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STANDARD STOCHASTIC COALESCENCE WITH SUM KERNELS

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Abstract
We build a Markovian system of particles entirely characterized by their masses, in which each pair of particles with masses x and y coalesce at rate $K(x,y) \simeq x^\lambda + y^\lambda$, for some $\lambda \in (0,1)$, and such that the system is initially composed of infinitesimally small particles.

1 Introduction

A stochastic coalescent is a Markovian system of macroscopic particles entirely characterized by their masses, in which each pair of particles with masses x and y merge into a single particle with mass $x+y$ at some given rate $K(x,y)$. This rate K is called the coagulation kernel. We refer to the review of Aldous [3] on stochastic coalescence, on its links with the Smoluchowski coagulation equation.

When the initial state consists of a finite number of particles, the stochastic coalescent obviously exists without any assumption on K, and is known as the Marcus-Lushnikov process [9, 8]. When there are initially infinitely many particles, stochastic coalescence with constant, additive, and multiplicative kernels have been extensively studied, see Kingman [7], Aldous-Pitman [2], Aldous [1]. The case of general coagulation kernels has first been studied by Evans-Pitman [4], and their results have recently been completed in [5, 6].

Of particular importance seems to be the *standard version* of the stochastic coalescent, that is the stochastic coalescent which starts from *dust*. By *dust* we mean a state with positive total mass in which all the particles have an infinitesimally small mass. Indeed, such a version of stochastic coalescence should describe a sort of typical behaviour, since it starts with a non really specified initial datum. There are also some possible links between such a standard stochastic coalescent and the Smoluchowski coagulation equation for large times, see Aldous [3, Open Problem 14].

The well-known Kingman coalescent [7] is a stochastic coalescent with constant kernel $K(x,y) = 1$ starting from dust at time $t = 0$. The standard additive (resp. multiplicative) coalescent , see [2] (resp. [1]) is a stochastic coalescent with kernel $K(x,y) = x + y$ (resp. $K(x,y) = xy$) starting from dust at time $t = -\infty$. It seems that no result is available for general kernels. Our aim in this paper is to show the existence of a standard coalescent for kernels $K(x,y) \simeq x^\lambda + y^\lambda$, for $\lambda \in (0,1)$, by using a refinement of the methods introduced in [5].

We have no uniqueness result, but the process we build is however a Markov process. The method we use is very restrictive: we are not able to study, for example, the case $K(x,y) = (xy)^{\lambda/2}$, for $\lambda \in (0,1)$.

In Section 2, we introduce our notations, recall the main result of [5], and state our result. The proofs are handled in Section 3.

2 Main result

We denote by $\mathcal{S}^\downarrow$ the set of non-increasing sequences $m = (m_k)_{k\geq 1}$ with values in $[0,\infty)$. A state $m \in \mathcal{S}^\downarrow$ represents the ordered masses in a particle system. For $\alpha > 0$ and $m \in \mathcal{S}^\downarrow$, we denote $||m||_\alpha := \sum_{k=1}^\infty m_k^\alpha$. Remark that the total mass of a state $m \in \mathcal{S}^\downarrow$ is simply given by $||m||_1$.

We will use, for $\lambda \in (0,1]$, the set of states with total mass 1 and with a finite moment of order λ:

$$\ell_\lambda = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^\downarrow, \ ||m||_1 = 1, \ ||m||_\lambda < \infty \right\}. \tag{2.1}$$

We also consider the sets of finite particle systems with total mass 1:

$$\ell_0 = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^\downarrow, \ ||m||_1 = 1, \ \inf\{k \geq 1, \ m_k = 0\} < \infty \right\}. \tag{2.2}$$

Remark that for $0 < \lambda_1 < \lambda_2$, the inclusions $\ell_0 \subset \ell_{\lambda_1} \subset \ell_{\lambda_2}$ hold.

