The Effect of Internal Wave Strain on Vertical Spectra of Fine-Structure

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ABSTRACT

Vertical spectra of temperature perturbations are due to (i) internal wave displacement in the mean profile and (ii) fine-structure that would still be measured in the absence of internal waves. The former is probably confined to low wavenumbers, but at a lower energy level could be responsible for some of the reported fine-structure. The latter is distorted by the internal waves and differs from the true fine-structure spectrum by an amount proportional to the mean-square strain of internal waves.

1. Introduction

The vertical structure of temperature and salinity in the oceans shows features on scales down to about 1 cm (Cox *et al.*, 1969), less for salinity (Gregg and Cox, 1971). The temperature fine-structure is largely one-dimensional (in basically horizontal layers) for vertical wavelengths $\gtrsim 30$ cm (Hacker, 1972).

As a result of this fine-structure, temperature (and salinity or current) variations at a fixed depth due to internal waves can be very different from the signals that would be associated with the same vertical displacements in a smooth temperature gradient. This fine-structure "contamination" has been extensively studied (Wunsch and Dahlen, 1970; Phillips, 1971; Reid, 1971; Garrett and Munk, 1971; Gould, 1971; Neshyba *et al.*, 1972) and its possible effects on frequency spectra at a fixed position or on the coherence between fluctuations at positions separated in the vertical are reasonably well understood.

An important part of these studies has been a knowledge of, or assumption about, the basic vertical temperature structure, i.e., the temperature structure *in the absence of internal waves*. But all temperature soundings are made in the presence of internal waves, and so are distorted by the associated vertical displacements. The undisturbed temperature profile can be resolved by multiple soundings in space or time (Neshyba *et al.*, 1972; Lazier, 1972), and indeed Lazier has produced convincing evidence that the temperature "fine-structure" apparent in any single sounding in a fresh-water lake was in fact very largely due to the distortion by internal waves of a perfectly smooth temperature profile! This may be an extreme case, but it does seem important to disentangle such a contribution from genuine fine-structure, and also to understand the extent to which the vertical wavenumber spectrum of fine-structure is changed by the vertical strain of internal waves.

The purpose of this paper is to extend, in a straightforward manner, the statistical approach of Garrett and Munk (1971) in an attempt to put these factors into perspective. The problem will be formulated in terms of temperature, but could equally well apply to salinity or some other scalar variable.

2. Temperature covariance

Given a temperature record $T(y)$ as a function of depth y (positive upward), centered on $y=0$, then the covariance with depth of the temperature with mean and trend removed is

$$R_T(\eta)=\langle[T(y)-y\bar{T}'][T(y+\eta)-(y+\eta)\bar{T}']\rangle, \quad (1)$$

where $\langle\ \rangle$ represents the average over y, $\bar{T}'$ is the mean temperature gradient, and the temperature is normalized so that $\langle T(y)\rangle=0$.

Suppose that $T(y)$ is due to vertical displacement $\zeta(y)$ in a basic temperature profile $T_0(y)=y\bar{T}'+\theta(y)$,[1] composed of the linear gradient $\bar{T}'$ plus a fine-structure $\theta(y)$. Thus, $T(y)=T_0(y_0)$, where y_0 is the solution of $y_0+\zeta(y_0)=y$. Now the typical internal wave displacement is much less than the distance over which ζ varies significantly, and so to a good approximation we may write $y_0=y-\zeta(y)$ and obtain

$$R_T(\eta)=\langle\{-\zeta(y)\bar{T}'+\theta[y-\zeta(y)]\}\times\{-\zeta(y+\eta)\bar{T}'+\theta[y+\eta-\zeta(y+\eta)]\}\rangle, \quad (2)$$

$$=\bar{T}'^2R_\zeta(\eta)+\langle\theta[y-\zeta(y)]\theta[y+\eta-\zeta(y+\eta)]\rangle, \quad (3)$$

[1] This partition is for analytical convenience, including with the fine-structure such features as the curvature of the basic profile.

