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THE TIME CONSTANT AND CRITICAL PROBABILITIES IN PERCOLATION MODELS

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Abstract

We consider a first-passage percolation (FPP) model on a Delaunay triangulation $\mathcal{D}$ of the plane. In this model each edge $\mathbf{e}$ of $\mathcal{D}$ is independently equipped with a nonnegative random variable $\tau_{\mathbf{e}}$, with distribution function $\mathbb{F}$, which is interpreted as the time it takes to traverse the edge. Vahidi-Asl and Wierman [9] have shown that, under a suitable moment condition on $\mathbb{F}$, the minimum time taken to reach a point $\mathbf{x}$ from the origin $\mathbf{0}$ is asymptotically $\mu(\mathbb{F})|\mathbf{x}|$, where $\mu(\mathbb{F})$ is a nonnegative finite constant. However the exact value of the time constant $\mu(\mathbb{F})$ still a fundamental problem in percolation theory. Here we prove that if $\mathbb{F}(0) < 1 - p_c^*$ then $\mu(\mathbb{F}) > 0$, where p_c^* is a critical probability for bond percolation on the dual graph $\mathcal{D}^*$.

Introduction

First-passage percolation theory on periodic graphs was presented by Hammersley and Welsh [4] to model the spread of a fluid through a porous medium. In this paper we continue a study of planar first-passage percolation models on random graphs, initiated by Vahidi-Asl and Wierman [9], as follows. Let $\mathcal{P}$ denote the set of points realized in a two-dimensional homogeneous Poisson point process with intensity 1. To each $\mathbf{v} \in \mathcal{P}$ corresponds an open polygonal region $\mathbf{C_v} = \mathbf{C_v}(\mathcal{P})$, the Voronoi tile at $\mathbf{v}$, consisting of the set of points of $\mathbb{R}^2$ which are closer to $\mathbf{v}$ than to any other $\mathbf{v}' \in \mathcal{P}$. Given $\mathbf{x} \in \mathbb{R}^2$ we denote by $\mathbf{v_x}$ the almost surely unique point in $\mathcal{P}$ such that $\mathbf{x} \in \mathbf{C}_{\mathbf{v_x}}$. The collection $\{\mathbf{C_v} : \mathbf{v} \in \mathcal{P}\}$ is called the Voronoi Tiling of the plane based on $\mathcal{P}$.

The Delaunay Triangulation $\mathcal{D}$ is the graph where the vertex set $\mathcal{D}_v$ equals $\mathcal{P}$ and the edge set $\mathcal{D}_e$ consists of non-oriented pairs $(\mathbf{v}, \mathbf{v}')$ such that $\mathbf{C_v}$ and $\mathbf{C_{v'}}$ share a one-dimensional edge (Figure 1). One can see that almost surely each Voronoi tile is a convex and bounded polygon, and the graph $\mathcal{D}$ is a triangulation of the plane [7]. The Voronoi Tessellation $\mathcal{V}$ is the graph where the vertex set $\mathcal{V}_v$ is the set of vertices of the Voronoi tiles and the edge set $\mathcal{V}_e$ is the set

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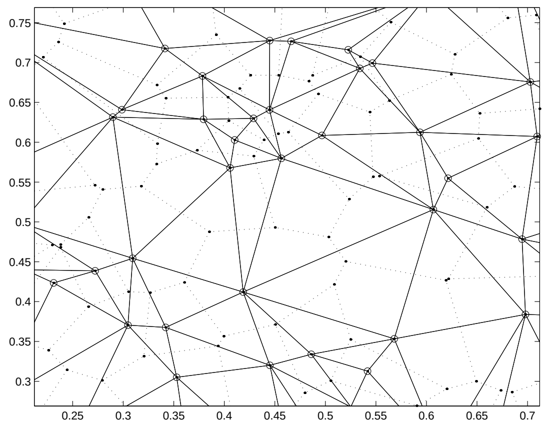

Figure 1: The Delaunay Triangulation and the Voronoi Tessellation.