For $i < j$, the coalescence between the i-th and j-th particles is described by the map $c_{ij} : \mathcal{S}^\downarrow \mapsto \mathcal{S}^\downarrow$, with

$$c_{ij}(m) = \text{reorder}(m_1, ..., m_{i-1}, m_i + m_j, m_{i+1}, ..., m_{j-1}, m_{j+1}, ...). \tag{2.3}$$

A coagulation kernel is a function K on $[0,\infty)^2$ such that $0 \leq K(x,y) = K(y,x)$.

Remark 2.1 *Consider a coagulation kernel K. For any $m \in \ell_0$, there obviously exists a unique (in law) strong Markov ℓ_0-valued process $(M(m,t))_{t\geq 0}$ with infinitesimal generator $\mathcal{L}$ defined, for all $\Phi : \ell_0 \mapsto \mathbb{R}$, all $\mu \in \ell_0$, by*

$$\mathcal{L}\Phi(\mu) = \sum_{1\leq i<j<\infty} K(\mu_i, \mu_j) \left[\Phi(c_{ij}(\mu)) - \Phi(\mu) \right]. \tag{2.4}$$

The process $(M(m,t))_{t\geq 0}$ is known as the Marcus-Lushnikov process.

Notice that (2.4) is well-defined for *all* functions Φ since the sum is actually finite. Indeed, $c_{ij}(\mu) = \mu$ as soon as $\mu_j = 0$. We refer to Aldous [3] for many details on this process.

To state our main result, we finally need to introduce some notations: for $\lambda \in (0,1)$, and for $m, \tilde{m} \in \ell_\lambda$, we consider the distance

$$d_\lambda(m,\tilde{m}) = \sum_{k\geq 1} \left| m_k^\lambda - \tilde{m}_k^\lambda \right|. \tag{2.5}$$

Remark that for m^n, m in ℓ_λ,

$$\lim_n d_\lambda(m^n, m) = 0 \Longleftrightarrow \lim_n \sum_{i\geq 1} |m_i^n - m_i|^\lambda = 0. \tag{2.6}$$

The main result of [5, Corollary 2.5] (see also [6, Theorem 2.2]) is the following.

Theorem 2.2 *Let K be a coagulation kernel satisfying, for some $\lambda \in (0,1]$ and some $a \in (0,\infty)$, for all $x,y,z \in [0,1]$,*

$$|K(x,y) - K(x,z)| \leq a|y^\lambda - z^\lambda|. \tag{2.7}$$

Endow ℓ_λ with the distance d_λ.
(i) For any $m \in \ell_\lambda$, there exists a unique (in law) strong Markov process $(M(m,t))_{t\geq 0} \in \mathbb{D}([0,\infty),\ell_\lambda)$ enjoying the following property. For any sequence of initial states $m^n \in \ell_0$ such that $\lim_{n\to\infty} d_\lambda(m^n,m) = 0$, the sequence of Marcus-Lushnikov processes $(M(m^n,t))_{t\geq 0}$ converges in law, in $\mathbb{D}([0,\infty),\ell_\lambda)$, to $(M(m,t))_{t\geq 0}$.
(ii) The obtained process is Feller in the sense that for all $t \geq 0$, the application $m \mapsto Law(M(m,t))$ is continuous from ℓ_λ into $\mathcal{P}(\ell_\lambda)$.

Notation 2.3 *Under the assumptions of Theorem 2.2, we will denote by $(P_t^K)_{t\geq 0}$ the Markov semi-group of $(M(m,t))_{t\geq 0, m\in\ell_\lambda}$: for $t \geq 0$, for $\Phi : \ell_\lambda \mapsto \mathbb{R}$ measurable and bounded, and for $m \in \ell_\lambda$, $P_t^K \Phi(m) := E[\Phi(M(m,t))]$.*

The result we will prove in the present paper is the following.

