\int_0^m h(t)F_2(t, x)dt}{\int_0^m h(t)F_1(t, x)dt} \leq \frac{\int_0^m F_2(t, x)dt}{\int_0^m F_1(t, x)dt},$$

where $F_1(t, x) = f(x-t) + f(x+t)$ and $F_2(t, x) = (x-t)f(x-t) + (x+t)f(x+t)$. The inequality (3.5) can be expressed by

$$(3.6) \quad E^\# \left[h(t) \frac{F_2(t, x)}{F_1(t, x)} \right] \leq E^\# [h(t)] \times E^\# \left[\frac{F_2(t, x)}{F_1(t, x)} \right],$$

where $E^\#[\cdot]$ denotes the expectation with respect to the probability density function of t given by $F_1(t, x)/\int_0^m F_1(t, x)dt$. Note that $F_2(t, x)/F_1(t, x) = G(t, x)$ for $G(t, x)$ defined by (3.4). From Lemma 3.1, $G(t, x)$ is nonincreasing in t . Also from the condition (a), the function $h(t)$ is nondecreasing. We thus get the inequality (3.6), which means the condition (c) is satisfied.

To check the condition (b), we shall show that $(d/dx)\phi_h^U(x) \geq 0$. Since $\phi_h^U(x)$ in (3.2) is rewritten by

$$\phi_h^U(x) = \int_{x-m}^{x+m} uh(x-u)f(u)du \Big/ \int_{x-m}^{x+m} h(x-u)f(u)du,$$

the derivative $(d/dx)\phi_h^U(x)$ is proportional to

$$\begin{aligned} & \left[\{(x+m)f(x+m) - (x-m)f(x-m)\}h(m) \int_{x-m}^{x+m} h(x-u)f(u)du \right. \\ & \quad \left. - \{f(x+m) - f(x-m)\}h(m) \int_{x-m}^{x+m} uh(x-u)f(u)du \right] \\ & + \left[\int_{x-m}^{x+m} uh'(x-u)f(u)du \int_{x-m}^{x+m} h(x-u)f(u)du \right. \\ & \quad \left. - \int_{x-m}^{x+m} uh(x-u)f(u)du \int_{x-m}^{x+m} h'(x-u)f(u)du \right] \\ & = I_1 + I_2, \quad (\text{say}). \end{aligned}$$

The same argument as in (2.8) shows that $I_1 \geq 0$. On the other hand, the inequality $I_2 \geq 0$ is expressed by

$$(3.7) \quad E^{**} \left[u \frac{h'(x-u)}{h(x-u)} \right] \geq E^{**}[u] \times E^{**} \left[\frac{h'(x-u)}{h(x-u)} \right],$$

where $E^{**}[\cdot]$ denotes the expectation with respect to the probability density function of u on $[x-m, x+m]$ given by $h(x-u)f(u)/\int_{x-m}^{x+m} h(x-u)f(u)du$. From the condition (b) of Proposition 3.1, $h'(x-u)/h(x-u)$ is nondecreasing in u , so that the inequality (3.7) holds, and it is seen that $I_2 \geq 0$. Therefore, the condition (b) of Theorem 2.2 is satisfied. \square

Instead of finding a function $h(\mu)$ satisfying the conditions of Proposition 3.1, we can consider the form

$$h(\mu) = \exp\{-k(\mu)\},$$

and the conditions on $h(\mu)$ in Proposition 3.1 are replaced with

- (a') $k(\mu)$ is nonincreasing for $\mu > 0$,
- (b') $k(\mu)$ is symmetric and convex.

For example, let us consider the form $k_1(\mu) = a_0/(1+|\mu|)^\ell$ for positive constants a_0 and ℓ . It is easily seen that $k_1(\mu)$ satisfies the conditions (a') and (b'), so that the function

$$h_1(\mu) = \exp \left\{ -\frac{a_0}{(1+|\mu|)^\ell} \right\}$$

is an example of $h(\mu)$ satisfying the conditions of Proposition 3.1. Another example is given by letting $k_2(\mu) = -\ell \log\{c_0(b_0 + |\mu|)\}$ for $\ell > 0$, $b_0 \geq 0$ and $c_0 > 0$, which yields

$$h_2(\mu) = \{c_0(b_0 + |\mu|)\}^\ell.$$

This satisfies the conditions of Proposition 3.1. Since $c_0(b_0 + |\mu|) \leq c_0(b_0 + m)$, we observe that if $c_0(b_0 + m) = 1$, then

$$\lim_{\ell \rightarrow \infty} h_2(\mu)I(|\mu| \leq m) = \begin{cases} 0 & \text{if } |\mu| < m, \\ 1 & \text{if } |\mu| = m, \end{cases}$$

which is a two-point uniform prior putting mass on $\{-m, m\}$, and results in the Bayes estimator (2.11). If m satisfies the inequality $m \leq 1/c_0$, we can choose a nonnegative b_0 such that $c_0(b_0 + m) = 1$. This suggests that the dominance property of the Bayes estimator (2.11) may be provided under the condition that m is bounded above. In the distributional assumption of normality, Casella and Strawderman (1981) proved that the Bayes estimator (2.11) is minimax if $m \leq 1.0567$, and Marchand and Perron (2001) showed that it dominates the maximum likelihood estimator $\hat{\mu}^{TR}$ if $m \leq 1$. For other distributions, similar conditions on the boundary m are required for guaranteeing the dominance results as discussed below.

For the general symmetric density $f(x - \mu)$, the Bayes estimator $\hat{\mu}^{BU}$ against the two-point boundary uniform prior putting mass on the endpoints $\{-m, m\}$ is expressed as

$$\hat{\mu}^{BU} = \frac{mf(X - m) - mf(X + m)}{f(X - m) + f(X + m)} = X - \phi^B(X),$$

which we call here *the boundary uniform prior Bayes estimator*, where

$$\phi^B(x) = \frac{(x - m)f(x - m) + (x + m)f(x + m)}{f(x - m) + f(x + m)}.$$

The following assumption for the density $f(x - \mu)$ guarantees the monotonicity of $\phi^B(x)$:

(A.3) For nonnegative x ,

$$f'(x - m)/f(x - m) - f'(x + m)/f(x + m) \leq 2/m.$$

Under the assumption (A.2), the inequality in the assumption (A.3) can be guaranteed by the following.

(A.3') The boundary m satisfies the inequality

$$(3.8) \quad 1 + m \inf_{x>0} f'(x + m)/f(x + m) \geq 0.$$

PROPOSITION 3.2. *Assume that the symmetric density $f(x - \mu)$ satisfies the assumptions (A.1), (A.2) and (A.3). Then the boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ belongs to the class provided by Theorem 2.2, namely, $\hat{\mu}^{BU}$ dominates X under the restriction A relative to the squared error loss.*

PROOF. The condition (a) is clearly satisfied since $\phi^B(0) = 0$. From Lemma 3.1 and (3.1), we observe that for $x > 0$,

$$\phi^U(x) \geq \inf_{0 < t < m} G(t, x) = G(m, x),$$

where $G(t, x)$ is defined by (3.4). This inequality also implies that for $x < 0$, $\phi^U(x) \leq G(m, x)$ since $G(m, x) = -G(m, -x)$ and $\phi^U(x) = -\phi^U(-x)$. Thus, the condition (c) is verified.

