Deep Water Exchanges in a Sill Fjord: A Stochastic Process

Herman G. Gade
Geophysical Institute, University of Bergen, Norway

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ABSTRACT

A statistical theory for the behavior of a fjord system with a sill is formulated and leads to the establishment of conditional and marginal distribution functions for the homogeneous deep water density of the enclosed part of the fjord basin. From these frequency distributions, the probability functions for influx to the fjord basin are determined. Numerical solutions for these functions are derived on the basis that the external input is a Gaussian (normal) variate. Application of the theory is made to two Norwegian fjords with characteristic influx intervals from 3–10 years. The theory is also applicable to prognostic problems in connection with proposed engineering measures.

1. Introduction

Investigations of deep water exchanges in Norwegian fjords with sills have revealed that the replacement of resident water is characterized by some sort of regularity, i.e., the influxes tend to take place at the same time of the year, thus establishing a recurring phenomenon with time intervals being essentially multiples of a full year. In the innermost basin of the Oslofjord, the Bonnefjord, this event normally occurs during the late winter for intervals from 3–6 years (Gade, 1970). In other Norwegian west coast fjords like the Hardangerfjord and the Sognefjord major influxes are known to be less frequent and mostly confined to the summertime. In some of the more sheltered fjord basins deep water renewals may be as rare as once every 50 years or more.

In most cases for which data are available the deep water renewal of a fjord basin is a relatively rapid event, which is often completed within the course of a few weeks. The influx itself constitutes a gravity flow following the appearance of a denser water, in the adjacent coastal areas at sill depth and above, than that existing within the fjord basin. In the case of a major renewal, virtually 100% of the resident water below sill level may become replaced, although fractional replacements are known to occur. The latter may be particularly the case with very large fjords requiring an extended period of influx to complete the renewal. Nevertheless, the possibility of a complete or near complete replacement of the resident water remains as one of the outstanding properties of such exchanges and will be retained here.

The principal mechanisms governing these exchanges are linked with the sustained density reduction of the resident water through vertical eddy diffusion and the naturally occurring density variations in the adjacent coastal water. As will be shown in the following, these processes constitute a statistical problem which may have a counterpart in analogous physical discharge processes in nature or in the laboratory.

A remarkable feature of the deep water masses of the sill fjord is the relatively high degree of uniformity[1] in comparison with the adjacent coastal water. This applies particularly to the density where the local variation may be within the accuracy of common determinations. Nevertheless, a vertical gradient of density does always exist in this surface layer and allows a density reduction through eddy diffusion.

2. Theory

In the following a statistical model is presented in which the event of influx is related to the statistical properties of the density variations in the adjacent coastal water. The theory rests on the following three fundamental assumptions:

1) The density S of the resident deep water in the fjord basin is homogeneous and decreases linearly with time. The corresponding annual density decrement is denoted by D.

2) Influxes may occur at intervals equal to integral multiples of a full year. The influx is assumed to be instantaneous.

3) When an influx takes place all the resident water in the fjord is replaced by a water mass of density S_0 which is the same as that of the coastal water present at sill depth.

It is furthermore assumed that the density S_u of the coastal water available for influx is a stochastic variable with a probability density function $g(S_u)$.

[1] The uniformity of the deep water in the fjord basin may be seen as a direct consequence of the lower boundary condition of a vanishing density gradient and the nonlinear coupling between the static stability and the eddy diffusivity, such as demonstrated for the Oslofjord (Gade, 1970).

Thus, the water entering the basin will be "drawn" from this ensemble, its density having the property of a stochastic variable with a probability density function $f(S)$.

We may then formulate the following statements:

(i) There corresponds to any initial value of $S=S_0$ a conditional frequency distribution $q(S|S_0)$ for the density S at the next influx.

(ii) The joint probability density for (S,S_0) at two consecutive influxes (not necessarily in consecutive years) is $f(S_0)q(S|S_0)$.

(iii) It follows from condition (ii) that the probability density for S regardless of the value of the density S_0 of the previous influx is the marginal probability density function

$$\int_{-\infty}^{\infty} f(S_0)q(S|S_0)dS_0.$$

(iv) If stationary conditions are presumed, by definition

$$\int_0^{\infty} f(S_0)q(S|S_0)dS_0=f(S), \quad 0<S<\infty. \tag{1}$$

Having given $g(S_u)$, the first aim of the present analysis is to find $f(S)$. The problem is therefore, in principle, reduced to that of determining $q(S|S_0)$.