of edges of the Voronoi tiles. The edges of $\mathcal{V}$ are segments of the perpendicular bisectors of the edges of $\mathcal{D}$. This establishes duality of $\mathcal{D}$ and $\mathcal{V}$ as planar graphs: $\mathcal{V} = \mathcal{D}^*$.
To each edge $\mathbf{e} \in \mathcal{D}_e$ is independently assigned a nonnegative random variable $\tau_{\mathbf{e}}$ from a common distribution $\mathbb{F}$, which is also independent of the Poisson point process that generates $\mathcal{P}$. From now on we denote $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space induced by the Poisson point process $\mathcal{P}$ and the passage times $(\tau_{\mathbf{e}})_{\mathbf{e}\in\mathcal{D}_e}$. The passage time $t(\gamma)$ of a path γ in the Delaunay Triangulation is the sum of the passage times of the edges in γ. The first-passage time between two vertices $\mathbf{v}$ and $\mathbf{v}'$ is defined by

$$T(\mathbf{v}, \mathbf{v}') := \inf\{t(\gamma)\,;\, \gamma \in \mathcal{C}(\mathbf{v}, \mathbf{v}')\}\,,$$

where $\mathcal{C}(\mathbf{v}, \mathbf{v}')$ the set of all paths connecting $\mathbf{v}$ to $\mathbf{v}'$. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ we define $T(\mathbf{x}, \mathbf{y}) := T(\mathbf{v}_{\mathbf{x}}, \mathbf{v}_{\mathbf{y}})$.
To state the main result of this work we require some definitions involving a bond percolation model on the Voronoi Tessellation $\mathcal{V}$. Such a model is constructed by choosing each edge of $\mathcal{V}$ to be open independently with probability p. An open path is a path composed of open edges. We denote $\mathbb{P}^*_p$ the law induced by the Poisson point process and the random state (open or not) of an edge. Given a planar graph $\mathcal{G}$ and $\mathbf{A}, \mathbf{B} \subseteq \mathbb{R}^2$ we say that a self-avoiding path $\gamma = (\mathbf{v}_1, ..., \mathbf{v}_k)$ is a path connecting $\mathbf{A}$ to $\mathbf{B}$ if $[\mathbf{v}_1, \mathbf{v}_2] \cap \mathbf{A} \neq \emptyset$ and $[\mathbf{v}_{k-1}, \mathbf{v}_k] \cap \mathbf{B} \neq \emptyset$ ($[\mathbf{x}, \mathbf{y}]$ denotes the line segment connecting $\mathbf{x}$ to $\mathbf{y}$). For $L > 0$ let A_L be the event that there exists an open path $\gamma = (\mathbf{v}_j)_{1\leq j\leq h}$ in $\mathcal{V}$, connecting $\{0\} \times [0, L]$ to $\{3L\} \times [0, L]$, and with $\mathbf{v}_j \in [0, 3L] \times [0, L]$ for all $j = 2, \ldots, h-1$. In this case we also say that γ crosses the rectangle $[0, 3L] \times [0, L]$. Define the function

$$\eta^*(p) := \liminf_{L\to\infty} \mathbb{P}^*_p(A_L)\,,$$

and consider the percolation threshold,

$$p^*_c := \inf\{p > 0 \,:\, \eta^*(p) = 1\}\,. \tag{1}$$

We have that $p^*_c \in (0, 1)$, which follows by standard arguments in percolation theory. For more in percolation thresholds on Voronoi tilings we refer to [1, 2, 11].

Theorem 1 *If $\mathbb{F}(0) < 1 - p_c^*$ then there exist constants $c_j > 0$ such that for all $n \geq 1$*

$$\mathbb{P}\big(T(\mathbf{0}, \mathbf{n}) < c_1 n\big) \leq c_2 \exp(-c_3 n)\,, \tag{2}$$

where $\mathbf{0} := (0,0)$ and $\mathbf{n} := (n, 0)$.