For checking the condition (b) for $\phi^B(x)$, we need to evaluate the derivative $(d/dx)\phi^B(x)$, which is proportional to the quantity

$$\begin{aligned}
 (3.9) \quad & \{f(x-m) + f(x+m) + (x-m)f'(x-m) + (x+m)f'(x+m)\} \\
 & \times \{f(x-m) + f(x+m)\} \\
 & - \{(x-m)f(x-m) + (x+m)f(x+m)\} \\
 & \times \{f'(x-m) + f'(x+m)\} \\
 & = \{f(x-m) + f(x+m)\}^2 \\
 & + 2m\{f'(x+m)f(x-m) - f'(x-m)f(x+m)\}.
 \end{aligned}$$

Since $\{f(x-m) + f(x+m)\}^2 \geq 4f(x-m)f(x+m)$, from (3.9), we see that $(d/dx)\phi^B(x) \geq 0$ if

$$2 + mf'(x+m)/f(x+m) - mf'(x-m)/f(x-m) \geq 0,$$

which is guaranteed by the assumption (A.3), and the condition (b) is satisfied. \square

To establish the dominance result provided in Proposition 3.2, the density function $f(u)$ is required to satisfy the assumptions (A.1), (A.2) and (A.3), which can be applied to logistic and double exponential distributions.

Example 3.1 (Logistic distribution). Let X be a random variable having a logistic distribution whose density is

$$f_L(x-\mu) = e^{-(x-\mu)} / \{1 + e^{-(x-\mu)}\}^2.$$

The assumption (A.1) is satisfied since $f'_L(u) \leq 0$ for $u > 0$. The assumption (A.2) follows from the fact that $f'_L(u)/f_L(u) = -1 + 2/(e^u + 1)$ is decreasing in u . The inequality in the assumption (A.3) is written by

$$(e^{x-m} + 1)^{-1} - (e^{x+m} + 1)^{-1} \leq 1/m,$$

equivalently expressed as

$$y^2 + \{(1-m)e^m + (1+m)e^{-m}\}y + 1 \geq 0,$$

where $y = e^x \geq 1$ for nonnegative x . This inequality is guaranteed if m satisfies the inequality

$$(3.10) \quad (1-m)e^m + (1+m)e^{-m} + 2 \geq 0.$$

Let m_0 be the unique solution to the equation

$$(1-m_0)e^{m_0} + (1+m_0)e^{-m_0} + 2 = 0.$$

The constant m_0 is evaluated numerically as about 1.5434. Then it can be seen that the inequality (3.10) holds for $m \leq m_0$. Hence the boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$, given by

$$(3.11) \quad \hat{\mu}^{BU} = m \frac{e^m(1 + e^{-X-m})^2 - e^{-m}(1 + e^{-X+m})^2}{e^m(1 + e^{-X-m})^2 + e^{-m}(1 + e^{-X+m})^2},$$

dominates X if $m \leq m_0$ from Proposition 3.2.

COROLLARY 3.1. *For the logistic distribution $f_L(x - \mu)$, the boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ given by (3.11) belongs to the class of improved estimators provided by Theorem 2.2 if $m \leq m_0$.*

Example 3.2 (Double exponential distribution). Another example of satisfying the assumptions (A.1), (A.2) and (A.3) is a double exponential distribution whose density is given by

$$f_{DE}(x - \mu) = (2\sigma_0)^{-1} \exp\{-\sigma_0^{-1}|x - \mu|\}$$

for known $\sigma_0 > 0$. The assumption (A.1) is satisfied since $f'_{DE}(u) \leq 0$ for $u > 0$. The assumption (A.2) follows since $f'_{DE}(u)/f_{DE}(u) = \sigma_0^{-1}I(u < 0) - \sigma_0^{-1}I(u > 0)$ is nonincreasing in u . The inequality in (A.3) is written by

$$2 - \frac{m}{\sigma_0} + \frac{m}{\sigma_0} \{I(x - m > 0) - I(x - m \leq 0)\} \geq 0,$$

which holds for $m \leq \sigma_0$. Hence the assumption (A.3) is satisfied when $m \leq \sigma_0$. The boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ has the form

$$(3.12) \quad \hat{\mu}^{BU} = \{me^{-|X-m|/\sigma_0} - me^{-|X+m|/\sigma_0}\} / \{e^{-|X-m|/\sigma_0} + e^{-|X+m|/\sigma_0}\}$$

$$= \begin{cases} mH(-m) & \text{if } X < -m, \\ mH(X) & \text{if } |X| \leq m, \\ mH(m) & \text{if } X > m, \end{cases}$$

where $H(x) = \tanh(x/\sigma_0) = \{e^{x/\sigma_0} - e^{-x/\sigma_0}\} / \{e^{x/\sigma_0} + e^{-x/\sigma_0}\}$.

COROLLARY 3.2. *For the double exponential distribution $f_{DE}(x - \mu)$, the boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ given by (3.12) belongs to the class of improved estimators provided by Theorem 2.2 if $m \leq \sigma_0$.*

3.2. Scale mixtures of normal distributions

The scale mixtures of normal distributions are important examples of the symmetric distributions. Let X be a single random variable having a scale mixture of normal distributions whose density is given by

$$(3.13) \quad f_{NM}(x - \mu) = \int \sqrt{v/2\pi} \exp\left\{-\frac{v}{2}(x - \mu)^2\right\} d\Lambda(v),$$

where $\Lambda(v)$ denotes a continuous or discrete distribution. The scale mixtures of normal distributions can be decomposed into two parts: a conditional distribution of X given $V = v$ and a marginal distribution of the scaling random variable V , described as

$$(3.14) \quad \begin{aligned} X | V = v &\sim \mathcal{N}(\mu, 1/v), \\ V &\sim \Lambda(v). \end{aligned}$$

When the mean μ is restricted to the bounded interval $A = \{\mu \mid |\mu| \leq m\}$, the simple estimator X is improved on by the truncated, the shrinkage and the fully uniform prior Bayes estimators corresponding to (2.9), (2.10) and (2.7), respectively, given by $\hat{\mu}^{TR} = (X/|X|) \min(|X|, m)$, $\hat{\mu}^S = \{m^2/(R_0 + m^2)\}X$ for $R_0 = E[V^{-1}]$, and

$$\hat{\mu}^{FU} = \int_{-m}^m \mu f_{NM}(X - \mu) d\mu \Big/ \int_{-m}^m f_{NM}(X - \mu) d\mu.$$

We here provide some examples of the scale mixture of normal distributions satisfying the assumptions (A.1), (A.2) and (A.3), which imply that X is dominated by the boundary uniform prior Bayes estimator. Although (A.1) is clearly satisfied for the density (3.13), the other assumptions (A.2) and (A.3) need to be checked for specific distributions.

