On the Detection of "Inertial" Waves with Pycnocline Followers[1]

CLAES ROOTH AND WALTER DÜING
Rosenstiel School of Marine and Atmospheric Sciences, University of Miami, Fla.

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ABSTRACT

The vertical displacements associated with oscillations near the local inertial frequency f, in a stratified ocean, are found to be significant, even very close to f. For a frequency, $\omega=(1+\epsilon)f$, the ratio of rms vertical velocity to horizontal velocity is $O(\epsilon^{\frac{1}{2}})$. Observations near Hawaii, made with a recently developed pycnocline follower, show inertial oscillation events similar to those found by Webster in current meter records, and discussed theoretically by Crepon and by Pollard.

1. Introduction

Observations of the oceanic density fields still constitute our main source of information about large-scale motions, although efforts to measure currents directly have increased greatly in recent years. The interpretation of direct observations of horizontal velocity components is complicated by the relatively high energy level in motions of short period, especially around the tidal and inertial period (e.g., Webster, 1968). In the presence of baroclinic motion components of substantial amplitude, the use of sensors at a fixed depth may introduce complications in the record which render it hard to interpret (Fig. 1).

This particular complication does not appear if one is able to follow the vertical displacement of a particular density surface. Devices based on this concept, known as thermocline or pycnocline followers, have been developed (e.g., Kullenberg, 1935; LaFond, 1961; Düing, 1969). It should be noted that data from fixed depth arrays, such as thermistor chains, are frequently represented by diagrams showing isotherm depth variations with time derived by interpolation techniques. Only when the vertical resolution of the array, or the amplitude of the vertical displacement, is greater than the vertical density fine structure scale, will such a presentation bear a quasi-linear relation to the actual displacement field (Fig. 2).

The purpose of this paper is to present a discussion of the relation of vertical to horizontal motions for baroclinic waves of frequencies near the local inertial frequency. Thus, we attempt to establish a framework for the interpretation of existing observations and the planning of future experiments with pycnocline followers.

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2. Vertical motions associated with quasi-inertial waves

The equation of continuity provides a direct link between the horizontal and vertical amplitudes and spatial scales of any motion pattern. It is easily seen that only in the physically unrealizable situation of a laterally unlimited domain, with a constant Coriolis parameter, can the classical inertial wave with purely horizontal motions exist. Constraints on the horizontal scale may arise for several reasons.

When dealing with *free waves*, one has mainly to consider three types of constraints. In an unlimited domain, with constant parameters, the dispersion relation determines the relative magnitude of the horizontal and vertical scales of the motion, as a function of wave frequency. This paper deals with this case in particular. When external parameter variations are introduced, such as β-effects or large-scale topographic variations, a geometric scale is imposed. A recent study by Munk and Phillips (1968) exemplifies this case.

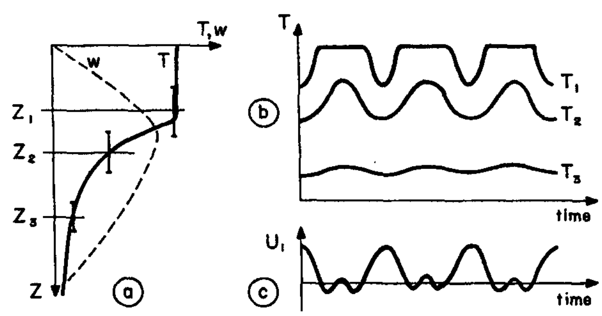

FIG. 1. Schematic response of sensors at fixed depths z_1, z_2, z_3. a) Distribution of temperature and vertical velocity with depth. The bars indicate the range of the vertical displacements. b) Schematic time series T_1, T_2, T_3 observed at fixed depths z_1, z_2, z_3. c) Schematic time series of horizontal current component observed at level z_1.

Only for waves substantially shorter than such externally imposed scales is the locally evaluated dispersion relation useful. A third constraint is provided by the vicinity of side boundaries or small-scale bottom irregularities. Examples for this category are the Kelvin wave and a study by Crepon (1967) of the response of a semi-infinite basin to intermittent forcing near the boundary.