Theorem 2.4 *Let K be a coagulation kernel satisfying for some $\lambda \in (0,1)$, some $a \in (0,\infty)$ and some $\varepsilon > 0$, for all $x,y,z \in [0,\infty)$,*

$$|K(x,y) - K(x,z)| \leq a|y^\lambda - z^\lambda| \quad \text{and} \quad K(x,y) \geq \varepsilon(x^\lambda + y^\lambda). \tag{2.8}$$

There exists a Markov process $(M^*(t))_{t\in(0,\infty)}$ with semi-group $(P_s^K)_{s\geq 0}$, belonging a.s. to $\mathbb{D}((0,\infty),\ell_\lambda)$, such that a.s., $\lim_{t\to 0+} M_1^*(t) = 0$, where $M^*(t) = (M_1^*(t), M_2^*(t), ...) \in \ell_\lambda$.

This result is not obvious, because clearly, $M^*(t)$ goes out of ℓ_λ as t decreases to $0+$. Indeed, we have $\sum_i M_i^*(t) = 1$ for all $t > 0$, and $\lim_{t\to 0+} \sup_i M_i^*(t) = 0$, so that necessarily, since $\lambda \in (0,1)$ $\limsup_{t\to 0+} ||M^*(t)||_\lambda = \infty$.

The main ideas of the proof are the following: first, there is a *regularization* of the moment of order λ. This means that in some sense, even if the moment of order λ is infinite at time 0, it becomes finite for all positive times.
Next, we prove a refined version of the Feller property obtained in [5], which shows that the map $m \mapsto Law(M(m,t))$ is actually continuous for the distance d_1 on the level sets $\{m \in \ell_\lambda, ||m||_\lambda \leq A\}$. We conclude using that these level sets are compact in ℓ_1 endowed with d_1.

3 Proof

Let us first recall the following easy compactness result.

Lemma 3.1 *For any $A > 0$, any $\lambda \in (0,1)$, the set*

$$\ell_\lambda(A) = \{m \in \ell_\lambda,\ \ ||m||_\lambda \leq A\} \tag{3.1}$$

is compact in (ℓ_1, d_1): for any sequence $(m^n)_{n\geq 1}$ of elements of $\ell_\lambda(A)$, we may find $m \in \ell_\lambda(A)$ and a subsequence $(m^{n_k})_{k\geq 1}$ such that $\lim_k d_1(m^{n_k}, m) = 0$.

We now check that under a suitable lowerbound assumption on the coagulation kernel, there is a *regularization* property for the moment of order λ of the stochastic coalescent.

Lemma 3.2 *Let $\lambda \in (0,1)$ be fixed, and consider a coagulation kernel K satisfying, for some $\varepsilon \in (0,\infty)$, for all $x, y \in [0,1]$,*

$$K(x,y) \geq \varepsilon(x^\lambda + y^\lambda). \tag{3.2}$$

For each $m \in \ell_0$, consider the Marcus-Lushnikov process $(M(m,t))_{t\geq 0}$. There exists a constant C, depending only on λ and ε, such that for all $t > 0$,

$$\sup_{m\in\ell_0}\left[\sup_{s\geq t}||M(m,s)||_\lambda\right] \leq C\left(1 \vee \frac{1}{t}\right) \tag{3.3}$$

Proof First of all notice that since $\lambda \in (0,1)$, we have for all $1 \leq i < j$, for all $m \in \ell_\lambda$, $||c_{ij}(m)||_\lambda = ||m||_\lambda + (m_i + m_j)^\lambda - m_i^\lambda - m_j^\lambda \leq ||m||_\lambda$. Hence the moment of order λ of $M(m,t)$ decreases a.s. at each coalescence. Thus for any $m \in \ell_0$, the map $t \mapsto ||M(m,t)||_\lambda$ is a.s. non-increasing. It thus suffices to check that for some constant $C > 0$, for all $t > 0$, $\sup_{m\in\ell_0} E\left[\Phi(M(m,t))\right] \leq C\left(1 \vee \frac{1}{t}\right)$, where $\Phi : \ell_\lambda \mapsto \mathbb{R}_+$ is defined by $\Phi(m) = ||m||_\lambda$.
An easy computation shows that for $0 \leq y \leq x$,

$$\begin{aligned} x^\lambda + y^\lambda - (x+y)^\lambda &= x\left[x^{\lambda-1} - (x+y)^{\lambda-1}\right] + y\left[y^{\lambda-1} - (x+y)^{\lambda-1}\right] \\ &\geq y\left[y^{\lambda-1} - (y+y)^{\lambda-1}\right] \geq (1 - 2^{\lambda-1})y^\lambda. \end{aligned} \tag{3.4}$$

