On Spectra Measured in an Undulating Layered Medium[1]

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ABSTRACT

Evidence has recently accumulated that stably stratified regions of the ocean and atmosphere often consist of a series of layers of nearly uniform density separated by steps in which the gradient is large. It is shown that the motion of this structure relative to a measuring instrument results in a spectral density proportional to (frequency)$^{-2}$, over a range which is not limited by the overall value of the stability frequency N. Similarly, the spectra obtained by transversing such a structure is found to be proportional to (wavenumber)$^{-2}$. Spectral forms of this type cannot necessarily be associated with spectral densities of either internal gravity waves or turbulent eddies.

1. Introduction

It now appears to be established reasonably well that in the strongly stratified regions of both the oceans and the atmosphere, the density profile, though decreasing monotonically with height, frequently does so irregularly in a sequence of steps. A series of very beautiful observations by Woods (1968) and by Woods and Fosberry (1966, 1967), making use of dye tracers in the summer thermocline of the Mediterranean Sea near Malta, have shown that the temperature gradient was large in sheets a few centimeters thick and of considerable horizontal extent, some tens of miles at least. Between these sheets, whose vertical separation was found to be of the order 4 m, were layers in which the temperature gradient was very much less. The motion in the layers was found to be weakly turbulent and three-dimensional; the diffusion of dye did not appear to be markedly different from that which would be found in a homogeneous turbulent fluid. In the circumstances studied by Woods, the motion in the sheets was generally laminar, the strong local static stability suppressing turbulence despite the velocity shear often found across the sheets. If the whole structure was disturbed by an internal gravity wave, the incremental shear across the sheets was sometimes sufficient to induce a local instability of the Kelvin-Helmholtz type that led to outbursts of more vigorous but very localized and small-scale turbulence, rather as predicted by Phillips (1967).

Similar structures have been detected in the atmosphere both by airborne instruments (Wickerts, 1970) and by the backscattering of high intensity radar (Lane, 1967; Katz and Randall, 1968; a valuable survey is given by Ottersten, 1969). The radar backscattering is dependent on the existence of small-scale fluctuations in refractive index, dominated in the atmosphere by moisture fluctuations. The layering observed by radar appears to be associated with the local breakdown of structures like Woods' "sheets" which separate layers of differing moisture content. Ottersten cites evidence that pronounced vertical wind shear is frequently associated with the sheets[2] of strong echoes precisely as Woods finds in the ocean. The atmospheric scales are, of course, different; in an inversion region, the sheets may be separated by 100 or perhaps a few hundred meters. Wickerts (1969) reports radar observations in which 50% of the sheets were less than 30 m thick, and 25% between 10 and 20 m. It should be remembered, however, that these thickness estimates are, if anything, on the high side, since radar echoes are associated with sheets that are already turbulent and have broken down possibly as Woods observed in the ocean; sheets across which the shear is insufficient to cause breakdown would generally be thinner and not observed at all by the radar backscattered echoes.

The *reasons* for this ubiquitous structure in stably stratified regions may not be altogether clear yet; the present contribution is concerned with some of the *consequences* of its existence, particularly in connection with spectra measured in its presence. The problem is illustrated most simply by considering the temperature spectra measured at a fixed point in the oceanic thermocline which may be disturbed by internal gravity waves. The whole structure of sheets and layers may heave and subside past the observation point, and the record obtained will reflect this. If the undisturbed temperature gradient were continuous and uniform, then the vertical displacement ζ of a fluid element would be

[1] *Invited paper for inaugural issue.*

[2] Radar meteorologists customarily use the word "layer" to describe the regions of high radar return; these correspond to Woods' "sheets." We will adopt Woods' usage, calling the intervening regions "layers."

proportional to the variation in temperature θ, according to the relation

$$\zeta=\left(\frac{\partial\bar{\theta}}{\partial z}\right)^{-1}\theta.$$

In a layered structure, however, this is clearly not so. In fact, the spectra of temperature fluctuations observed in this way in a layered sea may have *little directly to do with the spectrum of internal waves* as we will see; only direct measurement of vertical velocities can establish an internal wave spectrum unambiguously.

Similar comments apply to measurements of horizontal velocity fluctuations at a fixed point or measurements of either temperature or horizontal velocity along a horizontal traverse in stably stratified regions either of the ocean or the atmosphere. The velocity shear appears to be concentrated across the sheets and the velocity trace will record the traverse of the instrument across them. Superimposed are, of course, the turbulent fluctuations in the intervening layers and the results obtained will be a composite of the two. If we calculate spectra from these records, how are we to interpret them?