On the other hand, *forced waves* will, in general, have the spatial scale of the forcing mechanism. Lateral boundaries may generate edge effects of substantially different scale, enhancing or damping the motion relative to the response far from boundaries dependent on the particular circumstances at hand. The variety of problems that could be considered in this class is very great, including, for example, the tides. Several investigations relating to flows induced by atmospheric processes exist in the literature. We will only mention here Belyaev (1967), Tomczak (1968) and Kielmann *et al.* (1970) as particularly relevant to the present problem. We will restrict ourselves to a discussion of the free wave case in some detail.

Following Munk and Phillips, but with f-plane modeling introduced, our equations are

$$\frac{\partial u}{\partial t}-fv+\frac{\partial p}{\partial x}=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+fu+\frac{\partial p}{\partial y}=0, \tag{2}$$

$$\frac{\partial w}{\partial t}-b+\frac{\partial p}{\partial z}=0, \tag{3}$$

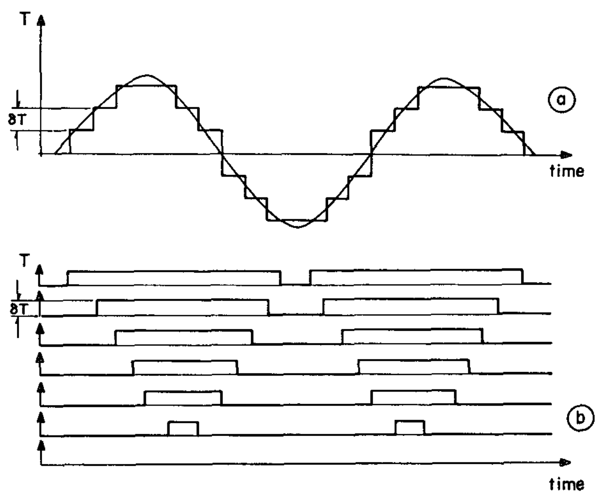

FIG. 2. Schematic response in the case of microstructure. a) Response of a single sensor at a fixed depth to a wave of large amplitude relative to microstructure. Several steps move past the sensor. The continuous curve represents the response for a smooth temperature profile. b) Response of a high-resolution sensor array to vertical motion of amplitude less than the step structure. A single step moves through the sensor array. The step amplitude is δT.

$$\frac{\partial b}{\partial t}+N^2(z)w=0, \tag{4}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{5}$$

where p is the perturbation pressure divided by the mean density, b the buoyancy force per unit mass, and N the Brunt-Väisälä frequency.

Variable elimination in the system (1)–(5) leads for constant N to the operator L,

$$L\equiv\left(f^2+\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2}{\partial z^2}+\left(N^2+\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right), \tag{6}$$

whose properties determine the character of the motion patterns. The vertical motion, for example, is governed by

$$L(w)=0. \tag{7}$$

For our present purpose it is most effective to consider two-dimensional motion patterns characterized by a streamfunction in the plane of propagation of the wave. We then have

$$L(\psi)=0, \tag{8}$$

with the boundary conditions $\psi=0$ at all rigid boundaries. The sea surface can for the present purpose be considered a rigid boundary.

This equation is generally approached either through the use of normal modes or by the method of characteristics. The preference for one or the other will depend on the boundary conditions imposed on the problem. The respective type solutions are

$$\psi_n(\omega)=\mathrm{Re}\left\{\sum_{\nu=1}^{\infty}A_\nu\exp[i\nu(\kappa x+lz)]\right\}, \tag{9}$$

$$\psi_c(\omega)=\psi_1(\kappa x+lz)+\psi_2(\kappa x-lz). \tag{10}$$

Here A_ν is a set of arbitrary complex numbers, while ψ_1 and ψ_2 are arbitrary real-valued functions, subject to the boundary conditions of the particular problem considered. For example, for a domain limited by parallel horizontal planes, ψ_1 and ψ_2 will be periodic in their arguments with the period lH where H is the vertical separation of the boundaries.