Example 3.3 (Normal distribution). The normal distribution $\mathcal{N}(\mu, \sigma_0^2)$ for known σ_0^2 is provided by putting $P[V = 1/\sigma_0^2] = 1$ in the model (3.14). Since $f'(u)/f(u) = -u/\sigma_0^2$ for the normal density $f(u)$, the assumption (A.2) is clearly satisfied. The inequality in (A.3) is represented as

$$2 - m(x + m)/\sigma_0^2 + m(x - m)/\sigma_0^2 \geq 0,$$

which is satisfied for $m \leq \sigma_0$. The boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ is given by

$$(3.15) \quad \begin{aligned} \hat{\mu}^{BU} &= m(e^{mX/\sigma_0^2} - e^{-mX/\sigma_0^2}) / (e^{mX/\sigma_0^2} + e^{-mX/\sigma_0^2}) \\ &= m \tanh(mX/\sigma_0^2) \\ &= \frac{m^2 X}{\sigma_0^2} \sum_{j=0}^{\infty} \frac{(mX/\sigma_0^2)^{2j}}{(2j + 1)!} \Big/ \sum_{j=0}^{\infty} \frac{(mX/\sigma_0^2)^{2j}}{(2j)!}, \end{aligned}$$

which, from Proposition 3.2, dominates X for $m \leq \sigma_0$.

COROLLARY 3.3. *For the normal distribution $\mathcal{N}(\mu, \sigma_0^2)$, the boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ given by (3.15) belongs to the class of improved estimators provided by Theorem 2.2 if $m \leq \sigma_0$.*

In the normal distribution, it is noted that Casella and Strawderman (1981) established the minimaxity of $\hat{\mu}^{BU}$ for $m/\sigma_0 \leq 1.0567$, and Marchand and Perron

(2001) showed the stronger result that $\hat{\mu}^{BU}$ dominates the maximum likelihood estimator for $m/\sigma_0 \leq 1$.

Example 3.4 (T-distribution). The t -distribution \mathcal{T}_r with r degrees of freedom is provided by letting rV follow a chi-squares distribution χ_r^2 with r degrees of freedom in the model (3.14). The density of \mathcal{T}_r is described by

$$f_T(x - \mu) = c[1 + (x - \mu)^2/r]^{-(r+1)/2}$$

for $c = \Gamma((r + 1)/2)(r\pi)^{-1/2}/\Gamma(r/2)$. The assumption (A.1) is clearly satisfied. It is noted that $f(x - \mu)$ does not have the monotone likelihood ratio property, so that we need to evaluate the inequality (3.3) directly for (A.2). Since $f'_T(t)/f_T(t) = -(r + 1)u/(r + u^2)$, the inequality (3.3) is represented as

$$-\frac{x - \mu}{(x - \mu)^2 + r} - \frac{x + \mu}{(x + \mu)^2 + r} \leq 0,$$

which can be simplified by

$$(3.16) \quad 2x(x^2 - \mu^2 + r) \geq 0.$$

Since $x^2 - \mu^2 + r \geq -m^2 + r$, the inequality (3.16) holds, and (A.2) is satisfied for any nonnegative x and $\mu \in [0, m]$ if $m \leq \sqrt{r}$.

The inequality in the assumption (A.3) is expressed by

$$1 - \frac{m(r + 1)(x + m)}{2\{(x + m)^2 + r\}} + \frac{m(r + 1)(x - m)}{2\{(x - m)^2 + r\}} \geq 0,$$

which is, after some calculations, rewritten as

$$(x^2 - m^2)^2 + 2r(x^2 + m^2) + r^2 + m^2(r + 1)(x^2 - m^2 - r) \geq 0,$$

or

$$(3.17) \quad x^4 + \{(r - 1)m^2 + 2r\}x^2 + r(r + m^2)(1 - m^2) \geq 0.$$

Since $x > 0$, the inequality (3.17) is satisfied if $r \geq 1$ and $m \leq 1$, that is, under these conditions the assumption (A.3) holds for the \mathcal{T}_r -distribution.

The boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ is expressed as

$$(3.18) \quad \hat{\mu}^{BU} = \frac{m[1 + (X - m)^2/r]^{-(r+1)/2} - m[1 + (X + m)^2/r]^{-(r+1)/2}}{[1 + (X - m)^2/r]^{-(r+1)/2} + [1 + (X + m)^2/r]^{-(r+1)/2}},$$

which, from Proposition 3.2, belongs to the class of Theorem 2.2.

COROLLARY 3.4. *For the \mathcal{T}_r -distribution with r degrees of freedom, the boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ given by (3.18) belongs to the class of improved estimators provided by Theorem 2.2 if $r \geq 1$ and $m \leq 1$.*

In the t -distribution, it is noted that Marchand and Perron (2005) recently showed the stronger result that $\hat{\mu}^{BU}$ dominates the truncated or ML estimator $\hat{\mu}^{TR}$ if $r \geq 1$ and $m \leq 1$.

We next want to get general conditions on the distribution $\Lambda(v)$ for the density $f_{NM}(x - \mu)$ in (3.13) to satisfy the assumptions (A.1), (A.2) and (A.3). Although it may be difficult to derive exact conditions, we can get rough sufficient conditions on $\Lambda(v)$ and m . Let m_1 be a positive constant satisfying the inequality

$$(3.19) \quad \frac{1}{4m_1^2} \geq \frac{E[V^{5/2}]}{E[V^{3/2}]} - \frac{E[V^{3/2}e^{-2m_1^2V}]}{E[V^{1/2}e^{-2m_1^2V}]},$$

where V is a random variable having the distribution $\Lambda(v)$. Also let m_2 be a constant such that

$$(3.20) \quad m_2^2 \leq E[V^{1/2}]/E[V^{3/2}].$$

It is noted that there exists such positive constants m_1 and m_2 , although the condition (3.19) is restrictive.

PROPOSITION 3.3. *The scale mixture of normal distributions $f_{NM}(x - \mu)$ satisfies the assumptions (A.1), (A.2) and (A.3) if $m \leq \min(m_1, m_2)$. That is, the boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ dominates X for $m \leq \min(m_1, m_2)$.*

PROOF. We shall verify that $f_{NM}(x - \mu)$ satisfies (A.2') and (A.3). Since $f_{NM}(u) = (2\pi)^{-1/2}E[V^{1/2}e^{-Vu^2/2}]$ and $f'_{NM}(u) = -(2\pi)^{-1/2}E[uV^{3/2}e^{-Vu^2/2}]$, the assumption (A.2') is expressed as $g(u) = E[uV^{3/2}e^{-Vu^2/2}]/E[V^{1/2}e^{-Vu^2/2}]$ is nondecreasing in u for $0 < u \leq 2m$. The derivative $g'(u)$ is nonnegative if

$$E[(u^{-2} - V)V^{3/2}e^{-Vu^2/2}]E[V^{1/2}e^{-Vu^2/2}] + \{E[V^{3/2}e^{-Vu^2/2}]\}^2 \geq 0,$$

or

$$(3.21) \quad \frac{1}{u^2} - \frac{E[V^{5/2}e^{-Vu^2/2}]}{E[V^{3/2}e^{-Vu^2/2}]} + \frac{E[V^{3/2}e^{-Vu^2/2}]}{E[V^{1/2}e^{-Vu^2/2}]} \geq 0,$$

for $0 < u \leq 2m$. Letting $h(t, a) = E[V^{a+1}e^{-Vt}]/E[V^a e^{-Vt}]$, we can express the inequality (3.21) as

$$(3.22) \quad u^{-2} - h(u^2/2, 3/2) + h(u^2/2, 1/2) \geq 0.$$

We here show that $h(t, a)$ is decreasing in t . In fact, $(d/dt)h(t, a) \leq 0$ if

$$-E[V^{a+2}e^{-Vt}]E[V^a e^{-Vt}] + \{E[V^{a+1}e^{-Vt}]\}^2 \leq 0,$$

which is equivalent to

$$E^*[V^2] - \{E^*[V]\}^2 \geq 0,$$

where $E^*[I_A] = E[I_A V^a e^{-Vt}] / E[V^a e^{-Vt}]$ for the indicator function I_A . Since this inequality is true, $h(t, a)$ is decreasing in t . Since $0 < u^2 \leq 4m^2$, from the monotonicity of $h(t, a)$, the inequality (3.22) holds if

$$(4m^2)^{-1} - h(0, 3/2) + h(2m^2, 1/2) \geq 0,$$

which is given by the condition (3.19). Hence, (A.2') is satisfied.