This paper is directed toward an answer to this question. For the sake of simplicity, we will first neglect the turbulent fluctuations in the layers and suppose that the property measured, be it velocity, temperature or density directly, is uniform in each of the layers but varies rapidly across the thin sheets between them. The influence of random fluctuations inside the layers will be considered later. For the sake of definiteness, we will perform the analysis in terms of density variations (or potential density in the atmosphere), keeping in mind that the results will be pertinent to variations in temperature, horizontal velocity, and either salinity in the ocean or humidity in the atmosphere.

2. Density spectra associated with internal waves

Suppose that in the undisturbed state the density field, say, is specified by $\rho_0(z)$, a function which on the large scale decreases monotonically with height. On the small scale, though, $\rho_0(z)$ decreases in a series of "steps," being relatively constant in segments (the layers) separated by thin regions (the sheets) where the magnitude of the gradient is much larger. An internal wave motion results in the pattern of sheets and layers being displaced from equilibrium so that a fixed probe observes this pattern as it drifts slowly up and down. In addition to the displacement, the pattern is also, of course, strained; the spacing of the sheets varies somewhat throughout the wave cycle. It is not difficult to show, however, that this strain is of the same order as the rms slope of the isopycnal surfaces in the internal wave, which will be presumed to be small. To a first approximation, then, the pattern can be assumed to be carried rigidly up and down past the probe, the fluid elements conserving their density and the sheets their spacing.

If the vertical displacement from equilibrium of an isopycnal surface at some given horizontal point is $\zeta(t)$, then the density $\rho(z)=\rho_0(z-\zeta)$, where ρ is the undisturbed density distribution. A probe at a fixed depth z then measures

$$\rho(t)=\rho_0[z-\zeta(t)]. \tag{2.1}$$

It is the spectral characteristics of this signal that we wish to establish.

Consider a record spanning a large but finite time interval T of observation. For the purpose of computing the spectrum, the density measured is presumed to be periodic with period T and so can be expanded as a Fourier series

$$\rho(t)=\sum_{-\infty}^{\infty} C_n \exp(in\omega_0 t), \tag{2.2}$$

where $\omega_0=2\pi/T$ and the coefficients are

$$C_n=\frac{\omega_0}{2\pi}\int_{-T/2}^{T/2}\rho(t)\exp(-in\omega_0 t)dt. \tag{2.3}$$

The zero order coefficient, $C_0=\bar{\rho}$, is mean density measured by the probe. For any other coefficient, an integration by parts gives

$$\begin{aligned} C_n&=\frac{i}{2\pi n}\left\{[\rho(t)\exp(-in\omega_0 t)]_{-T/2}^{T/2}\right. \\ &\qquad\left.-\int_{-T/2}^{T/2}\frac{\partial\rho}{\partial t}\exp(-in\omega_0 t)dt\right\}, \\ &=\frac{-i}{2\pi n}\int_{-T/2}^{T/2}\frac{\partial\rho}{\partial t}\exp(-in\omega_0 t)dt, \end{aligned} \tag{2.4}$$

the first term vanishing by virtue of the assumed periodicity of the sample record.

In this integral, $\partial\rho/\partial t$ is negligible except when a sheet drifts past the observation point. Suppose first that the sheets are, in fact, step discontinuities in density, the derivative $\partial\rho/\partial t$ being represented by a series of Dirac delta functions

$$\frac{\partial\rho}{\partial t}=\sum_r (\Delta\rho)_r\delta(t-t_r),$$

where $(\Delta\rho)_r$ is the difference in density before and after the passage of the discontinuity at time t_r. The Fourier coefficient is then, from (2.4),

$$C_n = \frac{-i}{2\pi n} \sum_r (\Delta\rho)_r \exp(-in\omega_0 t_r),$$

$$C_n C_n^* = \frac{1}{4\pi^2 n^2} \sum_r \sum_s (\Delta\rho)_r (\Delta\rho)_s \times \exp[-in\omega_0(t_r - t_s)], \quad (2.5)$$

where C_n^* is the complex conjugate of C_n. The times t_r of passage of the discontinuities are distributed randomly with a characteristic interval τ_l between successive crossings; moreover, if n is sufficiently large that

$$n\omega_0 \tau_l = 2\pi n \tau_l / T \gg 1,$$

and if the occurrence of a discontinuity is uncorrelated with its magnitude, then the random phase factor in (2.5) ensures that the average values of the individual terms vanish except when $r = s$. Consequently, if we average over a large number of realizations,

$$\overline{C_n C_n^*} = \frac{1}{4\pi^2 n^2} \overline{\sum (\Delta\rho)_r^2},$$

$$= \frac{\nu T}{4\pi^2 n^2} \overline{(\Delta\rho)_r^2}, \quad (2.6)$$

where ν is the average number of crossings per unit time. In this instance, therefore, the mean square value of the Fourier coefficients decreases asymptotically as n^{-2}.