Consider now the ratio of amplitudes of horizontal to vertical velocities. We have for the normal mode solutions

$$\frac{\partial\psi_N}{\partial x}=\mathrm{Re}\left\{\Sigma i\nu\kappa A_\nu\exp[i\nu(\kappa x+lz)]\right\}=w, \tag{11}$$

$$\frac{\partial\psi_N}{\partial z}=\mathrm{Re}\left\{\Sigma i\nu lA_\nu\exp[i\nu(\kappa x+lz)]\right\}=-u, \tag{12}$$

from which we have

$$\frac{\overline{w^2}}{\overline{u^2}}=\frac{\kappa^2\Sigma\nu^2A_\nu A_\nu^*}{l^2\Sigma\nu^2A_\nu A_\nu^*}=\frac{\kappa^2}{l^2}. \tag{13}$$

For the alternative representation on characteristics, we get analogously

$$\frac{\overline{w^2}}{\overline{u^2}}=\frac{\kappa^2}{l^2}\frac{(\overline{\psi_1'^2}+\overline{\psi_2'^2}-\overline{2\psi_1'\psi_2'})}{(\overline{\psi_1'^2}+\overline{\psi_2'^2}+\overline{2\psi_1'\psi_2'})}, \tag{14}$$

with the prime indicating differentiation with respect to the argument, and the overbar an average over the spatial domain considered.

Here we find that the ratio is the same as for the normal mode wave, provided that ψ_1' and ψ_2' are orthogonal to each other over the domain considered. A detailed discussion of this condition would carry too far in this context. We shall only mention that it seems to be fulfilled except in physically unreasonable circumstances, such as infinite horizontal wavelength. Now the ratio κ^2/l^2 can be directly evaluated from the form of the operator L; thus,

$$\frac{\kappa^2}{l^2}=\frac{\omega^2-f^2}{N^2-\omega^2}. \tag{15}$$

Since we are concerned with behavior near the inertial frequency in particular, we define a relative deviation of frequency ϵ, i.e.,

$$\omega=f(1+\epsilon). \tag{16}$$

We have then

$$\frac{\kappa^2}{l^2}=\frac{f^2[(1+\epsilon)^2-1]}{N^2-\omega^2}, \tag{17}$$

or allowing for $N\gg f$, $\epsilon\ll 1$, approximately

$$\frac{\kappa^2}{l^2}\approx 2\epsilon\frac{f^2}{N^2}, \tag{18}$$

or

$$\frac{\overline{w^2}^{\frac{1}{2}}}{\overline{u^2}^{\frac{1}{2}}}\approx(2\epsilon)^{\frac{1}{2}}\frac{f}{N}. \tag{19}$$

To give an order-of-magnitude estimate (related to the observations discussed in the next section), we take $f=5\times10^{-5}$ sec^{-1}, roughly corresponding to the latitude of Hawaii, and $N=10^{-2}$ sec^{-1}, or a Brunt-Väisälä period of $\sim$10 min. This gives us

$$\frac{\overline{w^2}^{\frac{1}{2}}}{\overline{u^2}^{\frac{1}{2}}}=5\times10^{-3}x(2\epsilon)^{\frac{1}{2}}. \tag{20}$$

A graph of (20) is given in Fig. 3, with an indication of approximate corresponding deviations of oscillation

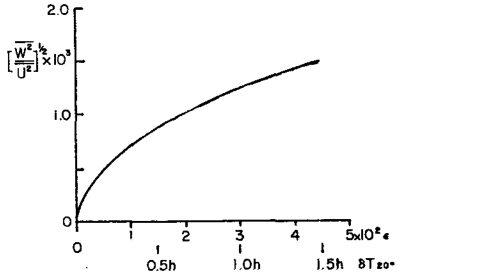

FIG. 3. Ratio of vertical to horizontal velocities as a function of small deviations from the local inertial frequency.

period from the local inertial period at the latitude of Hawaii, or approximately 20N. Note that the velocity ratio stays in the order-of-magnitude range of 10^{-3} within most of the frequency range indicated. Even at $\epsilon=0.005$, corresponding to only a 15-min deviation from the inertial period at the latitude of Hawaii, a value of 0.5×10^{-3} for the velocity ratio is indicated.

Thus, for an rms horizontal velocity of 10 cm sec^{-1}, vertical displacements of approximately ±1.5 m would occur, a displacement easily detected by a pycnocline follower.

3. Observations

We will now briefly discuss a characteristic time series that was obtained by using a density follower. The results not only demonstrate that "inertial" oscillations with considerable vertical amplitudes can occur, but they also display some further characteristics of these transient motions.