Using furthermore (2.4) and (3.2), we get, for any $m \in \ell_0$, since for $i < j$, $0 \leq m_j \leq m_i$,

$$\begin{aligned} \mathcal{L}\Phi(m) &= -\sum_{i<j} K(m_i, m_j)[m_i^\lambda + m_j^\lambda - (m_i + m_j)^\lambda] \\ &\leq -\varepsilon(1 - 2^{\lambda-1})\sum_{i<j}(m_i^\lambda + m_j^\lambda)m_j^\lambda. \end{aligned} \tag{3.5}$$

Setting $c = \varepsilon(1 - 2^{\lambda-1})/2$, we obtain, using that $m_i \leq 1$ for all i,

$$\begin{aligned} \mathcal{L}\Phi(m) &\leq -2c\sum_{i<j} m_i^\lambda m_j^\lambda = c\sum_{i\neq j} m_i^\lambda m_j^\lambda = -c\left\{\left(\sum_{i\geq 1} m_i^\lambda\right)^2 - \sum_{i\geq 1} m_i^{2\lambda}\right\} \\ &\leq -c\left(\Phi^2(m) - \Phi(m)\right). \end{aligned} \tag{3.6}$$

Using the Cauchy-Schwarz inequality, we deduce that for all $t \geq 0$, all $m \in \ell_0$,

$$\begin{aligned} \frac{d}{dt} E\left[\Phi(M(m,t))\right] &= E\left[\mathcal{L}\Phi(M(m,t))\right] \\ &\leq -c\left(E\left[\Phi(M(m,t))\right]^2 - E\left[\Phi(M(m,t))\right]\right). \end{aligned} \tag{3.7}$$

Using finally that $E\left[\Phi(M(m,0))\right] = ||m||_\lambda \in [1,\infty)$ since $m \in \ell_0$, we easily deduce from this differential inequality that for all $t > 0$, $E[\Phi(M(m,t))] \leq (1-e^{-ct})^{-1}$. For some constant $C > 0$ depending only on c, we obtain the bound $E[\Phi(M(m,t))] \leq C(1 \vee \frac{1}{t})$. □

We next prove a sort of refined version of the Feller property obtained in [5].

Lemma 3.3 *Let $\lambda \in (0,1)$ be fixed, and consider a coagulation kernel K satisfying, for some $a \in (0,\infty)$, for all $x,y,z \in [0,1]$,*

$$|K(x,y) - K(x,z)| \leq a|y^\lambda - z^\lambda|. \tag{3.8}$$

It is possible to build simultaneously all the processes $(M(m,t))_{t\geq 0}$, for all $m \in \ell_\lambda$, in such a way that for all $t \geq 0$, all $m, \tilde{m} \in \ell_\lambda$,

$$\left[\sup_{[0,t]} d_1(M(m,s), M(\tilde{m},s))\right] \leq d_1(m,\tilde{m}) e^{8a(||m||_\lambda + ||\tilde{m}||_\lambda)t}. \tag{3.9}$$