The influx event will take place if $S_u>(S_0-D)$ after one year, $S_u>(S_0-2D)$ after two years, or $S_u>(S_0-nD)$ after n years. Accordingly, the probability for influx after one year is

$$P(1)=\int_{S_0-D}^{\infty} g(S)dS. \tag{2}$$

Similarly, the probability for influx after n years, provided no influx has occurred during the last $n-1$ years, is

$$P(n)=\int_{S_0-nD}^{\infty} g(S)dS. \tag{3}$$

On the other hand, the probability for no influx the first year (after one year) is

$$0(1)=\int_0^{S_0-D} g(S)dS, \tag{4}$$

whereas the probability for no influx after n years is

$$0(n)=\int_0^{S_0-nD} g(S)dS=1-P(n). \tag{5}$$

Accordingly, the probability of an influx occurring for the first time after n years is

$$Q(n|S_0)=0(1)\cdot 0(2)\cdots 0(n-1)\cdot P(n). \tag{6}$$

In the event of influx after n years, the sample belongs to an ensemble with a conditional probability density defined by

$$p_n(S|S_0)=\frac{g_n(S)}{\int_{S_0-nD}^{\infty} g(S)dS}=\frac{g_n(S)}{P(n)}, \tag{7}$$

where

$$\left.\begin{aligned} g_n(S)&=g(S), && \text{for } S>(S_0-nD) \\ g_n(S)&=0, && \text{for } S\leqslant(S_0-nD) \end{aligned}\right\}.$$

As the probability density for (S,S_0) is given by the conditional probability density function $q(S|S_0)$ regardless of the time interval between the first and the second influx, the conditional probability density $q(S|S_0)$ must be the weighted sum of the conditional probability densities $p_n(S|S_0)$ for all values of n. Thus,

$$\begin{aligned} q(S|S_0)&=\sum_{n=1}^{\infty} p_n(S|S_0)Q(n|S_0), \\ &=\sum_{n=1}^{\infty} g_n(S)\frac{Q(n|S_0)}{P(n)}, \\ &=\sum_{n=1}^{m} g_n(S)\prod_{i=1}^{n-1} 0(i), \quad m\geqslant S_0/D. \end{aligned} \tag{8}$$

We are now in a position to determine $f(S)$ by means of (1). This task may appear quite formidable, but is nevertheless feasible by means of electronic computers. In Fig. 1 $g(S)$ and $f(S)$ are demonstrated for various values of D/σ, where σ is the standard deviation of $g(S)$, here conveniently chosen as a normal probability density function.

As is to be expected, $f(S)$ is not symmetrical. Its mean value is also displaced with respect to that of $g(S)$, increasing with diminishing values of D/σ. The latter property agrees with the common observation that even well-aerated fjord basins often exhibit a water density higher than that seen at corresponding levels outside the sill.

Another parameter of interest is the marginal probability of influx events. Let $f(S_0)Q(n|S_0)$ be the joint probability density for an influx of density S_0, combined with another influx after n years. Then

$$Q(n)=\int_0^{\infty} f(S_0)Q(n|S_0)dS_0 \tag{9}$$

will be the probability that the two consecutive influx events are separated by n years.

In all the examined cases in which $g(S_u)$ are assumed to be normal probability density functions, it appears that $Q(n)$ is a steadily decreasing function of n. In other words, if the state of the fjord is not known, the most probable time interval between two consecutive influxes is one year, regardless of the magnitude of the yearly decrement D. This does not mean that this is necessarily a very probable event, although more probable than any other specified time interval.

Rather than dealing with the marginal probability function $Q(n)$, evaluation of its mean value by

$$\bar{n}=\sum_{n=1}^{\infty} Q(n)n, \tag{10}$$

together with the standard deviation of n, may yield useful information of the statistical properties of a fjord basin. The quantity $\bar{n}$ is relevant in studies of long-term exchange statistics of a fjord system, and may be particularly applicable in connection with studies of the recent history of sedimentation in a fjord basin.