We next verify the assumption (A.3), which is written by

$$-(x-m) \frac{E[V^{3/2} e^{-V(x-m)^2/2}]}{E[V^{1/2} e^{-V(x-m)^2/2}]} + (x+m) \frac{E[V^{3/2} e^{-V(x+m)^2/2}]}{E[V^{1/2} e^{-V(x+m)^2/2}]} \leq \frac{2}{m},$$

or

$$(3.23) \quad -(x-m)h((x-m)^2/2, 1/2) + (x+m)h((x+m)^2/2, 1/2) \leq 2/m.$$

Since $h(t, a)$ is decreasing in $t > 0$ and $(x-m)^2 \leq (x+m)^2$ for $x > 0$, it is noted that $h((x+m)^2/2, 1/2) \leq h((x-m)^2/2, 1/2) \leq h(0, 1/2)$, so that the l.h.s. in the inequality (3.23) is evaluated as

$$\begin{aligned} & -(x-m)h((x-m)^2/2, 1/2) + (x+m)h((x+m)^2/2, 1/2) \\ & \leq 2mh((x-m)^2/2, 1/2) \leq 2mh(0, 1/2). \end{aligned}$$

Hence the inequality (3.23) holds if

$$m^2 \leq E[V^{1/2}] / E[V^{3/2}],$$

which is guaranteed by (3.20), and (A.3) is satisfied. Therefore, the proof of Proposition 3.3 is complete. \square

Example 3.5 (Finite mixture normal distribution). Let $\Lambda(v)$ be a discrete distribution on $\{v_1, \dots, v_k\}$ and $\Lambda(v_i) = p_i$ for $i = 1, \dots, k$. Then

$$f_{NM}(x - \mu) = \sum_{i=1}^k p_i \sqrt{v_i/2\pi} \exp\{-v_i(x - \mu)^2/2\}.$$

Let $v_{\min} = \min_{1 \leq i \leq k} v_i$ and

$$m_0 = \min \left\{ \left(\frac{\sum_{i=1}^k p_i v_i^{1/2}}{\sum_{i=1}^k p_i v_i^{3/2}} \right)^{1/2}, \frac{1}{2} \left[\frac{\sum_{i=1}^k p_i v_i^{5/2}}{\sum_{i=1}^k p_i v_i^{3/2}} - v_{\min} \right]^{-1/2} \right\}.$$

Hence, from Proposition 3.2, the boundary uniform prior Bayes estimator $\hat{\mu}^{BU}$ belongs to the class of improved estimators provided by Theorem 2.2 if $m \leq m_0$, namely, the Bayes estimator $\hat{\mu}^{BU}$ dominates X .

4. An extension to the scale family

The same arguments as in Section 2 allow us to extend the results to the scale family of the density $\sigma^{-n}f(\mathbf{x}/\sigma)$ for scale parameter $\sigma > 0$, where \mathbf{x}/σ means $(x_1/\sigma, \dots, x_n/\sigma)$. It is supposed that the scale σ is estimated by estimator $\hat{\sigma}$ relative to the entropy loss function

$$(4.1) \quad L_s(\hat{\sigma}/\sigma) = \hat{\sigma}/\sigma - \log \hat{\sigma}/\sigma - 1,$$

referred to as the Stein loss as well. The best scale-equivariant estimator $\hat{\sigma}_0$ is given by

$$\hat{\sigma}_0 = \hat{\sigma}_0(\mathbf{X}) = \int_0^\infty \sigma^{-n-1} f(\mathbf{X}/\sigma) d\sigma \bigg/ \int_0^\infty \sigma^{-n-2} f(\mathbf{X}/\sigma) d\sigma.$$

This is the unrestricted generalized Bayes estimator against the measure $\sigma^{-1}d\sigma$ over positive real line \mathbf{R}_+ . Assume that the scale σ is restricted to the bounded interval

$$B = \{\sigma \mid a \leq \sigma \leq b\}$$

for known positive values a and b .

To improve on the best scale-equivariant estimator $\hat{\sigma}_0$ by using the restriction B , consider a class of the estimators

$$(4.2) \quad \hat{\sigma}_\phi = \hat{\sigma}_\phi(\hat{\sigma}_0, \mathbf{Z}) = \hat{\sigma}_0 \phi(\hat{\sigma}_0, \mathbf{Z}), \quad \mathbf{Z} = \mathbf{X}/\hat{\sigma}_0$$

where $\phi(w, \mathbf{z})$ is an absolutely continuous function.

THEOREM 4.1. *Assume that $\phi(w, \mathbf{z})$ satisfies the following conditions:*

- (a) *There exists a function $c(\mathbf{z})$ such that $\phi(c(\mathbf{z}), \mathbf{z}) = 1$,*
- (b) *$\phi(w, \mathbf{z})$ is nonincreasing in w ,*
- (c) *$\phi(w, \mathbf{z})$ is bounded as*

$$\phi(w, \mathbf{z}) \begin{cases} \geq \phi_{w/b, \infty}(w, \mathbf{z}) & \text{if } w \geq c(\mathbf{z}), \\ \leq \phi_{0, w/a}(w, \mathbf{z}) & \text{if } w < c(\mathbf{z}), \end{cases}$$

where

$$\phi_{w/b, \infty}(w, \mathbf{z}) = \frac{\int_{w/b}^\infty v^{n-1} f(\mathbf{z}v) dv}{\int_{w/b}^\infty v^n f(\mathbf{z}v) dv}, \quad \phi_{0, w/a}(w, \mathbf{z}) = \frac{\int_0^{w/a} v^{n-1} f(\mathbf{z}v) dv}{\int_0^{w/a} v^n f(\mathbf{z}v) dv}.$$

Then $\hat{\sigma}_\phi$ given by (4.2) dominates the best scale-equivariant estimator $\hat{\sigma}_0$ relative to the L_s -loss.