In the ocean, of course, the sheets are not discontinuities but merely thin regions where the gradients are large. If the center of the rth sheet passes the probe at the instant t_r, then

$$\frac{\partial \rho}{\partial t} = \sum_r (\Delta\rho)_r f_r(t - t_r),$$

where $f_r(t - t_r)$ is a continuous function, nonzero only within a short time interval surrounding $t = t_r$ and such that over this interval

$$\int f_r(t - t_r) dt = 1.$$

In the integral (2.4), contributions arise only from the intervals surrounding the crossing times t_r. For any one of these, we have

$$(\Delta\rho)_r \int f_r(t - t_r) \exp(-in\omega_0 t) dt$$

$$= (\Delta\rho)_r \exp(-in\omega_0 t_r) \times \int f_r(t - t_r) \exp[-in\omega_0(t - t_r)] dt$$

$$= (\Delta\rho)_r \exp(-in\omega_0 t_r) \times \left[1 - in\omega_0 \int (t - t_r) f_r(t - t_r) dt - \tfrac{1}{2} n^2 \omega_0^2 \int (t - t_r)^2 f_r(t - t_r) dt + \cdots \right],$$

provided the time τ_s that a *sheet* characteristically takes to cross the probe is such that $n\omega_0 \tau_s \ll 1$, or, equivalently, that the frequency $n\omega_0$ of the harmonic concerned is small compared with $(\tau_s)^{-1}$.

Similar contributions arise from the passage of each sheet past the observation point and it follows as before that

$$\overline{C_n C_n^*} \approx \frac{1}{4\pi^2 n^2} \overline{\sum_r (\Delta\rho)_r [1 - n^2 \omega_0^2 (I_1 + I_2)]}, \quad (2.7)$$

where

$$\left. \begin{aligned} I_1 &= \left[\int (t - t_r) f_r(t - t_r) dt \right]^2 \\ I_2 &= \frac{1}{2} \int (t - t_r)^2 f_r(t - t_r) dt \end{aligned} \right\}.$$

If the density gradient in a sheet is symmetrical about its mid-point, then f_r is similarly symmetrical and $I_1 = 0$. The integral I_2, on the other hand, defines the width of the signal associated with the passage of the sheet, i.e., $\tau_s \sim I_2^{\frac{1}{2}}$. Consequently,

$$\overline{C_n C_n^*} = \frac{1}{4\pi^2 n^2} \overline{\sum_r (\Delta\rho)_r^2} [1 - O(n^2 \omega_0^2 \tau_s^2)],$$

$$\overline{C_n C_n^*} = \frac{1}{4\pi^2 n^2} \overline{\sum (\Delta\rho)_r^2}, \quad (2.8)$$

as before, where

$$(\tau_s)^{-1} \gg n\omega_0 \gg (\tau_l)^{-1}. \quad (2.9)$$

Clearly, the existence of a small but finite width of the sheets does not influence the dependence of the mean-square value of the Fourier component on the harmonic number n, but does place an upper limit upon the number n for which the relation remains valid.

The transformation is immediate from the values of $\overline{C_n C_n^*}$, for the record of finite length, to the spectral density $\Phi(\omega)$ of the ensemble of stationary random functions of which the record is a sample. The frequency ω corresponding to the harmonic number n is $n\omega_0$, and the interval $d\omega$ between successive frequencies is simply ω_0. Thus,

$$\Phi(\omega)d\omega=\overline{C_n C_n^*},$$

$$\Phi(\omega)=\frac{1}{4\pi^2}\frac{\omega_0}{\omega^2}\overline{\sum_r (\Delta\rho)_r^2},$$

$$=\frac{1}{2\pi}\left[\frac{1}{T}\overline{\sum_r (\Delta\rho)_r^2}\right]\omega^{-2}, \qquad (2.10)$$

where

$$(\tau_s)^{-1}\gg\omega\gg(\tau_l)^{-1}.$$

If, as seems to be the case in the lower part of the ocean thermocline, the thickness of the sheets is very much less than that of the layers, this ω^{-2} spectral distribution may span several decades. Note particularly that the high frequency (cut-off) of this form is governed not by the Brunt-Väisälä or stability frequency N, but by the inverse of the time that it takes a sheet to traverse the probe, or possibly, by the resolution frequency of the probe itself. The magnitude of the spectral density at a fixed frequency is determined by the average contribution to $\overline{(\Delta\rho)^2}$ at the probe position per unit time; if all the steps are of equal magnitude $\Delta\rho$, the term in the brackets of (2.10), reduces simply to $(\Delta\rho)^2$ times the number of sheet crossings per unit time.