where $R_\zeta(\eta)=\langle\zeta(y)\zeta(y+\eta)\rangle$ is the displacement covariance. The cross terms from (2) vanish (at least from an ensemble average) as the basic fine-structure is uncorrelated with the internal wave displacement. The second contribution in (3) may be written

$$R_T{}^{fs}(\eta)=\iint_{-\infty}^{\infty}\langle\theta(y-\zeta_1)\theta(y+\eta-\zeta_2)\rangle p(\zeta_1,\zeta_2;\eta)d\zeta_1 d\zeta_2, \quad (4)$$

$$=\iint_{-\infty}^{\infty} r_\theta(\zeta_1-\zeta_2+\eta)p(\zeta_1,\zeta_2;\eta)d\zeta_1 d\zeta_2, \quad (5)$$

where $r_\theta(\xi)$ is the fine-structure covariance and $p(\zeta_1,\zeta_2;\eta)$ is the joint probability density of displacements ζ_1, ζ_2 at a vertical separation η. (We assume that the temperature measurement is sufficiently rapid that we do not need to take into account the variation of ζ with time.) In the particular case of no internal waves $p(\zeta_1,\zeta_2;\eta)=\delta(\zeta_1)\delta(\zeta_2)$ and $R_T{}^{fs}(\eta)=r_\theta(\eta)$ as required. To proceed further we must specify $p(\zeta_1,\zeta_2;\eta)$.

3. Gaussian wave field

If the displacement ζ comes from many different independent wavenumbers and frequencies, $\zeta(y)$ and $\zeta(y+\eta)$ are jointly normal:

$$p(\zeta_1,\zeta_2;\eta)=\frac{1}{2\pi Z^2(1-\rho^2)^{\frac{1}{2}}}\exp\left[-\frac{\zeta_1{}^2-2\rho\zeta_1\zeta_2+\zeta_2{}^2}{2Z^2(1-\rho^2)}\right], \quad (6)$$

where $\rho(\eta)=R_\zeta(\eta)/Z^2$ is the autocorrelation of the displacement and $Z^2=\overline{\zeta^2}=R_\zeta(0)$ is the displacement variance. As in Garrett and Munk (1971) we now change one variable in (5) to $\zeta_1-\zeta_2=z$ and integrate over the other to obtain

$$R_T{}^{fs}(\eta)=[4\pi(1-\rho)Z^2]^{-\frac{1}{2}}$$

$$\times\int_{-\infty}^{\infty} r_\theta(z+\eta)\exp\left[-\frac{z^2}{4(1-\rho)Z^2}\right]dz, \quad (7)$$

which may be written alternatively as

$$R_T{}^{fs}(\eta)=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} r_\theta[2(1-\rho)^{\frac{1}{2}}Zx+\eta]e^{-x^2}dx. \quad (8)$$

Either (7) or (8) may, in principle, be Fourier-transformed to obtain the fine-structure spectrum.

4. Approximations

The average number of zeros of the vertical displacement (if Gaussian) per unit vertical distance is K/π, where

$$K^2=\int_0^\infty k^2F_\zeta(k)dk\Big/\int_0^\infty F_\zeta(k)dk$$

$$=\overline{(\partial\zeta/\partial z)^2}/Z^2=-\rho''(0), \quad (9)$$

and $F_\zeta(k)$ is the vertical wavenumber spectrum of the vertical displacement. Thus, $\rho(\eta)=1-\frac{1}{2}K^2\eta^2+O(\eta^4)$, and so, provided that $KZ\ll 1$, $2(1-\rho)^{\frac{1}{2}}Zx\ll\eta$ for $x=O(1)$, which gives the main part of the integral in (8). Thus, we may expand r_θ in (8) in a Taylor series, and

$$R_T{}^{fs}(\eta)=r_\theta(\eta)+(1-\rho)Z^2r_\theta''(\eta)+\cdots. \quad (10)$$

The corresponding power spectrum is

$$F_T{}^{fs}(k)=\frac{2}{\pi}\int_0^\infty R_T{}^{fs}(\eta)\cos k\eta d\eta$$

$$\approx F_\theta(k)+\frac{2Z^2}{\pi}\int_0^\infty(1-\rho)r_\theta''(\eta)\cos k\eta d\eta, \quad (11)$$

where

$$F_\theta(k)=\frac{2}{\pi}\int_0^\infty r_\theta(\eta)\cos k\eta d\eta$$

is the fine-structure spectrum in the absence of internal waves.

