An Algebraic Treatment of the Leibniz Law.

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"Eadem sunt quorum unum potest substitui alteri salva veritate". This famous dictum was stated by Gottfried Wilhelm Leibniz in his *"Discourse on Methaphysics"* and is now termed *Leibniz Law of the Identity of Indiscernibles* (PII) after him. This principle is a fragment of Leibniz's analytical ontology. A version of Leibniz Law can be written in a second order language as:

$$\forall F(F(a) \leftrightarrow F(b)) \rightarrow a = b$$

where a and b denote individuals and F is a variable running over properties of an individual. Also in a second order logic one can state the converse of this principle, to arrive at the so-called the *Principle of Indiscernibility of Identicals* (II), namely:

$$a = b \rightarrow \forall F(F(a) \leftrightarrow F(b)).$$

The conjunction of these two principles gives *Leibniz Law*. The meaning of this law is ordinarily understood to mean that identity (i.e., being the same object, symbolically $a = b$) is defined by way of indiscernibility (i.e., agreement with respect to all properties).

In classical logic and mathematics we are forced to deal with the indistinguishability relation in the framework of certain structures whereby the relations and functions available within this structure can not individuate between two distinct objects. In the domain of these relational systems it means that such structures are not *rigid*, i.e., there are automorphisms defined on these structures other than the identity functions. As a simple example of such a structure we can consider the additive group of integers $\mathcal{Z} = \langle Z, + \rangle$ where one can not distinguish between two integers n and $-n$ since the function $f(x) = -x$ is a automorphism of this structure other than identity. But there is a theorem of ZFC stating that any *non-rigid* structure can be extended to become a rigid one. For instance, $\mathcal{Z} = \langle Z, + \rangle$ can be extended to $\mathcal{Z}' = \langle Z', +, < \rangle$ where one can surely distinguish between $-n$ and n (i.e., $-n < n$). Generalizing the above considerations one can assume that the indiscernibility of the objects is an internal property of the given relational structure, i.e., two distinct objects are indiscernible inside this structure but from outside of this structure they can be distinguished by the adequate added predicate. In the whole well-founded set-theoretical universe $\mathcal{V} = \langle V, \in \rangle$ every object is an individual thing and Leibniz's Law is applicable to it. Hence, any two distinct things a and b can be individuated by the property *"being the identical to x"* (where $x = a$ or $x = b$) namely $a = (\exists! x)N_a(x)$ where $N_a(.)$ is a *naming predicate "to be an a"*. Denoting the truth of a proposition A by $1 \leq A$ and its falsity by $A \leq 0$ we can define the so-called *"uniqueness property"* for any naming predicate:

$$1 \leq N_a(a) \wedge \forall a'(N_a(a') \rightarrow a = a').$$

Using Russell's theory of *definite description* one can introduce *Russell's formula* for any other predicate B, i.e.,

$$B(a) = \exists! x \left\{ N_x(x) \wedge B(x) \wedge \forall y N_x(y) \rightarrow x = y \right\}.$$