The observational site was located 1.5 mi off the southwest coast of the Island of Oahu, Hawaii, at a water depth of 220 m. Since the Island base consists of an extinct volcano, the nearshore topography is characterized by a fairly uniform slope. In the area of observations, the bottom slope is approximately 10–15°. The method of observation was described in more detail by Düing (1969). It basically consists of a taut-wire mooring to which a sliding density-balanced float is attached. Vertical depth variations of the float around the main pycnocline are recorded with a self-contained pressure recorder. Fig. 4a presents observations obtained during May 1968 at the described position. It is necessary to note that the slight cut-offs in the upper left-hand portion of the record are artificial. They were caused by an electrical zero-suppression of the pressure recorder, not anticipating that such large vertical displacements would occur.

The power spectrum for this time series is presented in Fig. 5. The largest peak in the spectrum at 12.5 hr represents semidiurnal internal waves. The implications involved here have been discussed by Düing. In this paper we will focus our attention on the peak at 34.1 hr, since it is close to the local inertial period. The exact period for the observation site is $T_i=32.9$ hr (latitude

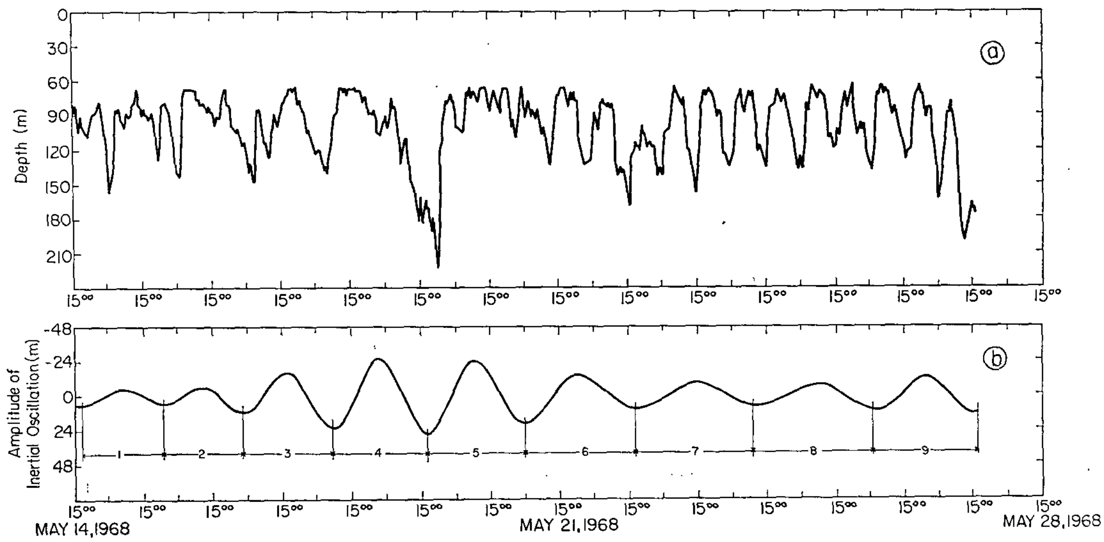

FIG. 4. Internal waves of large amplitudes 1.5 mi off the southwest coast of Oahu, Hawaii, at a water depth of 220 m. a) Observations were made with an anchored density-follower balanced to the density near the largest gradient of the pycnocline. b) Band-passed time series around the inertial period. Numbers 1–9 correspond to periods t in Table 1.

21°26′N). The original time series was band-passed around the period of 34.1 hr, using a window between 28 and 40 hr (Fig. 4b). It is important to note that the energy levels around the 34.1-hr peak are low, which ensures a relatively unbiased band-passed time series.

The interpretation of a single record like this must be handled with some caution. It can, on the other hand, be related to the current meter observations discussed by Webster (1968), and to some recent theoretical studies.

Webster finds, in general, that observations of "inertial" oscillations tend to be biased toward frequencies somewhat above the local inertial frequency. This does not mean, however, that observations of lower than inertial frequency are uncommon. Two different theoretical approaches have been taken toward the interpretation of the various observations of almost inertial oscillations. Munk and Phillips (1968) derive model equations for the latitudinal amplitude variation of inertial gravity waves. They find significant amplitudes over a substantial range of lattitudes around the inertial latitude (where the inertial frequency equals the wave frequency), with a bias toward lower latitudes in agreement with Webster's findings.

Crepon (1967) and Pollard (1969), on the other hand, deal with the response to intermittent forcing. In their studies finite wave trains are generated.