To show the importance of Theorem 1 we recall two fundamental results proved by Vahidi-Asl and Wierman [9, 10]. Consider the growth process

$$\mathbf{B_x}(t) := \{\mathbf{y} \in \mathbb{R}^2 \,:\, \mathbf{y} \in c(\mathbf{C_v}) \text{ with } \mathbf{v} \in \mathcal{D}_v \text{ and } T(\mathbf{v_x}, \mathbf{v}) \leq t\}\,.$$

where $c(\mathbf{C})$ denotes the closure of $\mathbf{C} \in \mathbb{R}^2$. Set

$$\mu(\mathbb{F}) := \inf_{n>0} \frac{\mathbb{E}T(\mathbf{0}, \mathbf{n})}{n} \in [0, \infty]\,.$$

and let τ_1, τ_2, τ_3 be independent random variables with distribution $\mathbb{F}$. If

$$\mathbb{E}\big(\min_{j=1,2,3}\{\tau_j\}\big) < \infty \tag{3}$$

then $\mu(\mathbb{F}) < \infty$ and for all unit vectors $\vec{\mathbf{x}} \in S^1$ ($|\vec{\mathbf{x}}| = 1$) $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{T(\mathbf{0}, n\vec{\mathbf{x}})}{n} = \lim_{n\to\infty} \frac{\mathbb{E}T(\mathbf{0}, \mathbf{n})}{n} = \mu(\mathbb{F})\,. \tag{4}$$

Further, if

$$\mathbb{E}\big(\min_{j=1,2,3}\{\tau_j\}^2\big) < \infty \tag{5}$$

and $\mu(\mathbb{F}) > 0$ then for all $\epsilon > 0$ $\mathbb{P}$-a.s. there exists $t_0 > 0$ such that for all $t > t_0$

$$(1-\epsilon)t\mathbf{D}(1/\mu) \subseteq \mathbf{B_0}(t) \subseteq (1+\epsilon)t\mathbf{D}(1/\mu)\,, \tag{6}$$

where $\mathbf{D}(r) := \{\mathbf{x} \in \mathbb{R}^2 \,:\, |\mathbf{x}| \leq r\}$.
We note here that the asymptotic shape is an Euclidean ball due to the statistical invariance of the Poisson point process. Unfortunately the exact value of the time constant $\mu(\mathbb{F})$, as a functional of $\mathbb{F}$, still a basic problem in first-passage percolation theory. Our result provides a sufficient condition on $\mathbb{F}$ to ensure $\mu(\mathbb{F}) > 0$.

Corollary 1 *Under assumption (3), if $\mathbb{F}(0) < 1 - p_c^*$ then $\mu(\mathbb{F}) \in (0, \infty)$.*

Proof. Together with the Borel-Cantelli Lemma, Theorem 1 and (4) imply

$$0 < c_1 \leq \liminf_{n\to\infty} \frac{T(\mathbf{0}, \mathbf{n})}{n} = \lim_{n\to\infty} \frac{T(\mathbf{0}, \mathbf{n})}{n} = \mu(\mathbb{F}) < \infty\,,$$

which is the desired result. □

For FPP models on the $\mathbb{Z}^2$ lattice Kesten (1986) has shown that $\mathbb{F}(0) < 1/2 = p_c(\mathbb{Z}^2)$ (the critical probability for bond percolation on $\mathbb{Z}^2$) is a sufficient condition to get (2) by using a stronger version of the BK-inequality. Here we follow a different method and we apply a simple renormalization argument to obtain a similar result. We expect that our condition to get (2) is equivalent to

$$\mathbb{F}(0) < p_c := \inf\{p > 0\,;\, \theta(p) = 1\}\,,$$

where $\theta(p)$ is the probability that bond percolation on $\mathcal{D}$ occurs with density p, since it is conjectured that $p_c + p_c^* = 1$ (duality) for many planar graphs. In fact, by combining Corollary 1 with (6) we have:

Corollary 2

$$1 \le p_c + p_c^* .$$

Proof. To see this assume we have a first-passage percolation model on $\mathcal{D}$ with

$$\mathbb{P}(\tau_{\mathbf{e}} = 0) = 1 - \mathbb{P}(\tau_{\mathbf{e}} = 1) = \mathbb{F}(0) = 1 - p > p_c^* . \tag{7}$$

Then $\mathbb{P}$-a.s. there exists an infinite cluster $\mathcal{W} \subseteq \mathcal{D}$ composed by edges $\mathbf{e}$ with $\tau_{\mathbf{e}} = 0$. Denote by $T(\mathbf{0}, \mathcal{W})$ the first-passage time from $\mathbf{0}$ to $\mathcal{W}$. Then for all $t > T(\mathbf{0}, \mathcal{W})$ we have that $\mathbf{B_0}(t)$ is an unbounded set. By (6) (since such a distribution satisfies (3) and (5)), this implies that $\mu(\mathbb{F}) = \mu(p) = 0$ if $1 - p > p_c$. On the other hand, by Corollary 1, $\mu(p) > 0$ if $1 - p < 1 - p_c^*$, and so (2) must hold. $\square$

Other passage times have been considered in the literature such as $T(\mathbf{0}, \mathbf{H}_n)$, where $\mathbf{H}_n$ is the hyperplane consisting of points $\mathbf{x} = (x^1, x^2)$ so that $x_1 = n$, and $T(\mathbf{0}, \partial[-n, n]^2)$. The arguments in this article can be used to prove the analog of Theorem 1 when $T(\mathbf{0}, \mathbf{n})$ is replaced by $T(\mathbf{0}, \mathbf{H}_n)$ or $T(\mathbf{0}, \partial[-n, n]^2)$. For site versions of FPP models the method works as well if we change the condition on $\mathbb{F}$ to $\mathbb{F}(0) < 1 - \bar{p}_c$, where now $\bar{p}_c$ is the critical probability for site percolation. Similarly to Corollary 2, in this case one can also obtain the inequality $1/2 \le \bar{p}_c$. For more details we refer to [8].

1 Renormalization

For the moment we assume that $\mathbb{F}$ is Bernoulli with parameter p. Let $L \ge 1$ be a parameter whose value will be specified later. Let $\mathbf{z} = (z^1, z^2) \in \mathbb{Z}^2$ and

$$|\mathbf{z}|_\infty := \max_{j=1,2} \{|z^j|\} .$$

Denote $\mathbf{C}_z$ the circuit composed by sites $\mathbf{z}' \in \mathbb{Z}^2$ with $|\mathbf{z} - \mathbf{z}'|_\infty = 2$. For each $\mathbf{A} \subseteq \mathbb{R}^2$, we denote by $\partial \mathbf{A}$ its boundary. For each $\mathbf{z} \in \mathbb{Z}^2$ and $r \in \{j/2 : j \in \mathbb{N}\}$ consider the box

$$\mathbf{B}_z^{rL} := Lz + [-rL, rL]^2 .$$

Divide $\mathbf{B}_{\mathbf{z}}^{L/2}$ into thirty-six sub-boxes with the same size and declare that $B_{\mathbf{z}}^{L/2}$ is a **full box** if all these thirty-six sub-boxes contain at least one point of $\mathcal{P}$. Let

$$H_{\mathbf{z}}^L := \left[\mathbf{B}_{\mathbf{z}'}^{L/2} \text{ is a full box } \forall\, \mathbf{z}' \in \mathbf{C}_{\mathbf{z}}\right] .$$

Let $\mathcal{C}_L$ be the set of all self-avoiding paths $\gamma = (\mathbf{v}_j)_{1 \le j \le h}$ in $\mathcal{D}$, connecting $\partial \mathbf{B}_{\mathbf{z}}^{L/2}$ to $\partial \mathbf{B}_{\mathbf{z}}^{3L/2}$ and with $\mathbf{C}_{\mathbf{v}_j} \cap \mathbf{B}_{\mathbf{z}}^{3L/2}$ for all $j = 2, \ldots, h-1$. Let

$$G_z^L := \left[t(\gamma) \ge 1 \,\forall\, \gamma \in \mathcal{C}_L\right] .$$

We say that $\mathbf{B}_{\mathbf{z}}^{L/2}$ is a **good box** (or that $\mathbf{z}$ is a good point) if

$$Y_{\mathbf{z}}^L := \mathbb{I}\big(H_{\mathbf{z}}^L \cap G_{\mathbf{z}}^L\big) = 1 ,$$

where $\mathbb{I}\big(E\big)$ denotes the indicator function of the event E.