Proof We use here [5, Definition 2.1] and [5, Theorem 2.4]. We set $\bar{K} = \sup_{(x,y)\in[0,1]^2} K(x,y)$. We consider a Poisson measure $N(dt, d(i,j), dz)$ on $[0,\infty) \times \{(i,j) \in \mathbb{N}^2, i < j\} \times [0,\bar{K}]$ with intensity measure $dt \left(\sum_{1\leq k<l<\infty} \delta_{(k,l)}\right) dz$, and denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the associated canonical filtration.
For $m \in \ell_\lambda$, we know from [5] that there exists a unique ℓ_λ-valued càdlàg $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process $(M(m,t))_{t\geq 0}$ such that a.s., for all $t \geq 0$,

$$\begin{aligned} M(m,t) = m + \int_0^t \int_{i<j} \int_0^{\bar{K}} &\left[c_{ij}(M(m,s-)) - M(m,s-)\right] \\ &\mathbb{1}_{\{z \leq K(M_i(m,s-), M_j(m,s-))\}} N(ds, d(i,j), dz). \end{aligned} \tag{3.10}$$

Furthermore, this process $(M(m,t))_{t\geq 0}$ is a Markov process starting from m with semi-group $(P_s^K)_{s\geq 0}$ defined in Notation 2.3.
Remark here that the processes $(M(m,t))_{t\geq 0}$ for different initial data are coupled, in the sense that they are all built with the same Poisson measure N.
Before handling the computations, let us recall the following estimates, that can be found in [5, Corollary 3.2]: for all $1 \leq i < j$, all $m, \tilde{m} \in \ell_1$,

$$d_1(c_{ij}(m), c_{ij}(\tilde{m})) \leq d_1(m,\tilde{m}), \tag{3.11}$$

$$d_1(c_{ij}(m), \tilde{m}) \leq d_1(m,\tilde{m}) + 2m_j. \tag{3.12}$$

We may now compute. Let thus $m, \tilde{m} \in \ell_\lambda$. We have, for any $t \geq 0$,

$$d_1(M(m,t), M(\tilde{m},t)) = d_1(m,\tilde{m}) + A_t + B_t^1 + B_t^2, \tag{3.13}$$

where, setting $\Delta_{ij}(s) := d_1(c_{ij}(M(m,s)), c_{ij}(M(\tilde{m},s))) - d_1(M(m,s), M(\tilde{m},s))$,

$$A_t = \int_0^t \int_{i<j} \int_0^{\bar{K}} \mathbb{1}_{\{z \le K(M_i(m,s-),M_j(m,s-)) \wedge K(M_i(\tilde{m},s-),M_j(\tilde{m},s-))\}} \Delta_{ij}(s-) N(ds, d(i,j), dz), \tag{3.14}$$

while, setting $\Gamma_{ij}(s) := d_1(c_{ij}(M(m,s)), M(\tilde{m},s)) - d_1(M(m,s), M(\tilde{m},s))$,

$$B_t^1 = \int_0^t \int_{i<j} \int_0^{\bar{K}} \mathbb{1}_{\{K(M_i(\tilde{m},s-),M_j(\tilde{m},s-)) \le z \le K(M_i(m,s-),M_j(m,s-))\}} \Gamma_{ij}(s-) N(ds, d(i,j), dz), \tag{3.15}$$

and where B_t^2 is the same as B_t^1 permuting the roles of m and $\tilde{m}$.
Due to (3.11), we know that $A_t \le 0$ for all $t \ge 0$ a.s. Next, using (3.12) and (3.8), we conclude that, setting $(x)_+ = \max(x, 0)$,

$$\begin{aligned} E\left[\sup_{[0,t]} B_s^1\right] &\le \int_0^t ds E\Big[\sum_{i<j} 2M_j(m,s) \\ &\quad \Big(K(M_i(m,s), M_j(m,s)) - K(M_i(\tilde{m},s), M_j(\tilde{m},s))\Big)_+\Big] \\ &\le 2a \int_0^t ds E\Big[\sum_{i<j} M_j(m,s) \\ &\quad \Big(|M_i(m,s)^\lambda - M_i(\tilde{m},s)^\lambda| + |M_j(m,s)^\lambda - M_j(\tilde{m},s)^\lambda|\Big)\Big]. \end{aligned} \tag{3.16}$$