In general, the scale of the fine-structure is much less than the scale of the internal waves, so that in the correction term in (11) $r_\theta''(\eta)$ becomes very small for a sufficiently small η that $1-\rho$ may be approximated well by $\frac{1}{2}K^2\eta^2$. Hence,

$$F_T{}^{fs}(k)-F_\theta(k)=\pi^{-1}K^2Z^2\int_0^\infty r_\theta(\eta)\frac{d^2}{d\eta^2}(\eta^2\cos k\eta)d\eta, \quad (12)$$

$$=\tfrac{1}{2}\overline{(\partial\zeta/\partial z)^2}\frac{d^2}{dk^2}[k^2F_\theta(k)]. \quad (13)$$

So, subject to the two rather acceptable approximations (i) that the vertical strain caused by the internal waves is small and (ii) that the autocorrelation scale of the fine-structure is much less than the autocorrelation scale of the vertical displacement, we find a correction to the fine-structure spectrum proportional to the mean-square strain of the internal waves. The result is not surprising, but it is interesting to note that if $F_\theta(k)$ is proportional to k^{-n} locally, then so is the correction. If $F_\theta(k)\propto k^{-2}$, the correction vanishes. A possible interpretation of this is that a k^{-2} spectrum can correspond to the sheet/layer model of fine-structure, and such a structure is unchanged by the straining of internal waves.

Consideration of higher order terms in the Taylor series expansion of $R_T{}^{fs}(\eta)$ in (10) leads to terms proportional to K^4Z^4. Of more interest is a better approximation to ρ in (11):

$$\rho(\eta)=1-\tfrac{1}{2}K^2\eta^2+\tfrac{1}{12}L^4\eta^4+O(\eta^6), \quad (14)$$

where

$$L^4=\int_0^\infty k^4F_\zeta(k)dk \Big/ \int_0^\infty F_\zeta(k)dk$$

$$=\overline{(\partial^2\zeta/\partial z^2)^2}/Z^2=\rho^{iv}(0), \quad (15)$$

and the mean distance between turning points of $\zeta(y)$ is $\pi\, M/L^2$.

A better approximation to (13) is now

$$F_T{}^{fs}(k)=F_\theta(k)+\tfrac{1}{2}K^2Z^2\frac{d^2}{dk^2}[k^2F_\theta(k)]$$

$$-\tfrac{1}{12}L^4Z^2k^{-2}\frac{d^2}{dk^2}[k^4F_\theta''(k)]. \quad (16)$$

The validity of this approximation now depends on $L^4Z^2k^{-2}\ll 1$, i.e., $k\gg(MZ)(L^2/M)$ and the expansion may not be valid for wavenumbers corresponding to scales much larger than the distance between turning points of the displacement. This is not a serious restriction.

5. Numbers

The mean square strain $\overline{(\partial\zeta/\partial z)^2}$ depends on the richness of the vertical mode structure in the ocean, and this is not yet well-known. The best current estimate of a typical mid-ocean value is probably obtainable from the wavenumber-frequency spectrum of Garrett and Munk (1972) which gives $\overline{(\partial\zeta/\partial z)^2}=0.04n$. Here n is the local mean Brunt-Väisälä frequency divided by 3 cycles per hour (cph) (i.e., $n\approx 1$ in the main thermocline and generally decreases with depth). This value is more likely to under- than overestimate the strain, but does indicate that the straining effect of internal waves on the fine-structure spectrum is likely to be small.

The evaluation in this paper of the correction to the fine-structure spectrum depends on assuming the internal waves to be Gaussian, but is probably qualitatively correct for any distribution.