The motions are, therefore, necessarily characterized by a finite bandwidth. The shorter the wave train is, the greater the bandwidth. A qualitative agreement exists between our observations of the variation of apparent period with time during the passage of the wave group (see Table 1), and corresponding results for Pollard's computations, which show a trend of apparent period during the passage of the wave group. Two aspects of the tabulated data may seem puzzling, in the light of our discussion so far. First, the trend of apparent period begins with values safely within the range possible for free waves, but it progresses far past the inertial period, never quite returning to it. There are a variety of possible explanations for this deviation from the simple theory given above. Our analysis applies to oscillations relative to the mean motion of the

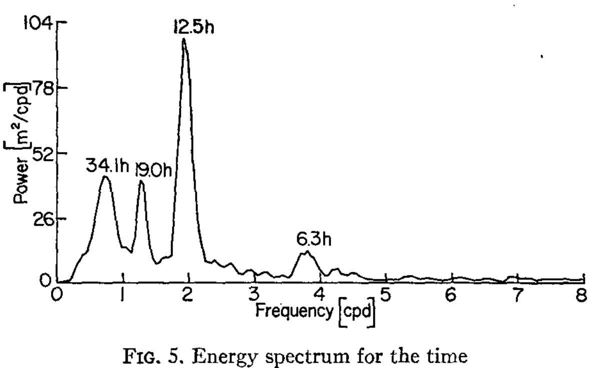

FIG. 5. Energy spectrum for the time series presented in Fig. 4a.

TABLE 1. Variation of period for observed inertial observations. (The number of cycles corresponds to the number in Fig. 4b.)

t (cycles)	T_{obs} (hr)	T_i/T_{obs}
1	27.0	1.22
2	27.5	1.20
3	31.0	1.06
4	*32.5**	*1.01**
5	34.0	0.97
6	38.5	0.86
7	40.5	0.82
8	41.0	0.80
9	34.0	0.97

* Closest match to inertial period of 32.9 hr.

body of water. The intrinsic frequency will be Doppler-shifted, if that motion is not steady. By the same token, it is the absolute vorticity of the water mass involved which determines the dispersion relation. An anticyclonic vorticity of $0.2f$ would be sufficient to shift the critical period to the limit of the observations. But such exercises are not very meaningful, in part because we are dealing with a short-term transient oscillation. Thus, the signal must have a broad bandwidth, i.e., the motion of free oscillations must be applied with caution. In addition, one must consider the spatial dependence of the wave modes, as discussed by Munk and Phillips (1968). Penetration of what might be called a forced oscillation considerably beyond the critical latitude is evident in their case. Since its decay with increasing latitude is exponential, essentially on the scale of the zonal wavelength, a wave critical at 17° latitude, with a 41-hr period, would still have a substantial effect at Hawaii (21.5N) provided its zonal wavelength $\gtrsim 1000$ km. Considering this variety of possibilities, it is impossible to discuss meaningfully the reasons for the particular form of the amplitude variation with time.

4. Conclusions

In considering various observations reported in the literature, one gets the impression that observers, who have used devices like thermistor chains or bathythermographs, have accepted that when a peak in their spectra of vertical displacement occurs at the local inertial period, they deal with an inertial oscillation (e.g., Gonella *et al.*, 1967). Authors like Webster, who deal primarily with observations of extremely low coherence over relatively small vertical separations, have tended to emphasize the horizontality of the motions to the point where an impression is created that no vertical displacements occur. We have shown that existing theories for inertial gravity waves can easily reconcile these points of view. The observations demonstrate the occurrence of large vertical amplitudes near the inertial frequency. Provided that inertial oscillations are generally characterized by a systematic drift of frequency, one might be able to use this phenomenon as a tool to sort inertial oscillations from other wave phenomena. In particular, it may be possible to distinguish astronomically forced diurnal internal motions from meteorologically generated inertial oscillations at the critical latitude of 30°, on the grounds that the energy peak associated with the latter is quite broad, and smooth in its structure.

In consideration of the problems caused by the temperature microstructure, as well as in the vertical distribution of horizontal velocity, one must conclude that the entire frequency range of nongeostrophic baroclinic motions in the ocean can be effectively studied with devices which follow the actual displacements of density surfaces.

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