But one easily checks that $(x^{1-\lambda} + y^{1-\lambda})|x^\lambda - y^\lambda| \le 2|x-y|$ for all $x, y \in [0, \infty)$, so that, since $M_j(m,s) \le M_i(m,s)$ for all $i < j$,

$$\begin{aligned} &\sum_{i<j} M_j(m,s)|M_i(m,s)^\lambda - M_i(\tilde{m},s)^\lambda| \\ &\le \sum_{i<j} M_j(m,s)^\lambda (M_i(m,s)^{1-\lambda} + M_i(\tilde{m},s)^{1-\lambda})|M_i(m,s)^\lambda - M_i(\tilde{m},s)^\lambda| \\ &\le 2||M(m,s)||_\lambda \times d_1(M(m,s), M(\tilde{m},s)). \end{aligned} \tag{3.17}$$

By the same way,

$$\begin{aligned} &\sum_{i<j} M_j(m,s)|M_j(m,s)^\lambda - M_j(\tilde{m},s)^\lambda| \\ &\le \sum_{i<j} M_i(m,s)^\lambda (M_j(m,s)^{1-\lambda} + M_j(\tilde{m},s)^{1-\lambda})|M_j(m,s)^\lambda - M_j(\tilde{m},s)^\lambda| \\ &\le 2||M(m,s)||_\lambda \times d_1(M(m,s), M(\tilde{m},s)). \end{aligned} \tag{3.18}$$

We conclude that for all $t \ge 0$,

$$E\left[\sup_{[0,t]} B_s^1\right] \le 8a \int_0^t E\left[||M(m,s)||_\lambda \times d_1(M(m,s), M(\tilde{m},s))\right] ds. \tag{3.19}$$

Using the same computation for B_t^2 and the fact that the maps $t \mapsto ||M(m,t)||_\lambda$ and $t \mapsto ||M(\tilde{m},t)||_\lambda$ are a.s. non-increasing, we finally get

$$E\left[\sup_{[0,t]} d_1(M(m,s),M(\tilde{m},s))\right] \leq d_1(m,\tilde{m}) \tag{3.20}$$
$$+8a\left(||m||_\lambda + ||\tilde{m}||_\lambda\right)\int_0^t ds E\left[d_1(M(m,s),M(\tilde{m},s))\right].$$

The Gronwall Lemma allows us to conclude. □

We may finally handle the

Proof of Theorem 2.4 We divide the proof into several steps.

Step 1. For each $n \in \mathbb{N}$, let $m^n = (1/n, ..., 1/n, 0, ...) \in \ell_0$, which is an approximation of dust. We then consider, for each $n \in \mathbb{N}$, the Marcus-Lushnikov process $(M(m^n,t))_{t\geq 0}$. For each $t > 0$, using Lemma 3.2 and the notations of Lemma 3.1, we obtain

$$\lim_{A\to\infty} \inf_{n\in\mathbb{N}} P\left(M(m^n,t) \in \ell_\lambda(A)\right) = 1. \tag{3.21}$$

Due to Lemma 3.1, we deduce that for each $t > 0$, we may find a subsequence n_k such that $(M(m^{n_k},t), ||M(m^{n_k},t)||_\lambda)$ converges in law in $\ell_1 \times [0,\infty)$, ℓ_1 being endowed with the distance d_1. By (3.3) and the Fatou Lemma, the limit belongs a.s. to $\ell_\lambda \times [0,\infty)$.

Step 2. Consider now a decreasing sequence $(t_l)_{l\geq 1}$ of positive numbers such that $\lim_{l\to\infty} t_l = 0$. Using a diagonal extraction, we deduce from Step 1 that we may find a subsequence n_k such that for all $l \geq 1$, $(M(m^{n_k},t_l), ||M(m^{n_k},t_l)||_\lambda)$ converges in law, in $\ell_1 \times [0,\infty)$. We denote by (M^l, X^l) the limit, which belongs a.s. to $\ell_\lambda \times [0,\infty)$. We thus may consider, using Theorem 2.2, the Markov process $(M^{l,*}(t))_{t\geq t_l}$, belonging a.s. to $\mathbb{D}([t_l,\infty),\ell_\lambda)$, with semi-group $(P_s^K)_{s\geq 0}$ starting at time t_l with the initial condition M^l.