Evaluation of $\bar{n}$ shows that for practical purposes

$$\bar{n}\approx 1+0.729\left(\frac{\sigma}{D}\right)^{\sqrt{3}/2}, \tag{11}$$

the error in the range $2^{\frac{1}{2}}/80<D/\sigma<2^{\frac{1}{2}}/2.5$ being less than 1%.

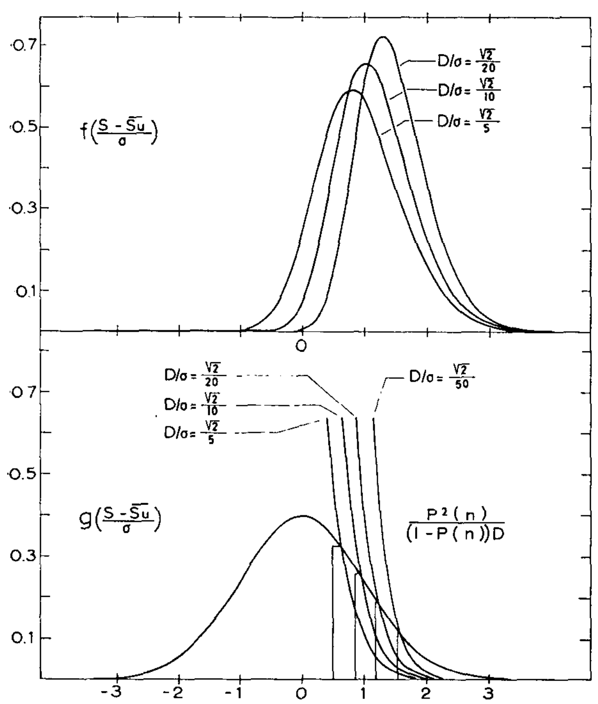


FIG. 1. Lower: Assumed normal probability distribution g of dimensionless external density. Also indicated are four solutions S_1 (intersections with curve family) and corresponding values of S_1' (foot point of normals), the states at which influx is most probable. Upper: Computed marginal probability density functions f for influxes to a fjord basin for three selected values of D/σ.

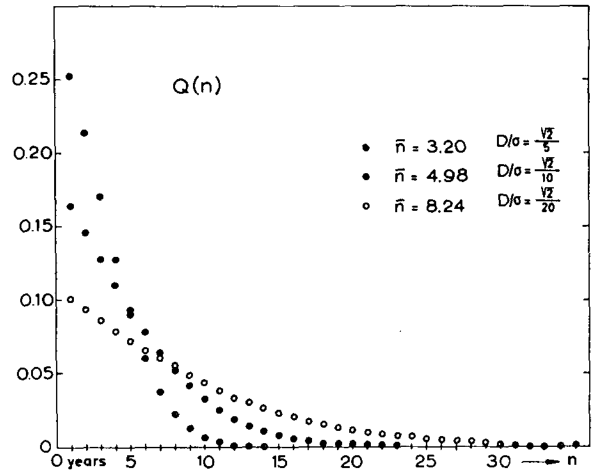


FIG. 2. Marginal probability distributions of the time interval between consecutive deep water exchanges.

In Fig. 2 $Q(n)$ is shown for the same three values of D/σ selected for the presentation of $f(S)$. It is evident that only in the case of the comparably large D/σ do exchanges with intervals of one year occur relatively frequently (about 26% of the cases). For the two other examples the probabilities of a one-year interval are 17 and 10%, respectively, while the interval expectancy (expectation value) ranges from 3.2 years in the first case to 8.24 years in the latter. The drawn examples are typical for many Norwegian fjords, although both higher and lower values of D/σ occur.

For problems concerning exchange frequencies in a fjord basin for which the state S_0 is known, the marginal probability function $Q(n)$ is rather inadequate. In this case it is preferable to deal with the conditional probability density function $q(S|S_0)$ or the conditional probability distribution $Q(n|S_0)$.

Assuming a state in which the initial density S_0 is arbitrarily high, simple reasoning leads to the argument that the cumulative probability for influx within a given time limit increases as time goes on. However, the probability for influx at a specific number of years after the last influx does not possess this property. It will be shown that $Q(n|S_0)$ may have a maximum for a particular year, the probability of influx being higher than at any other previous or later year.