PROOF. The same arguments as in the proof of Theorem 2.1 is used for the proof of this theorem, an outline of which is given here. The risk difference of the two estimators $\hat{\sigma}_0$ and $\hat{\sigma}_\phi$ is written by

$$\Delta = R(\sigma, \hat{\sigma}_0) - R(\sigma, \hat{\sigma}_\phi)$$

$$\begin{aligned}
 &= E \left[\int_1^{c(\mathbf{Z})/\hat{\sigma}_0} \frac{d}{dt} L_s(\hat{\sigma}_0 \sigma^{-1} \phi(t\hat{\sigma}_0, \mathbf{Z})) dt \right] \\
 &= \int \int_1^{c(\mathbf{z})/\hat{\sigma}_0} \left\{ \frac{\hat{\sigma}_0}{\sigma} - \frac{1}{\phi(t\hat{\sigma}_0, \mathbf{z})} \right\} \hat{\sigma}_0 \phi'(t\hat{\sigma}_0, \mathbf{z}) \sigma^{-n} f(\mathbf{x}/\sigma) dt d\mathbf{x} \\
 &= \int_{c(\mathbf{z}) \geq \hat{\sigma}_0} \int_1^{c(\mathbf{z})/\hat{\sigma}_0} \left\{ \frac{\hat{\sigma}_0}{\sigma} - \frac{1}{\phi(t\hat{\sigma}_0, \mathbf{z})} \right\} \hat{\sigma}_0 \phi'(t\hat{\sigma}_0, \mathbf{z}) \sigma^{-n} f(\mathbf{x}/\sigma) dt d\mathbf{x} \\
 &\quad - \int_{c(\mathbf{z}) < \hat{\sigma}_0} \int_{c(\mathbf{z})/\hat{\sigma}_0}^1 \left\{ \frac{\hat{\sigma}_0}{\sigma} - \frac{1}{\phi(t\hat{\sigma}_0, \mathbf{z})} \right\} \hat{\sigma}_0 \phi'(t\hat{\sigma}_0, \mathbf{z}) \sigma^{-n} f(\mathbf{x}/\sigma) dt d\mathbf{x} \\
 &= \Delta_1 + \Delta_2. \quad (\text{say}).
 \end{aligned}$$

For the proof that $\Delta_1 \geq 0$, note that $\hat{\sigma}_0$ is scale-equivariant. Also, note the set that $c(\mathbf{z})/\hat{\sigma}_0 \geq 1$ and $1 \leq t \leq c(\mathbf{z})/\hat{\sigma}_0$ is equivalent to the set that $1 \leq t < \infty$ and $c(\mathbf{z})/\hat{\sigma}_0 \geq t$. Making the transformations $\mathbf{u} = t\mathbf{x}$ and $v = \hat{\sigma}_0/(\sigma t)$ in turn with $d\mathbf{u} = t^n d\mathbf{x}$ and $dv = (\hat{\sigma}_0/(\sigma t^2))dt$, we can rewrite Δ_1 as

$$\begin{aligned}
 \Delta_1 &= \int_1^\infty \int_{c(\mathbf{z})/\hat{\sigma}_0(\mathbf{x}) \geq t} \left\{ \frac{\hat{\sigma}_0(\mathbf{x})}{\sigma} - \frac{1}{\phi(t\hat{\sigma}_0(\mathbf{x}), \mathbf{z})} \right\} \hat{\sigma}_0(\mathbf{x}) \phi'(t\hat{\sigma}_0(\mathbf{x}), \mathbf{z}) \sigma^{-n} \\
 &\quad \times f(\mathbf{x}/\sigma) d\mathbf{x} dt \\
 &= \int_1^\infty \int_{c(\mathbf{z}) \geq \hat{\sigma}_0(\mathbf{u})} \left\{ \frac{\hat{\sigma}_0(\mathbf{u})}{t\sigma} - \frac{1}{\phi(\hat{\sigma}_0(\mathbf{u}), \mathbf{z})} \right\} \frac{\hat{\sigma}_0(\mathbf{u})}{t} \phi'(\hat{\sigma}_0(\mathbf{u}), \mathbf{z}) (t\sigma)^{-n} \\
 &\quad \times f(\mathbf{u}/(t\sigma)) d\mathbf{u} dt \\
 &= \int_{c(\mathbf{z}) \geq \hat{\sigma}_0(\mathbf{u})} \int_0^{\hat{\sigma}_0(\mathbf{u})/\sigma} \left\{ v - \frac{1}{\phi(\hat{\sigma}_0(\mathbf{u}), \mathbf{z})} \right\} \phi'(\hat{\sigma}_0(\mathbf{u}), \mathbf{z}) \left(\frac{v}{\hat{\sigma}_0(\mathbf{u})} \right)^{n-1} \\
 &\quad \times f((\mathbf{u}/\hat{\sigma}_0(\mathbf{u}))v) dv d\mathbf{u}
 \end{aligned}$$

where $\mathbf{z} = \mathbf{u}/\hat{\sigma}_0(\mathbf{u})$. Since $\phi(w, \mathbf{z})$ is nonincreasing in w , it is sufficient to show that

$$\phi(w, \mathbf{z}) \leq \int_0^{w/\sigma} v^{n-1} f(\mathbf{z}v) dv \Big/ \int_0^{w/\sigma} v^n f(\mathbf{z}v) dv,$$

which is expressed as

$$(4.3) \quad \phi(\sigma w^*, \mathbf{z}) \leq \int_0^{w^*} v^{n-1} f(\mathbf{z}v) dv \Big/ \int_0^{w^*} v^n f(\mathbf{z}v) dv,$$

for $w^* = w/\sigma$. Again from the condition (b), we note that $\phi(\sigma w^*, \mathbf{z}) \leq \phi(aw^*, \mathbf{z})$ for $a \leq \sigma \leq b$. Hence, the inequality (4.3) holds if

$$\phi(aw^*, \mathbf{z}) \leq \int_0^{w^*} v^{n-1} f(\mathbf{z}v) dv \Big/ \int_0^{w^*} v^n f(\mathbf{z}v) dv.$$

Replacing w^* with w/a again yields the inequality in the condition (c) of Theorem 4.1.

The similar arguments give the expression that

$$\begin{aligned}
 \Delta_2 &= - \int_{c(\mathbf{z}) < \hat{\sigma}_0(\mathbf{u})} \int_{\hat{\sigma}_0(\mathbf{u})/\sigma}^\infty \left\{ v - \frac{1}{\phi(\hat{\sigma}_0(\mathbf{u}), \mathbf{z})} \right\} \phi'(\hat{\sigma}_0(\mathbf{u}), \mathbf{z}) \left(\frac{v}{\hat{\sigma}_0(\mathbf{u})} \right)^{n-1} \\
 &\quad \times f((\mathbf{u}/\hat{\sigma}_0(\mathbf{u}))v) dv d\mathbf{u},
 \end{aligned}$$

which, from the condition (b), leads to the sufficient condition that

$$\phi(w, \mathbf{z}) \geq \int_{w/\sigma}^{\infty} v^{n-1} f(\mathbf{z}v)dv \Big/ \int_{w/\sigma}^{\infty} v^n f(\mathbf{z}v)dv.$$

This inequality is guaranteed by the condition (c), and, therefore, the proof of Theorem 4.1 is complete. \square

Concerning the condition (c) of Theorem 4.1, we can get another condition corresponding to Theorem 2.2. It can be derived by noting that the functions

$$\frac{\int_{w/b}^s v^{n-1} f(\mathbf{z}v)dv}{\int_{w/b}^s v^n f(\mathbf{z}v)dv} \quad \text{and} \quad \frac{\int_s^{w/a} v^{n-1} f(\mathbf{z}v)dv}{\int_s^{w/a} v^n f(\mathbf{z}v)dv}$$

are decreasing in s , which implies that $\phi^U(w, \mathbf{z}) \geq \phi_{w/b, \infty}(w, \mathbf{z})$ and $\phi^U(w, \mathbf{z}) \leq \phi_{0, w/a}(w, \mathbf{z})$, respectively. Hence we obtain the following theorem.