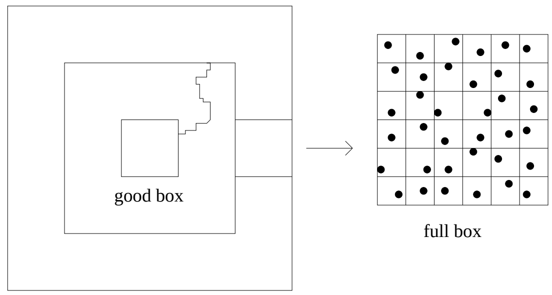

Figure 2: Renormalization

Lemma 1 *If* $\mathbb{P}(\tau_{\mathbf{e}} = 0) = 1 - p < 1 - p_c^*$ *then*

$$\lim_{L\to\infty} \mathbb{P}\big(Y_{\mathbf{0}}^L = 1\big) = 1\,.$$

Proof. First notice that

$$\mathbb{P}\big(Y_{\mathbf{0}}^L = 0\big) \le \mathbb{P}\big((H_{\mathbf{0}}^L)^c\big) + \mathbb{P}\big((G_{\mathbf{0}}^L)^c\big)\,. \tag{8}$$

By the definition of a two-dimensional homogeneous Poisson point process,

$$\lim_{L\to\infty} \mathbb{P}\big((H_{\mathbf{0}}^L)^c\big) = 0\,. \tag{9}$$

Now, let $X_{\mathbf{e}^*} := \tau_{\mathbf{e}}$, where $\mathbf{e}^*$ is the edge in $\mathcal{V}_e$ (the Voronoi tessellation) dual to $\mathbf{e}$. Then $\{X_{\mathbf{e}^*}\,;\,\mathbf{e}^* \in \mathcal{V}_e\}$ defines a bond percolation model on $\mathcal{V}$ with law $\mathbb{P}_p^*$. Consider the rectangles

$$R_L^1 := [L/2, 3L/2] \times [-3L/2, 3L/2]\,,\; R_L^2 := [-3L/2, 3L/2] \times [L/2, 3L/2]$$

$$R_L^3 := [-3L/2, -L/2] \times [-3L/2, 3L/2] \text{ and } R_L^4 := [-3L/2, 3L/2] \times [-3L/2, -L/2]\,.$$

We denote by A_L^i the event A_L (recall the definition of p_c^*) but now translate to the rectangle R_L^i, and by F_L the event that an open circuit σ^* in $\mathcal{V}$ which surrounds $B_{\mathbf{0}}^{L/2}$ and lies inside $B_{\mathbf{0}}^{3L/2}$ does not exist. Thus one can easily see that

$$\cap_{i=1}^4 A_L^i \subseteq (F_L)^c\,.$$

Notice that if there exists an open circuit σ^* in $\mathcal{V}$ which surrounds $B_{\mathbf{0}}^{L/2}$ and lies inside $B_{\mathbf{0}}^{3L/2}$, then every path γ in $\mathcal{C}_L$ has an edge crossing with σ^* and thus $t(\gamma) \ge 1$. Therefore,

$$\mathbb{P}\big((G_{\mathbf{0}}^L)^c\big) \le \mathbb{P}_p^*(F_L) \le 4\big(1 - \mathbb{P}_p^*(A_L)\big)\,. \tag{10}$$

Since $p > p_c^*$, by using (8), (9), (10) and the definition of p_c^*, we get Lemma 1. □