Consider the conditional probability function $Q(n|S_0)$. If this has a maximum for a value near $m=n+\frac{1}{2}$, then

$$Q(n|S_0)\approx Q(n+1|S_0) \tag{12}$$

for this value. Using the definition of $Q(n|S_0)$, it can be shown that

$$Q(n+1|S_0)=Q(n|S_0)\frac{0(n)P(n+1)}{P(n)}. \tag{13}$$

FIG. 6. Percentage oxygen saturation at 400 m in inner basin, Nordfjord (after Sælen, 1967).

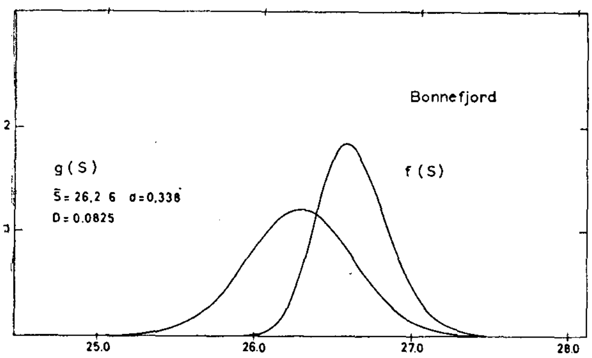


FIG. 7. Adapted normal frequency distribution $g(S)$ to sill-level (55 m) densities in the Vestfjord (Oslofjord), and computed marginal probability function $f(S)$ of the density of the influx water to the Bonnefjord.

In spite of the limitations indicated above, the computed frequency function $f(S)$, as shown in Fig. 5, does not appear unacceptable, except possibly for very high values of S. The recognized influxes have brought in water of density in the range of $\sigma_t = 27.26$–27.32, as marked by a band in Fig. 5. Corresponding to an S_0 value of 27.30, the most probable value of S is practically the same (27.30), with an interval expectancy of 5.3 years. The mean value of n for the marginal distribution is 10.4 years.

Observations from the Oslofjord are available in two series, one extending from 1946–51, the other from 1962–71, plus three individual years. These series cover a total of 19 years fairly well represented with observations, from which a distribution function $g(S)$ of the annual density maximum at sill level (55 m) to the Bonnefjord has been established.

The theoretical distribution function $f(S)$ of the density of the deep water influxes to the Bonnefjord, corresponding to an annual decrement of $D = 0.0825$ (σ_t units) is shown in Fig. 7. The marginal expectancy of S is 26.64, which is also near its most probable value. The corresponding marginal interval expectancy is 3.69 years.

There are but a few identified consecutive cases of deep water exchanges in the Bonnefjord. A major influx in 1963 ($\sigma_t = 26.73$) was followed by exchanges in 1966, 1969 and 1970 in all three new influxes over a period of eight years, or roughly one every three years. Corresponding to an initial value $S_0 = 26.73$, the conditional probabilities of influx $Q(n|S_0)$ are shown in Fig. 8a. The occasionally more applicable cumulative probabilities are given in Fig. 8b, from which it is seen that in this case there was a 50% probability of exchange before the end of three years, or a 99% probability of influx within eight years! As already stated, influx took place after three years.

4. Concluding remarks

The model presented here suffers from limitations inherent in the fundamental assumptions. The most prevalent of these are associated with partial exchanges, i.e., influxes in which only a part of the resident water in the fjord basin is replaced. Two cases can be distinguished depending upon whether the incoming water is heavy enough to replace the bottom water of the fjord basin, or not. In the former case, if substantial, the influx will normally count as an "exchange." In the latter case, however, it depends upon the nature of the problems involved whether a partial influx to higher levels is of significance. If not, one will normally observe a reduced annual decrement in the deeper layers following the introduction of heavier water above, thereby reducing the established density gradient. This behavior can to some extent be compensated in the model.