THEOREM 4.2. *Assume that $\phi(w, \mathbf{z})$ satisfies the following conditions:*

- (a) *There exists a function $c(\mathbf{z})$ such that $\phi(c(\mathbf{z}), \mathbf{z}) = 1$,*
- (b) *$\phi(w, \mathbf{z})$ is nonincreasing in w ,*
- (c) *$\phi(w, \mathbf{z})$ is bounded as*

$$\phi(w, \mathbf{z}) \begin{cases} \geq \phi^U(w, \mathbf{z}) & \text{if } w \geq c(\mathbf{z}), \\ \leq \phi^U(w, \mathbf{z}) & \text{if } w < c(\mathbf{z}), \end{cases}$$

where

$$(4.4) \quad \phi^U(w, \mathbf{z}) = \int_{w/b}^{w/a} v^{n-1} f(\mathbf{z}v)dv \Big/ \int_{w/b}^{w/a} v^n f(\mathbf{z}v)dv.$$

Then $\hat{\sigma}_\phi$ given by (4.2) dominates the best scale-equivariant estimator $\hat{\sigma}_0$ relative to the L_s -loss.

Some improved estimators can be derived by the same arguments as in Section 2.2. When the prior over the bounded interval, described by

$$\pi^U(\sigma) = (\log b - \log a)^{-1} \sigma^{-1} d\sigma I(a \leq \sigma \leq b),$$

is supposed, the resulting Bayes estimator is given by

$$(4.5) \quad \begin{aligned} \hat{\sigma}^{FU} &= \int_a^b \sigma^{-n-1} f(\mathbf{X}/\sigma) d\sigma \Big/ \int_a^b \sigma^{-n-2} f(\mathbf{X}/\sigma) d\sigma \\ &= \hat{\sigma}_0 \int_{\hat{\sigma}_0/b}^{\hat{\sigma}_0/a} v^{n-1} f((\mathbf{X}/\hat{\sigma}_0)v) dv \Big/ \int_{\hat{\sigma}_0/b}^{\hat{\sigma}_0/a} v^n f((\mathbf{X}/\hat{\sigma}_0)v) dv \\ &= \hat{\sigma}_0 \phi^U(\hat{\sigma}_0, \mathbf{Z}), \end{aligned}$$

where $\phi^U(w, \mathbf{z})$ is defined by (4.4). It can be verified that the Bayes estimator $\hat{\sigma}^{FU}$ belongs to the class provided by Theorem 4.2. For the condition (b), the derivative of $\phi^U(w, \mathbf{z})$ with respect to w is proportional to the quantity

$$\begin{aligned} & \left\{ \frac{1}{a} \left(\frac{w}{a} \right)^{n-1} f(\mathbf{z}w/a) - \frac{1}{b} \left(\frac{w}{b} \right)^{n-1} f(\mathbf{z}w/b) \right\} \int_{w/b}^{w/a} v^n f(\mathbf{z}v) dv \\ & \quad - \left\{ \frac{1}{a} \left(\frac{w}{a} \right)^n f(\mathbf{z}w/a) - \frac{1}{b} \left(\frac{w}{b} \right)^n f(\mathbf{z}w/b) \right\} \int_{w/b}^{w/a} v^{n-1} f(\mathbf{z}v) dv \\ & = \int_{w/b}^{w/a} v^{n-1} f(\mathbf{z}v) \left[(v - w/a) \frac{1}{a} \left(\frac{w}{a} \right)^{n-1} f(\mathbf{z}w/a) \right. \\ & \quad \left. + (w/b - v) \frac{1}{b} \left(\frac{w}{b} \right)^{n-1} f(\mathbf{z}w/b) \right] dv, \end{aligned}$$

which is not positive. Thus, $\phi^U(w, \mathbf{z})$ satisfies the condition (b). Since

$$\frac{w}{b} \int_{w/b}^{w/a} v^{n-1} f(\mathbf{z}v) dv \leq \int_{w/b}^{w/a} v^n f(\mathbf{z}v) dv \leq \frac{w}{a} \int_{w/b}^{w/a} v^{n-1} f(\mathbf{z}v) dv,$$

it is noted that $\phi^U(w, \mathbf{z}) \geq a/w > 1$ for $w < a$ and $\phi^U(w, \mathbf{z}) \leq b/w < 1$ for $w > b$. From the monotonicity of $\phi^U(w, \mathbf{z})$, it follows that there exists a function $c(\mathbf{z})$ such that $\phi^U(c(\mathbf{z}), \mathbf{z}) = 1$.

PROPOSITION 4.1. *The Bayes estimator $\hat{\sigma}^{FU}$ given by (4.5) dominates the best scale-equivariant estimator $\hat{\sigma}_0$ relative to the L_s -loss.*

Since the estimator $\hat{\sigma}_0$ takes values outside the parameter space $a \leq \sigma \leq b$, it is reasonable to truncate it at the endpoints a and b , and we get the truncated estimator

$$\begin{aligned} \hat{\sigma}^{TR} &= \max\{a, \min\{\hat{\sigma}_0, b\}\} \\ &= \hat{\sigma}_0 \phi^{TR}(\hat{\sigma}_0, \mathbf{Z}), \end{aligned}$$

where

$$\phi^{TR}(\hat{\sigma}_0, \mathbf{z}) = \max\{a/\hat{\sigma}_0, \min\{1, b/\hat{\sigma}_0\}\}.$$

The function $\phi^{TR}(w, \mathbf{z})$ satisfies all the conditions in Theorem 4.1, and $\hat{\sigma}^{TR}$ belongs to the class of improved estimators provided by Theorem 4.1.

A shrinkage estimator corresponding to (2.10) is provided by

$$(4.6) \quad \hat{\sigma}^S = \hat{\sigma}_0 \{\sqrt{ab}/\hat{\sigma}_0\}^A = \hat{\sigma}_0^{1-A} \times (ab)^{A/2},$$

where A is the solution of the equation

$$(4.7) \quad E[\{\log V \sqrt{a/b}\} V^{1-A} (b/a)^{A/2}] = E[\log V \sqrt{b/a}],$$

for $V = \hat{\sigma}_0/\sigma$. When $A \geq 0$, $\hat{\sigma}^S$ shrinks $\hat{\sigma}_0$ towards the geometric mean \sqrt{ab} of the two endpoints a and b .

PROPOSITION 4.2. *If $A > 0$, then the shrinkage estimator $\hat{\sigma}^S$ given by (4.6) dominates $\hat{\sigma}_0$ relative to the L_s -loss, and the estimator $\hat{\sigma}_0$ is not minimax.*

PROOF. The risk function of the estimator $\hat{\sigma}^S$ is written by

$$\begin{aligned} R(\sigma, \hat{\sigma}^S) &= E[\hat{\sigma}^S/\sigma - \log \hat{\sigma}^S/\sigma - 1] \\ &= E[V^{1-A}(\sqrt{ab}/\sigma)^A - (1 - A) \log V + A \log \sigma/\sqrt{ab} - 1]. \end{aligned}$$