To obtain some sort of independence between the random variables $Y_{\mathbf{z}}^L$ we shall study some geometrical aspects of Voronoi tilings. Given $\mathbf{A} \subseteq \mathbb{R}^2$, let $\mathcal{I}_{\mathcal{P}}(\mathbf{A})$ be the sub-graph of $\mathcal{D}$ composed of vertices $\mathbf{v}_1$ in $\mathcal{D}_v$ and edges $(\mathbf{v}_2, \mathbf{v}_3)$ in $\mathcal{D}_e$ so that $\mathbf{C}_{\mathbf{v}_i} \cap \mathbf{A} \neq \emptyset$ for all $i = 1, 2, 3$.

Lemma 2 *Let $L > 0$ and $\mathbf{z} \in \mathbb{Z}^2$. Assume that $\mathcal{P}$ and $\mathcal{P}'$ are two configurations of points so that $\mathcal{P} \cap \mathbf{B}_{\mathbf{z}}^{5L/2} = \mathcal{P}' \cap \mathbf{B}_{\mathbf{z}}^{5L/2}$ and that $\mathbf{B}_{\mathbf{z}'}^{L/2}$ is a full box with respect to $\mathcal{P}$, for all $\mathbf{z}' \in \mathbf{C}_{\mathbf{z}}$. Then $\mathcal{I}_{\mathcal{P}}(\mathbf{B}_{\mathbf{z}}^{3L/2}) = \mathcal{I}_{\mathcal{P}'}(\mathbf{B}_{\mathbf{z}}^{3L/2})$.*

Proof. By the definition of the Delaunay Triangulation, Lemma 2 holds if we prove that

$$\mathbf{C}_{\mathbf{v}}(\mathcal{P}) \cap \mathbf{B}_{\mathbf{z}}^{3L/2} \neq \emptyset \Rightarrow \mathbf{C}_{\mathbf{v}}(\mathcal{P}) = \mathbf{C}_{\mathbf{v}}(\mathcal{P}') \,. \tag{11}$$

To prove this we claim that

$$\mathbf{C}_{\mathbf{v}}(\mathcal{P}) \cap \mathbf{B}_{\mathbf{z}}^{3L/2} \neq \emptyset \Rightarrow \mathbf{C}_{\mathbf{v}}(\mathcal{P}) \subseteq \mathbf{B}_{\mathbf{z}}^{2L} \,. \tag{12}$$

If (12) does not hold then there exist $\mathbf{x}_1 \in \partial \mathbf{B}_{\mathbf{z}}^{3L/2} \cap \mathbf{C}_{\mathbf{v}}(\mathcal{P})$ and $\mathbf{x}_2 \in \partial \mathbf{B}_{\mathbf{z}}^{2L} \cap \mathbf{C}_{\mathbf{v}}(\mathcal{P})$ (by convexity of Voronoi tilings). Since every box $B_{\mathbf{z}'}^{L/2}$ with $|\mathbf{z}-\mathbf{z}'|_\infty = 2$ is a full box, there exist $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{P}$ so that

$$|\mathbf{v}_1 - \mathbf{x}_1| \leq \sqrt{2}L/6 \text{ and } |\mathbf{v}_2 - \mathbf{x}_2| \leq \sqrt{2}L/6 \,.$$

Although, $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to $\mathbf{C}_{\mathbf{v}}(\mathcal{P})$ and so

$$|\mathbf{v} - \mathbf{x}_1| \leq |\mathbf{v}_1 - \mathbf{x}_1| \text{ and } |\mathbf{v} - \mathbf{x}_2| \leq |\mathbf{v}_2 - \mathbf{x}_2| \,.$$