Notwithstanding the assumptions and limitations of the model, one can perceive certain areas of application of the theory:

1. Fjords which intermittently become anoxic will trace a record of the recent history in bottom deposits. With reference to $f(S)$ and $Q(n)$, we are in a position to judge the geological events on the assumption of stationary long-term oceanographic conditions, or otherwise to apply the geological records more discrimi-

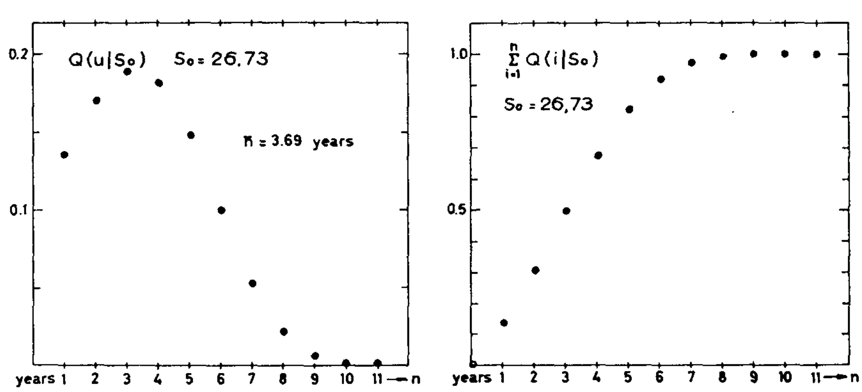


FIG. 8. Marginal probability distribution $Q(n|S_0)$ of influx interval to the Bonnefjord (left, a.) and conditional cumulative distribution of the influx interval to the Bonnefjord (right, b.).

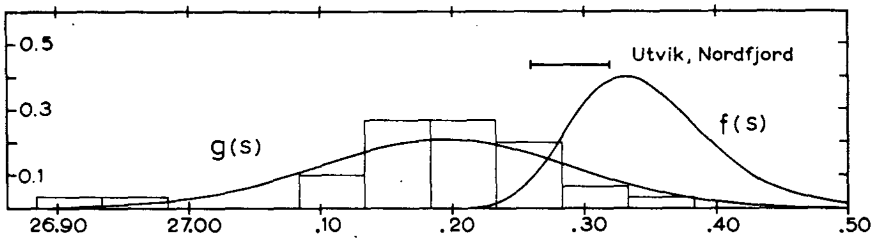


Fig. 5. Observed frequency distribution (1930–60) and adapted normal distribution $g(S)$ of sill-level densities in the Nordfjord, and computed marginal probability function $f(S)$ for the density of influx water to the inner basin. Observed influxes are within indicated range.

which indicates that the conditional probability function has a preserving form while its magnitude is controlled by the 0-function. For increasing values of S_0 the conditional probability functions approach an asymptotic form. In Fig. 3 are shown the conditional probability functions for equidistant values of S_0/σ referring to a normal distribution $g(S)$ with a standard deviation $\sigma = D(10/2^{\frac{1}{2}})$. Fig. 3 demonstrates a fundamental property of the conditional probability functions: A high value of S_0 yields a conditional probability function $Q(n|S_0)$ with a most probable influx interval of many years, whereas a low value of S_0 is likely to be followed by an influx interval of short duration.

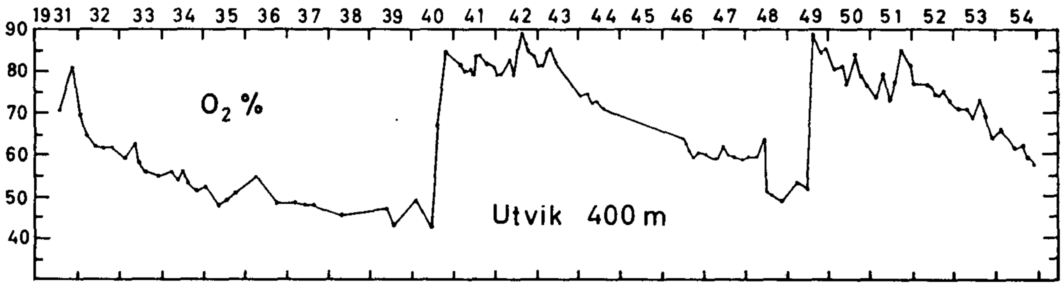
The most conspicuous properties of the conditional probability functions become evident by studying directly $q(S|S_0)$. In Fig. 4 the conditional probability density distributions are displayed for two selected values of S_0: $(S_0-\bar{S}_u)/\sigma=0.71$ and $(S_0-\bar{S}_u)/\sigma=1.84$. In both cases the annual relative decrement is the same, namely $D/\sigma = 2^{\frac{1}{2}}/10$. These functions demonstrate that for an arbitrarily high value of S_0 both the expectancy and the most probable value of S is relatively high, while for a lower value of S_0 the expectancy and the most probable value of S at the next influx is comparably lower. This behavior gives evidence of a tendency to persistency in the density of water influxes. In other words, two fjords with identical long-term statistical properties may differ radically in influx frequencies observed over a limited number of years in which both fjords may exhibit surprisingly regular deep water renewals.