Noting that $a \leq \sigma \leq b$ and $A > 0$, we see that

$$(4.8) \quad R(\sigma, \hat{\sigma}^S) \leq E[V^{1-A}(\sqrt{b/a})^A - (1 - A) \log V + A \log \sqrt{b/a} - 1],$$

which can be minimized when A is the solution of the equation (4.7). Hence, the risk difference of the two estimators $\hat{\sigma}_0$ and $\hat{\sigma}^S$ is evaluated as

$$\begin{aligned} \Delta &= R(\sigma, \hat{\sigma}_0) - R(\sigma, \hat{\sigma}^S) \\ &\geq E[V - \log V - 1] - E[V^{1-A}(\sqrt{b/a})^A - (1 - A) \log V + A \log \sqrt{b/a} - 1] \\ &= E[V^{1-A}(b/a)^{A/2} \{V^A(a/b)^{A/2} - \log V^A(a/b)^{A/2} - 1\}], \end{aligned}$$

which is positive, since $x - \log x - 1 > 0$ for $x \neq 1$. This shows that the risk of the estimator $\hat{\sigma}^S$ is bounded by the constant strictly smaller than the constant risk $R(\sigma, \hat{\sigma}_0)$, and the proof is complete. \square

Corresponding to (2.11), we can consider a Bayes estimator against a two-point prior. The prior putting mass on the two endpoints $\{a, b\}$ is provided by

$$\pi^B(\sigma) = pP[\sigma = a] + (1 - p)P[\sigma = b],$$

where p is a known constant in the interval $[0, 1]$. Then the Bayes estimator against the two-point prior is given by

$$(4.9) \quad \hat{\sigma}^{BU} = \frac{pa^{-n}f(\mathbf{X}/a) + (1 - p)b^{-n}f(\mathbf{X}/b)}{pa^{-n-1}f(\mathbf{X}/a) + (1 - p)b^{-n-1}f(\mathbf{X}/b)} = \hat{\sigma}_0\phi^B(\hat{\sigma}_0, \mathbf{Z}),$$

where for $\mathbf{Z} = \mathbf{X}/\hat{\sigma}_0$,

$$\phi^B(w, \mathbf{z}) = \frac{p(w/a)^n f(\mathbf{z}w/a) + (1 - p)(w/b)^n f(\mathbf{z}w/b)}{p(w/a)^{n+1} f(\mathbf{z}w/a) + (1 - p)(w/b)^{n+1} f(\mathbf{z}w/b)}.$$

Example 4.1 (Gamma distribution). Consider the estimation of the scale parameter of the gamma distribution $\mathcal{G}a(r, \sigma)$ whose density is described by $\{\Gamma(r)\}^{-1}\sigma^{-r}x^{r-1}\exp\{-x/\sigma\}$. The best scale equivariant estimator under the loss (4.1) is $\hat{\sigma}_0 = X/r$, which is unbiased. Let us consider the restriction $a \leq \sigma \leq b$ where a and b are positive constants satisfying the equation

$$ae^{1/a} = be^{1/b}.$$

The Bayes estimator against the prior $\pi^U(\sigma) = \sigma^{-1}d\sigma I(a \leq \sigma \leq b)$ is given by

$$\hat{\sigma}^{FU} = \int_a^b \sigma^{-r-1} e^{-X/\sigma} d\sigma \bigg/ \int_a^b \sigma^{-r-2} e^{-X/\sigma} d\sigma = \hat{\sigma}_0 \phi^U(X),$$

where the function $\phi^U(x)$ satisfies

$$(4.10) \quad \phi^U(x) = r \int_{x/b}^{x/a} v^{r-1} e^{-v} dv \bigg/ \int_{x/b}^{x/a} v^r e^{-v} dv.$$

The integration by parts gives the equation

$$\int_{x/b}^{x/a} v^r e^{-v} dv = [-v^r e^{-v}]_{x/b}^{x/a} + r \int_{x/b}^{x/a} v^{r-1} e^{-v} dv,$$

which implies that the function $\phi^U(x)$ in (4.10) is rewritten by

$$\phi^U(x) = \int_{x/b}^{x/a} v^{r-1} e^{-v} dv \bigg/ \left\{ g(x) + \int_{x/b}^{x/a} v^{r-1} e^{-v} dv \right\},$$

where

$$g(x) = r^{-1} [-v^r e^{-v}]_{x/b}^{x/a} = \frac{1}{r} \{ (x/b)^r e^{-x/b} - (x/a)^r e^{-x/a} \}.$$

Since $ae^{1/a} = be^{1/b}$, it follows that

$$g(r) = r^{r-1} \{ (b^{-1} e^{-1/b})^r - (a^{-1} e^{-1/a})^r \} = 0,$$

which shows that $\phi^U(r) = 1$, and the constant $c = c(\mathbf{z})$ given in Theorem 4.2 corresponds to r . From the theorem, it is seen that the estimator $\hat{\sigma}^{FU}$ dominates $\hat{\sigma}_0$.

It seems difficult to derive an improved two-points prior Bayes estimator corresponding to (4.9) even in the above simple example. However, the expectation of such an improved two-point Bayes estimator can be demonstrated in the following example.

Example 4.2 (Specific distribution related to F). Let X be a positive random variable having the density

$$\frac{1}{B((r-1)/2, (r+1)/2)} \frac{(x/\sigma)^{(r-1)/2-1}}{\sigma(1+x/\sigma)^r}, \quad x > 0,$$

where r is a constant such that $r > 1$. The best scale equivariant estimator is $\hat{\sigma}_0 = X$. When the scale σ is restricted to the interval $B = \{\sigma \mid 1/b \leq \sigma \leq b\}$ for $b > 1$, the Bayes estimator against the prior $\pi^U(\sigma) = \sigma^{-1}d\sigma I(1/b \leq \sigma \leq b)$ is given by

$$\hat{\sigma}^{FU} = \frac{\int_{1/b}^b \sigma^{-(r+1)/2} (1+X/\sigma)^{-r} d\sigma}{\int_{1/b}^b \sigma^{-(r+1)/2-1} (1+X/\sigma)^{-r} d\sigma} = X \phi^U(X),$$

where

$$\phi^U(x) = \int_{x/b}^{bx} v^{(r-1)/2-1} (1+v)^{-r} dv \bigg/ \int_{x/b}^{bx} v^{(r-1)/2} (1+v)^{-r} dv.$$

By the integration by parts, it is noted that

$$(4.11) \quad \int_{1/b}^b \left(\frac{v}{1+v} \right)^{(r-1)/2} \left(\frac{1}{1+v} \right)^{(r-1)/2+1} dv \\ = \frac{2}{r-1} \left[-\frac{v^{(r-1)/2}}{(1+v)^{r-1}} \right]_{1/b}^b + \int_{1/b}^b v^{(r-1)/2-1} (1+v)^{-r} dv,$$

and the first term in the r.h.s. of (4.11) is equal to zero, which implies that $\phi^U(1) = 1$.

The Bayes estimator against the boundary uniform prior $\pi^B(\sigma) = 2^{-1}P[\sigma = 1/b] + 2^{-1}P[\sigma = b]$ is

$$(4.12) \quad \hat{\sigma}^{BU} = \frac{b^{(r-1)/2}/(1+bX)^r + b^{-(r-1)/2}/(1+X/b)^r}{b^{(r+1)/2}/(1+bX)^r + b^{-(r+1)/2}/(1+X/b)^r} = X\phi^B(X),$$

where

$$\phi^B(x) = \frac{1(x+b)^r + b(1+bx)^r}{x b(x+b)^r + (1+bx)^r}.$$

It is noted that $\phi^B(1) = 1$. To get the dominance result of $\hat{\sigma}^{BU}$ over $\hat{\sigma}_0$, we need to check the conditions (b) and (c) of Theorem 4.2.