Step 3. We now prove that for all $l \geq 1$, $(M(m^{n_k},t))_{t\geq t_l}$ goes in law to $(M^{l,*}(t))_{t\geq t_l}$ as k tends to infinity, the convergence holding in $\mathbb{D}([t_l,\infty),\ell_1)$, ℓ_1 being equipped d_1.
Using the Skorokhod representation Theorem, we may assume that as k tends to infinity, $(M(m^{n_k},t_l), ||M(m^{n_k},t_l)||_\lambda)$ goes a.s. to (M^l, X^l) in $\ell_1 \times [0,\infty)$. This implies that a.s. for all $T > t_l$,

$$\lim_{k\to\infty} d_1\left(M(m^{n_k},t_l), M^l\right) e^{8a\left(||M(m^{n_k},t_l)||_\lambda + ||M^l||_\lambda\right)(T-t_l)} = 0. \tag{3.22}$$

Lemma 3.3 allows us to conclude.

Step 4. We deduce from Step 3, by uniqueness of the limit, that for $p > l \geq 1$ (so that $0 < t_p < t_l$), the processes $(M^{p,*}(t))_{t\geq t_l}$ and $(M^{l,*}(t))_{t\geq t_l}$ have the same law. We may thus define a process $(M^*(t))_{t>0}$ (using for example the Kolmogorov Theorem) in such a way that for all $l \geq 1$, the processes $(M^*(t))_{t\geq t_l}$ and $(M^{l,*}(t))_{t\geq t_l}$ have the same law. This process $(M^*(t))_{t\in(0,\infty)}$ is obviously a Markov process with semi-group $(P_s^K)_{s\geq 0}$ belonging a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$.

Step 5. It only remains to prove that a.s., $\lim_{t\to 0+} M_1^*(t) = 0$. By nature, the map $t \mapsto M_1^*(t)$ is a.s. non-decreasing, non-negative, and bounded by 1. It thus suffices to prove that $\lim_{t\to 0+} E[M_1^*(t)^2] = 0$. But for all $t > 0$, $M_1^*(t)$ is the limit in law, as $k \to \infty$, of $M_1(m^{n_k}, t)$. An easy computation, using (2.4) shows that for all $n \geq 1$,

$$\begin{aligned} E[M_1(m^n,t)^2] &= \frac{1}{n^2} + \int_0^t ds E\Big[\sum_{i<j} K(M_i(m^n,s), M_j(m^n,s)) \\ &\qquad \big([M_i(m^n,s)+M_j(m^n,s)]^2 - M_1(m^n,s)^2\big)_+\Big] \\ &\leq \frac{1}{n^2} + 3\bar{K}\int_0^t ds E\Big[\sum_{i<j} M_i(m^n,s)M_j(m^n,s)\Big] \\ &\leq \frac{1}{n^2} + 3\bar{K}t, \end{aligned} \tag{3.23}$$

where $\bar{K} = \sup_{[0,1]^2} K(x,y)$. We have used that for $1 \leq i < j$, $M_j(m^n,s) \leq M_i(m^n,s) \leq M_1(m^n,s)$ and that $\sum_{i\geq 1} M_i(m^n,s) = 1$. Thus for all $t > 0$, $E[M_1^*(t)^2] \leq 3\bar{K}t$, from which the conclusion follows. □

Remark 3.4 *For the kernel $K = (xy)^{\lambda/2}$ Lemma 3.3 still holds. However, instead of Lemma 3.2, we are only able to prove a regularization of the moment of order α, for any $\alpha \in (\lambda, 1)$. Since the continuity property stated in Lemma 3.3 involves the moment of order λ, the proof breaks down.*

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