In the algebraic approach to logic (mainly developed by the *Lvov-Warsaw School of Logic*) we may define the sentential language as an absolutely free algebra. **Fm** denotes the algebra of formulae (Fm being the universe of this algebra) which is supposed to be absolutely free algebra of type **L** over a denumerable set of generators $Var = \{p, q, r, ...\}$. The set of free generators is identical to the infinite countable set of propositional variables. The algebra of terms **Fm** is endowed with a finite number of finitary operations (counterparts of connectives) $F_1, F_2, ..., F_n$. The structure $\mathbf{Fm} = \langle Fm, F_1, F_2, ..., F_n \rangle$ is called the algebra of terms. The concept of logic or - more generally - the concept of deductive systems with the language of type **L** is defined as pair $\mathcal{S} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is a substitution-invariant consequence relation on **Fm**. Since **Fm** is absolutely free algebra, freely generated by a set of variables and its algebraic semantics, i.e., algebra **A** (in the case of classical propositional language (-CPL) is a two element Boolean algebra) is a structure of the same similarity types as **Fm** then there exists a function $f : Var \rightarrow \mathbf{A}$ and exactly one function $h^f : \mathbf{Fm} \rightarrow \mathbf{A}$ which is the extension of f, i.e., $h^f(p) = f(p)$ for each $p \in Var$. This function is the homomorphism from the algebra of terms to the algebra **A** constituting the algebraic models of sentential language. Using these tools one can identify the interpretation of a given formula $\varphi \in \mathbf{Fm}$ with $h(\varphi)$ where h is a homomorphism from **Fm** to **A** mapping each variable of φ into its assigned value. If a formula of propositional language is represented in the form $\varphi(x_0, x_1, ..., x_{n-1})$ indicating that each of its variables occur in the list $x_0, x_1, ..., x_{n-1}$ then $\varphi^{\mathbf{A}}(a_0, a_1, ..., a_{n-1})$ denotes the algebraic translation of this formula for a given homomorphism $h(\varphi)$ such that $h(x_i) = a_i$ for all $i < \omega$. Alternatively speaking, such homomorphisms from **Fm** to **A** are a possible semantic c*orrelate functions* of the sentential language. The universe of **A** is assumed to be a set of possible semantic correlates of sentences. On **Fm** one can define a binary relation θ having all formal properties of the congruence relation being a counterpart of the indistinguishability relation. This congruence is compatible with each theory $\mathcal{T}$ defined in this language, i.e.,

$$\text{for all } a, b \in \mathbf{A} \text{ if } \langle a, b \rangle \in \theta \text{ and } a \in \mathcal{T} \text{ then } b \in \mathcal{T}.$$

Every algebraic model of a given language has the largest congruence called the *Leibniz congruence* of this model and is denoted by $\Omega_{\mathbf{A}}\mathcal{T}$. This congruence is a first order analogue of Leibniz's second-order definition of identity:

$$\langle a, b \rangle \in \Omega_{\mathbf{A}}\mathcal{T} \text{ iff for every } \varphi(x, x_0, x_1, ..., x_n) \in \mathbf{Fm} \text{ and all } c_0, c_1, ..., c_{n-1} \in \mathbf{A}$$

$$\varphi^{\mathbf{A}}(a, c_0, c_1, ..., c_{n-1}) \in \mathcal{T} \text{ iff } \varphi^{\mathbf{A}}(b, c_0, c_1, ..., c_{n-1}) \in \mathcal{T}.$$

The Leibniz congruence $\Omega_{\mathbf{A}}T$ is the synonymy relation defined on the sentential language $\mathcal{S} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ relative to the theory T. Thus

$$\alpha \equiv \beta(\Omega_{\mathbf{A}}T) \leftrightarrow \forall\varphi \in \mathbf{Fm}\forall p \in Var(\varphi)(\varphi(p/\alpha) \in \mathcal{T} \leftrightarrow \varphi(p/\beta) \in \mathcal{T})$$

where $\equiv$ is an indiscernibility relation, $Var(\varphi)$ is the set of all variables occurring in φ and $\varphi(p/\alpha)$ is the results of simultaneously substituting the variable $p \in \varphi$ by the sentence α.

On an ontological level, the Leibniz congruence is interpreted as the relation of indiscernibility. Given a model for a first order propositional language $\mathcal{S} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ one can say that two objects of its domain a and b are indiscernible relative to the predicates expressible in this language, without using the equality symbol, even without necessarily being the same. Hence, two objects a and b are ontologically indiscernible if $\langle a, b \rangle \in \Omega_{\mathbf{A}}\mathcal{T}$. Hence two sentences are synonymous if they possess the same semantic correlates which are elements of $\Omega_{\mathbf{A}}\mathcal{T}$. Such sentences are interchangeable in any propositional context *salva veritate.*

Recalling Quine's thesis that *"no entity without identity"* we introduce in this short note the notion of the *Quine congruence* $Q_{\mathbf{A}}T$ when the Leibniz congruence is the identity relation. In this case $\Omega_{\mathbf{A}}\mathcal{T}$ is simply said to be Quinian. It is always the case that $Q_{\mathbf{A}}\mathcal{T} \subseteq \Omega_{\mathbf{A}}\mathcal{T}$.