A precisely similar analysis yields spectra for other observational situations of interest involving layered structures in either the atmosphere or the ocean. Suppose, for example, an aircraft is making a constant height traverse of such a structure disturbed by internal waves in the atmosphere, or a towed thermistor is making a traverse in the ocean. The records obtained in either case will have a similar "stepped" structure and the spectra, interpreted in terms of wavenumber, will have the form

$$\Phi(k)=\frac{\Gamma}{2\pi}k^{-2}, \qquad (2.11)$$

where

$$\Gamma=L^{-1}\overline{\sum_r (\Delta\rho)_r^2}$$

is the average contribution per unit length of traverse to the total $(\Delta\rho)^2$ measured. Likewise, if one component of the *horizontal* velocity field is measured rather than the temperature or density, a similar spectral distribution is likely to emerge. The uniformity of density in the layers provides little constraint on turbulent mixing in them, so that the horizontal velocity would be expected to be almost independent of height in any one layer. There may be significant velocity differences between adjacent layers, the shear in the sheet being generally stabilized by the density gradient. A constant level device will again record a stepped signal, giving an energy spectrum measured in a traverse of

$$E(k)=(2\pi)^{-1}\left[L^{-1}\overline{\sum_r (\Delta u)_r^2}\right]k^{-2}, \qquad (2.12)$$

where Δu is the magnitude of the velocity difference in the component measured across the sheet.

It must be emphasized that these spectra (2.10), (2.11) and (2.12) are in a sense artifacts of the analysis and reflect no more than the layered structure of the medium in which the observations are made. Although it is not possible to argue in reverse—the existence of a k^{-2} range in the spectrum does not *prove* that the medium is layered—such behavior can be taken at least as good prima facie evidence for a structure of this kind.

Generally the fluid in the layers themselves will be turbulent and the fluctuations associated with this motion will be superimposed on those produced by the relative motion of the layered structure and the recording device. The larger scales of the turbulent motion are likely to be influenced strongly by geometry, particularly the spacing between the sheets, but the smaller scales can be regarded as statistically independent of both the large scales and the geometry. The spectra of the true turbulent fluctuation would then be simply additive to the spectra derived above and have the forms given by the theory of local similarity, for example, the turbulent energy spectrum,

$$E(k)=(2\pi)^{-1}\left[L^{-1}\overline{\sum_r (\Delta u_r)^2}\right]k^{-2}+\alpha\epsilon_0^{\frac{2}{3}}k^{-5/3}, \qquad (2.13)$$

where α is the Kolmogoroff constant and ϵ_0 the rate of energy dissipation in the turbulent motion. When

$$k^{\frac{1}{3}}\gg\frac{L^{-1}\overline{\sum (\Delta u)_r^2}}{2\pi\alpha\epsilon_0^{\frac{2}{3}}}\equiv k_t^{\frac{1}{3}},$$

then the 5/3 spectrum will obtain; when $k\ll k_t$, we obtain the form (2.12). Similarly, if temperature fluctuations are measured, then in the inertial subrange

$$\Phi(k)=\chi\epsilon_0^{-\frac{1}{3}}k^{-5/3}, \qquad (2.14)$$

as shown by Obukhov (1949) and Corrsin (1951), where χ is the rate of destruction by diffusion of mean square temperature fluctuations. The spectrum (2.11) will then be limited at the high wavenumbers not by the thickness of the sheets but by the transition to the form (2.14) at a wavenumber of the order

$$(2\pi)^{-3}\Gamma^3\chi^{-3}\epsilon_0.$$

Finally, it should be noted that the coherence between signals obtained from two probes separated in the vertical is determined not by the coherence of the vertical motion but by the correlation between the thickness of neighboring layers. If the vertical separation is large compared with a characteristic layer thickness, this coherence would be expected to be small, even if the vertical motion at the two points is perfectly correlated.

3. Single frequency undulations

The spectral density (2.10) at a given frequency ω depends on the distribution of $(\Delta\rho)^2$ in the sheets that pass the probe position, together with the number of such passages per unit time. The former is determined by the structure of the undisturbed fluid and the latter by the nature of the motion producing the undulations.