We first show that $\phi^B(x)$ is nonincreasing in x . Differentiating $\phi^B(x)$ with respect to x , we see that $(d/dx)\phi^B(x)$ is proportional to the quantity

$$xr[\{(x+b)^{r-1} + b^2(1+bx)^{r-1}\} \times \{b(x+b)^r + (1+bx)^r\} \\ - \{(x+b)^r + b(1+bx)^r\} \times \{b(x+b)^{r-1} + b(1+bx)^{r-1}\}] \\ - \{(x+b)^r + b(1+bx)^r\} \times \{b(x+b)^r + (1+bx)^r\} \\ = I_1 - I_2, \quad (\text{say}).$$

It is easily seen that $I_1 = xr(b^2 - 1)^2(x+b)^{r-1}(1+bx)^{r-1}$. On the other hand, we observe that

$$I_2 = (b^2 + 1)(1+bx)(x+b)(x+b)^{r-1}(1+bx)^{r-1} \\ + b(x^2 + 2bx + b^2)(x+b)^{2r-2} + b(b^2x^2 + 2bx + 1)(1+bx)^{2r-2}.$$

It is noted that

$$b(x^2 + 2bx + b^2)(x+b)^{2r-2} + b(b^2x^2 + 2bx + 1)(1+bx)^{2r-2} \\ \geq 2b[bx^2 + 2bx + b](x+b)^{r-1}(1+bx)^{r-1},$$

which leads to the evaluation of I_2 as

$$(4.13) \quad I_2 \geq [(b^2 + 1)\{bx^2 + (b^2 + 1)x + b\} + 2b^2x^2 + 4b^2x + 2b^2] \\ \times (x+b)^{r-1}(1+bx)^{r-1}.$$

Combining I_1 and (4.13) gives that

$$\begin{aligned} &(I_1 - I_2)(x + b)^{1-r}(1 + bx)^{1-r} \\ &\leq xr(b^2 - 1)^2 - (b^2 + 1)\{bx^2 + (b^2 + 1)x + b\} - 2b^2(x + 1)^2 \\ &= -b(b + 1)^2x^2 - \{(b^2 + 1)^2 + 4b^2 - r(b^2 - 1)^2\}x - b(b + 1)^2, \end{aligned}$$

which, since the r.h.s. is a quadratic function of x , is not positive for all $x > 0$ if

$$(4.14) \quad r \leq \{(b^2 + 1)^2 + 4b^2\}/(b^2 - 1)^2,$$

or if $[(b^2 + 1)^2 + 4b^2 - r(b^2 - 1)^2]^2 - 4b^2(b + 1)^4 \leq 0$, which is equivalent to the inequality

$$(4.15) \quad \begin{aligned} &[(b^2 + 1)^2 + 4b^2 - r(b^2 - 1)^2 - 2b(b + 1)^2] \\ &\times [(b^2 + 1)^2 + 4b^2 - r(b^2 - 1)^2 + 2b(b + 1)^2] \leq 0. \end{aligned}$$

Since $r > 1$, combining (4.14) and (4.15) gives the condition

$$(4.16) \quad 1 < r \leq 1 + \frac{2b(b^2 + 6b + 1)}{(b^2 - 1)^2}.$$

Hence, the condition (b) is verified if r and b satisfy the condition (4.16).

We next show that $\phi^B(x) \geq \phi^U(x)$ for $x \geq 1$ and $\phi^B(x) \leq \phi^U(x)$ for $0 < x \leq 1$. By making the transformation $\theta = 1/\sigma$ for $1/b < \sigma < 1$, $\hat{\sigma}^{FU}$ can be rewritten as

$$\hat{\sigma}^{FU} = \frac{\int_1^b \{\theta^{-(r-1)/2}(1 + X/\theta)^{-r} + \theta^{(r-1)/2}(1 + \theta X)^{-r}\} \theta^{-1} d\theta}{\int_1^b \{\theta^{-(r+1)/2}(1 + X/\theta)^{-r} + \theta^{(r+1)/2}(1 + \theta X)^{-r}\} \theta^{-1} d\theta},$$

which implies that

$$\inf_{1 \leq \theta \leq b} G(\theta, X) \leq \hat{\sigma}^{FU} \leq \sup_{1 \leq \theta \leq b} G(\theta, X),$$

where

$$\begin{aligned} G(\theta, x) &= \frac{\theta^{-(r-1)/2}(1 + x/\theta)^{-r} + \theta^{(r-1)/2}(1 + \theta x)^{-r}}{\theta^{-(r+1)/2}(1 + x/\theta)^{-r} + \theta^{(r+1)/2}(1 + \theta x)^{-r}} \\ &= \frac{\theta(1 + \theta x)^r + (\theta + x)^r}{(1 + \theta x)^r + \theta(\theta + x)^r}. \end{aligned}$$

Hence it is sufficient to show that $\sup_{1 \leq \theta \leq b} G(\theta, x) = G(b, x)$ for $x \geq 1$ and $\inf_{1 \leq \theta \leq b} G(\theta, x) = G(b, x)$ for $0 < x \leq 1$. For the purpose, we shall show that $G(\theta, x)$ is increasing in θ for $x \geq 1$ and decreasing for $0 < x < 1$. The derivative $(d/d\theta)G(\theta, x)$ is proportional to the quantity

$$\begin{aligned} &\{(1 + \theta x)^r + r\theta x(1 + \theta x)^{r-1} + r(\theta + x)^{r-1}\} \times \{(1 + \theta x)^r + \theta(\theta + x)^r\} \\ &\quad - \{\theta(1 + \theta x)^r + (\theta + x)^r\} \\ &\quad \times \{rx(1 + \theta x)^{r-1} + (\theta + x)^r + \theta r(\theta + x)^{r-1}\} \\ &= (1 + \theta x)^{2r} - (\theta + x)^{2r} + (1 + \theta x)^{r-1}(\theta + x)^{r-1}r(\theta^2 - 1)(x^2 - 1). \end{aligned}$$

Since $\theta > 1$, note that $(1+\theta x)^{2r} \geq$ (resp. $<$) $(\theta+x)^{2r}$ if and only if $x \geq$ (resp. $<$) 1, so that we see that $(d/d\theta)G(\theta, x) \geq$ (resp. $<$) 0 if and only if $x \geq$ (resp. $<$) 1. This means that $\phi^B(x)$ satisfies the condition (c) of Theorem 4.2.

We thus conclude that the two-point boundary uniform Bayes estimator $\hat{\sigma}^{BU}$ given by (4.12) dominates $\hat{\sigma}_0 = X$ when r and b satisfy the condition (4.16), namely, $1 < r \leq 1 + q(b)$ for $q(b) = 2b(b^2 + 6b + 1)/(b^2 - 1)^2$. Since $q(b)$ is a decreasing function of b for $b > 1$ and $\lim_{b \rightarrow \infty} q(b) = 0$, for a fixed $r > 1$, there exists a constant $b_0(r)$ such that $b_0(r) > 1$ and $0 < r - 1 = q(b_0(r))$. Hence, the condition (4.16) can be rewritten as $1 < b \leq b_0(r)$. This condition means that the improvement of $\hat{\sigma}^{BU}$ over X holds when b is bounded above by the constant $b_0(r)$, which is the same property as observed in the estimation of the location parameter.

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