The contribution to the vertical temperature spectrum by internal wave displacement in the mean temperature profile is $F_T{}^{gd}(k)=\bar{T}'^2F_\zeta(k)$, where $F_\zeta(k)=(2/\pi)\int_0^\infty R_\zeta(\eta)\cos k\eta\, d\eta$ is the vertical spectrum of internal wave displacement. The Garrett and Munk (1972) spectrum gives this as

$$\left.\begin{aligned} F_\zeta(k)&=7n^{-2}, \quad \text{for } 0\leqslant k\leqslant 7.7n \\ &=0 \quad , \quad \text{otherwise.}\end{aligned}\right\}, \quad (17)$$

where the units on F_ζ are meters^2 per cycle per kilometer (cpkm) and cpkm on k.

(The top-hat shape is arbitrary and any other spectrum with the same vertical coherence distance is just as likely.) This level corresponds to $F_T{}^{gd}(k)\approx 10^{-3}$ (°C)^2 (cpkm)$^{-1}$ in the main thermocline, comparable with the spectral densities reported by Roden (1971) at a few cpkm, though Roden's spectra decrease rapidly with k, and it is not clear to what extent they merely represent the mean temperature profile (included with the fine-structure in the present formulation). Roden's spectra decrease to about 10^{-6} (°C)^2 (cpkm)$^{-1}$ at 100 cpkm, comparable with the low-frequency end of the spectrum published by Cox *et al.* (1969) which extends from 10^2 to 10^5 cpkm. Bennett (1972) reports spectral levels roughly 100 times higher for an active region of the northwest Atlantic. Much of these spectra could be due to internal wave displacements in the mean profile without changing the bandwidth required by Garrett and Munk (1972). It is not clear whether one can determine from a single sounding the wavenumber at which genuine fine-structure takes over from internal waves.

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REFERENCES

Bennett, A. S., 1972: Some observations of salinity and temperature with a variable depth towed body. Arctic Ice Dynamics Joint Experiment, Bull. No. 13, 353–356.

Cox, C. S., Y. Nagata and T. Osborn, 1969: Oceanic fine structure and internal waves. *Bull. Japan. Soc. Fisheries Oceanogr., Papers in Dedication to Prof. Michitaka Uda*, 67–71.

Garrett, C., and W. Munk, 1971: Internal wave spectra in the presence of fine-structure. *J. Phys. Oceanogr.*, **1**, 196–202.

——, and ——, 1972: Space-time scales of internal waves. *Geophys. Fluid Dyn.*, **2**, 225–264.

Gould, W. J., 1971: Spectral characteristics of some current records from the eastern North Atlantic. *Phil. Trans. Roy. Soc. London*, **A270**, 437–450.

Gregg, M. C., and C. S. Cox, 1971: Measurements of oceanic microstructure of temperature and electrical conductivity. *Deep-Sea Res.*, **18**, 925–934.

Hacker, P. W., 1972: The mixing of heat deduced from temperature fine-structure measurements in the Pacific and Lake Tahoe. Ph.D. thesis, University of California, San Diego.

Lazier, J. R. N., 1972: Temporal changes in some fresh water temperature structures. Submitted to *J. Phys. Oceanogr.*

Neshyba, S., V. T. Neal and W. W. Denner, 1972: Spectra of internal waves: In-situ measurements in a multiple layered structure. *J. Phys. Oceanogr.*, **2**, 91–95.

Phillips, O. M., 1971: On spectra measured in an undulating layered medium. *J. Phys. Oceanogr.*, **1**, 1–6.

Reid, R. O., 1971: A special case of Phillips' general theory of sampling statistics for a layered medium. *J. Phys. Oceanogr.*, **1**, 61–62.

Roden, G. I., 1971: Spectra of North Pacific temperature and salinity perturbations in the depth domain. *J. Phys. Oceanogr.*, **1**, 25–33.

Wunsch, C., and J. Dahlen, 1970: Preliminary results of internal wave measurements in the main thermocline at Bermuda. *J. Geophys. Res.*, **75**, 5899–5908.