For example, suppose that the vertical motions are associated with a single low-frequency sinusoidal wave, with amplitude a in the neighborhood of the probe depth and frequency Ω. Note that since currents may be present, Ω is not necessarily the intrinsic frequency determined by the dispersion relation in still water, but the apparent frequency of the vertical oscillation at the fixed probe position. If ν represents the number of sheets per unit depth within $\pm a$ of the probe position, then each layer is sampled just twice per cycle of the undulation, and the total number of layers sampled is $2\nu a$. The number per unit time is thus $2\nu a/(2\pi/\Omega)$ and in the long time interval $T\gg 2\pi/\Omega$, the total number entering the summation is $\pi^{-1}\nu a\Omega T$. Consequently, in (2.10),

$$\sum_r \overline{(\Delta\rho)_r{}^2}=\pi^{-1}\nu a\Omega T\,\overline{(\Delta\rho)^2}, \tag{3.1}$$

where $\overline{(\Delta\rho)^2}$ is the average over the sheets within $\pm a$ of the probe when the layered density structure is in its equilibrium position. The spectrum measured is then

$$\Phi(\omega)=[\tfrac{1}{2}\pi^{-2}\nu a\Omega\overline{(\Delta\rho)^2}]\omega^{-2}, \tag{3.2}$$

provided $(\tau_s)^{-1}\gg\omega\gg\nu a/\Omega$. If the motion in the layers is turbulent, the upper limit to the range is determined not by τ_s but by the transition to the equilibrium range spectrum as described in the previous section.

4. A brief reconsideration of some observations

Evidence for this kind of stepped structure can be seen at least sometimes in internal wave spectra. One of the first long series of observations of temperature fluctuations in a stably stratified region of the ocean was made by Haurwitz *et al.* (1959). These were taken

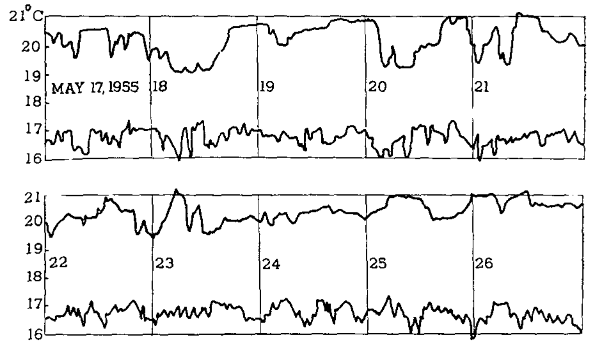

FIG. 1. Records taken by Haurwitz *et al.* (1959) showing temperatures as a function of time at a depth of 50 m (upper trace) and 500 m (lower trace) off Castle Harbor, Bermuda.

off Castle Harbor, Bermuda, between December 1954 and May 1957. Both the actual records and spectra for different intervals are presented by Haurwitz *et al.*, but unfortunately there is no interval over which both are given. A representative sample of the early summer records is shown in Fig. 1, the upper trace giving the temperature at a depth of 50 m and the lower at 500 m. During this interval (May 1955) the overall stable stratification of the upper hundred meters or so was fairly strong and the stepped structure in the upper trace is moderately clear. At the lower thermistor, it is perhaps less so since there appears to be a greater contribution of high-frequency components. Observations taken during the following winter (November 1955 to January 1956) were analyzed spectrally and found to decrease approximately as (frequency)$^{-3}$, but there was no indication of a high-frequency cut-off near the local value of the stability frequency N. Presumably under these circumstances the stepped structure, if present, was relatively weak since the rate of decrease of spectral density with frequency is faster than could be accounted for by the process described in this paper. It is possible, though not very likely as I now believe, that the mechanism of wave turning in a weak shear (Phillips, 1966) may describe the waves observed under the winter conditions; the process is almost certainly irrelevant to the records shown in Fig. 1.

Many measurements have been made of velocity and temperature spectra in the atmosphere under stable conditions. Vinnichenko (1970) describes some of these and gives a valuable list of references. The spectra are rather variable, but they characteristically decrease at low frequencies at some power between -2 and -3, tailing off to the $-5/3$ law at higher frequencies. It is difficult on the evidence now available to decide on those occasions in which the spectral behavior is a result simply of the layering, but if they form a significant set, then there is no necessity in these cases for a continuous cascade of energy between mesoscale and microscale motions. The energy flux may be discontinuous in spectral terms and sporadic in physical space, and result from the intermittent instability and breakdown of larger scale motions.

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