3. Application

The theory developed above has been applied to statistics from two Norwegian fjords, the Nordfjord and the Oslofjord. In both cases the "fjord basin" is defined as the innermost basin of the fjord, whereas the "external" conditions, from which $g(S)$ is derived, are taken from observations at sill level (or slightly above) to the connecting outer basins of the fjord.

Data from the Nordfjord are available in the form of an unbroken series of observations over 30 years. Referring to observations taken just outside the sill (137 m) of the inner basin, a normal distribution $g(S)$ has been fitted to the frequency histogram according to the principles of estimated variance (Fig. 5). In this case the goodness of fit is acceptable, with the normal distribution cutoff in both ends at about $\sigma_t = 26.8$, and 27.6 making it more realistic.

The situation in the inner basin (maximum depth 440 m) is not so clearly defined. The annual density decrement of $0.005\sigma_t$ units is determined over a period of 5 years, over which the total variation is still near the accuracy of the density determinations, which is dependent upon chloride titrations. During the entire period 1930–60, three major deep water renewals can be recognized with certainty from the oxygen records (Fig. 6). Although barely discernible, three additional minor influxes to the 400 m level can be identified from the observations of temperature and salinity. This leaves us with six influxes to the inner basin during this period, amounting to one every five years on the average.



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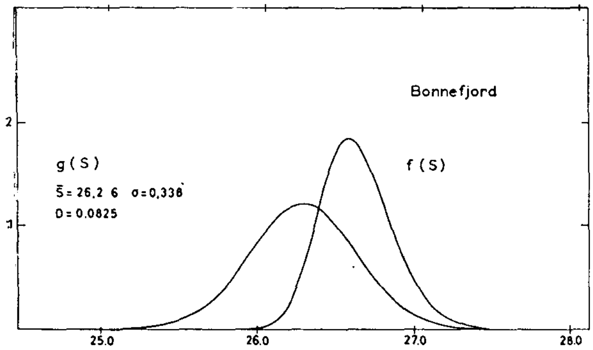

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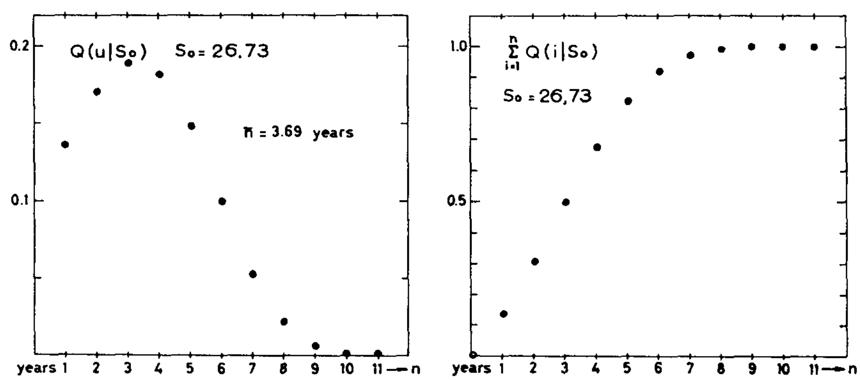

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nately in obtaining evidence of changes in the oceanographic "climate" of a fjord.

2. Observational programs in fjords with the objectives of establishing knowledge of deep water exchange frequencies can, in principle, be reduced to that of determining $g(S)$ and D. From these parameters, an evaluation of the fjord as a recipient of sewage or industrial wastes can be given.

3. It is possible to influence the statistical properties of the deep water exchange system of a fjord basin by introducing turbulence artificially through the injection of an air stream or through the continuous discharge of fresh water at great depths. In both cases it is possible to estimate the new value of D and therefore the exchange frequencies of the fjord basin.

4. Dredging may sometimes be an appropriate measure in deepening the entrance to a fjord through a shallow sill. This operation may have a significant effect on the appropriate $g(S)$, which in most cases can be established beforehand. Again, the theory allows estimation of the statistical properties of the modified fjord system.

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