Thus,

$$L/2 \leq |\mathbf{x}_1 - \mathbf{x}_2| \leq |\mathbf{x}_1 - \mathbf{v}| + |\mathbf{x}_2 - \mathbf{v}| \leq \sqrt{2}L/3 \,,$$

which leads to a contradiction since $\sqrt{2}/3 < 1/2$. By an analogous argument, one can prove that

$$\mathbf{C}_{\mathbf{v}'}(\mathcal{P}') \cap (\mathbf{B}_{\mathbf{z}}^{5L/2})^c \neq \emptyset \Rightarrow \mathbf{C}_{\mathbf{v}'}(\mathcal{P}') \subseteq (\mathbf{B}_{\mathbf{z}}^{2L})^c \,. \tag{13}$$

Now suppose (11) does not hold. Without lost of generality, we may assume that there exists $\mathbf{v} \in \mathcal{P}$ with $\mathbf{C}_{\mathbf{v}}(\mathcal{P}) \cap \mathbf{B}_{\mathbf{z}}^{3L/2} \neq \emptyset$ and $\mathbf{x} \in \mathbf{C}_{\mathbf{v}}(\mathcal{P})$ with $\mathbf{x} \notin \mathbf{C}_{\mathbf{v}}(\mathcal{P}')$. So $\mathbf{x} \in \mathbf{C}_{\mathbf{v}'}(\mathcal{P}')$ for some $\mathbf{v}' \in \mathcal{P}'$. Although, $\mathcal{P} \cap \mathbf{B}_{\mathbf{z}}^{5L/2} = \mathcal{P}' \cap \mathbf{B}_{\mathbf{z}}^{5L/2}$ and then $\mathbf{v}' \in (\mathbf{B}_{\mathbf{z}}^{5L/2})^c$, which is a contradiction with (12) and (13). $\square$

For each $l \geq 1$, we say that the collection of random variables $\{Y_{\mathbf{z}} : \mathbf{z} \in \mathbb{Z}^2\}$ is l-dependent if $\{Y_{\mathbf{z}} : \mathbf{z} \in \mathbf{A}\}$ and $\{Y_{\mathbf{z}} : \mathbf{z} \in \mathbf{B}\}$ are independent whenever

$$l < d_\infty(\mathbf{A}, \mathbf{B}) := \min\{|\mathbf{z} - \mathbf{z}'|_\infty : \mathbf{z} \in \mathbf{A} \text{ and } \mathbf{z}' \in \mathbf{B}\} \,.$$

Combining Lemma 2 with the translation invariance and the independence property of the Poisson point process we obtain:

Lemma 3 *For all $L > 0$, $\{Y_{\mathbf{z}}^L : \mathbf{z} \in \mathbb{Z}^2\}$ is a 5-dependent collection of identically distributed Bernoulli random variables.*

Denote $Y^L := \{Y_{\mathbf{z}}^L \,; \mathbf{z} \in \mathbb{Z}^2\}$ and let $M_m(Y^L)$ be the maximum number of pairwise disjoint good circuits in $\mathbb{Z}^2$, surrounding the origin and lying inside the box $[-m, m]^2$.

Lemma 4 *If $\mathbb{F}(0) < 1 - p_c^*$ then there exists $L_0 > 0$ and $c_j = c_j(L_0) > 0$ such that*

$$\mathbb{P}\big(M_m(Y^{L_0}) \leq c_1 m\big) \leq \exp(-c_2 m) \,.$$

PROOF. Combining Lemmas 1 and 3 with and Theorem 0.0 of Ligget, Schonman and Stacey [6], one gets that Y^L is dominated from below by a collection $X^L := \{X^L_z\,;\, z \in \mathbb{Z}^2\}$ of i.i.d. Bernoulli random variables with parameter $\rho(L) \to 1$ when $L \to \infty$. But for ρ_L sufficiently close to 1, we can chose $c > 0$ sufficiently small, so that the probability of the event that $M_m(X^L) < cm$ decays exponentially fast with m (see Chapter 3 of Grimmett [3]). Together with domination, this proves Lemma 4. □

The connection between the variable $M_m(Y^L)$ and the first-passage time $T(\mathbf{0}, \mathbf{n})$ is summarize by the following:

Lemma 5

$$\frac{M^L_{nL^{-1}}}{6} \leq T(\mathbf{0}, \mathbf{n})\,.$$

PROOF. We say that $(B^{L/2}_{\mathbf{z}_j})_{1\leq j\leq h}$ is a circuit of good boxes if $(\mathbf{z}_j)_{1\leq j\leq h}$ is a good circuit in $\mathbb{Z}^2$, and that $(B^{L/2}_{\mathbf{z}_j})_{1\leq j\leq h}$ and $(B^{L/2}_{\mathbf{z}'_j})_{1\leq j\leq h'}$ are l-distant if

$$d_\infty\big((\mathbf{z}_j)_{1\leq j\leq k}, (\mathbf{z}'_j)_{1\leq j\leq h'}\big) > l\,.$$

Denote $M^L_m := M_m(Y^L)$. Notice that there exist at least $(M^L_{nL^{-1}}/6)$ pairwise 5-distant circuits of good boxes surrounding the origin and lying inside $[-n, n]^2 \subseteq \mathbb{R}^2$. Therefore, every path γ between the origin and any point outside $[-n, n]^2$ must cross at least $(M^L_{nL^{-1}}/6)$ 5-distant circuits of good boxes. We claim this yields

$$\frac{M^L_{nL^{-1}}}{6} \leq t(\gamma)\,. \tag{14}$$

Indeed, assume we take two 5-distant good boxes, say $\mathbf{B}^{L/2}_{\mathbf{z}_1}$ and $\mathbf{B}^{L/2}_{\mathbf{z}_2}$, connected by a path γ in $\mathcal{D}$. Then γ must contain two sub-paths in $\mathcal{D}$, say $\bar{\gamma}_i = (\mathbf{v}^i_j)_{1\leq j\leq h_i}$ for $i = 1, 2$, connecting $\partial\mathbf{B}^{3L/2}_{\mathbf{z}_i}$ to $\partial\mathbf{B}^{5L/2}_{\mathbf{z}_i}$ and with $\mathbf{C}_{\mathbf{v}^i_j} \cap \mathbf{B}^{3L/2}_{\mathbf{z}_i}$ for all $j = 2, ..., h_i - 1$. Since $B^{L/2}_{\mathbf{z}_1}$ and $B^{L/2}_{\mathbf{z}_2}$ are 5-distant good boxes, by Lemma 2, these sub-paths must be edge disjoint. By the definition of a good box, $t(\bar{\gamma}_1) \geq 1$ and $t(\bar{\gamma}_2) \geq 1$, which yields

$$2 \leq t(\bar{\gamma}_1) + t(\bar{\gamma}_2) \leq t(\gamma)\,.$$

By repeating this argument inductively (on the number of good boxes which are crossed by γ) one can get (14). Lemma 5 follows directly from (14). □

Now we are ready to prove Theorem 1.
PROOF. Together with Lemma 5, Lemma 4 implies Theorem 1 under (7). For the general case, assume $\mathbb{F}(0) = \mathbb{P}(\tau_{\mathbf{e}} = 0) < 1 - p_1$. Fix $\epsilon > 0$ so that $\mathbb{F}(\epsilon) < 1 - p^*_c$ (we can do so since $\mathbb{F}$ is right-continuous). Define the auxiliary process $\tau^\epsilon_{\mathbf{e}} := \mathbb{I}(\tau_{\mathbf{e}} > \epsilon)$ and denote by T^ϵ the first-passage time associated to the collection $\{\tau^\epsilon_{\mathbf{e}} : \mathbf{e} \in \mathcal{D}_e\}$. Thus $T^\epsilon(\mathbf{0}, \mathbf{n}) \leq \epsilon^{-1} T(\mathbf{0}, \mathbf{n})$. Since $\tau^\epsilon_{\mathbf{e}}$ has a Bernoulli distribution with parameter $\mathbb{P}\big(\tau^\epsilon_{\mathbf{e}} = 0\big) = \mathbb{F}(\epsilon) < 1 - p^*_c$, together with the previous case this yields Theorem 1. □

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