Theorem 1. *The structure for sentential language is rigid if the Leibniz congruence of this structure is equal to its Quine congruence (i.e.,* $\Omega_{\mathbf{A}}\mathcal{T} = Q_{\mathbf{A}}\mathcal{T}$*).*

All elements which belong to the Leibniz congruence are called *monads*. Hence, $Mon(a) = \{b \mid \langle a, b \rangle \in \Omega_{\mathbf{A}}\mathcal{T}\}$. From the definition of a congruence relation it follows that if $\langle a, b \rangle \in \Omega_{\mathbf{A}}\mathcal{T}$ and $\langle b, c \rangle \in \Omega_{\mathbf{A}}\mathcal{T}$ then $\langle a, c \rangle \in \Omega_{\mathbf{A}}\mathcal{T}$.

We introduce the following definition:

Definition 2. The Leibniz congruence is termed *compact* if there are at least two such elements that $\langle a, b \rangle \in \Omega_{\mathbf{A}}T$ and $a \neq b$.

Corollary 3. *If the Leibniz congruence is compact then the structure on which this congruence is defined is non-rigid and hence* $\Omega_{\mathbf{A}}T \subset Q_{\mathbf{A}}T$ *(and this inclusion must be proper).*

Algebraically, one can define the lattice of the Leibniz congruences.

Definition 4. The sequence $\Omega^{n+1}_{\mathbf{A}}\mathcal{T} \subseteq \Omega^{n}_{\mathbf{A}}\mathcal{T}$ of the Leibniz congruences represents a sequence of still sharper and sharper *discernibility criteria* leading to the notion of a *discernibility horizon* represented by the Leibniz congruence $\Omega_{\mathbf{A}}\mathcal{T} = \bigcap\{\Omega^{n}_{\mathbf{A}}\mathcal{T} : n \in N\}$. The above sequence is called the *generating sequence* of the Leibniz congruence $\Omega_{\mathbf{A}}T$. It can be the case that $\Omega_{\mathbf{A}}\mathcal{T} = Q_{\mathbf{A}}\mathcal{T}$.

Fact 5. *The sequence* $(\Omega^{n}_{\mathbf{A}}\mathcal{T} : n \in N)$ *is an algebraic lattice.*

Theorem 6. *The limit* $\lim_{n \to x} \Omega^{n}_{\mathbf{A}}\mathcal{T} = Q_{\mathbf{A}}\mathcal{T}$ *(where* x *is the cardinality of* $\mathbf{A}$*,i.e., the set of semantic correlates) is equal to the Quine congruence. When this limit is attained then each object in the universe of discourse can be individuated by an adequate predicate.*

The unique predicate enabling the explicit individuation of each object is the naming predicate - "to be an x". This predicate can also be constructed algebraically. Namely, each property of a given object is represented univocally by the corresponding predicate. The fact that a property (F) can be prescribed to x is expressed by $F(x)$. The fact that a negation of this property $(\widetilde{F})$ is prescribed to x is expressed by $\neg F(x)$. Hence, in the case of a macroscopic (!) object obeying the classical logic, from the set of all possible properties (F) or $(\widetilde{F})$ must be ascribed to x. Hence, $F(x)$ or $\neg F(x)$. Kant expressed this simply as "if all possible predicates are taken together with their contradictory opposites then one of each pair of contradictory opposites must belong to it". It follows that x has each elementary property either positive (F) or negative $(\widetilde{F})$. On the metalogical level, the above facts are reflected by the *Lindenbaum theorem* stating that any semantically consistent set of terms admits a semantically consistent complete extension. Complete theory $(\mathcal{T}_{\max})$ is characterized by the following formal condition:

$$\forall\beta(\beta \in \mathcal{T}_{\max} \text{ or } \neg\beta \in \mathcal{T}) \text{ where } \beta \in \mathbf{Fm}.$$

The *Lindenbaum property* asserts the following fact:

$$\mathcal{T} \subseteq \mathcal{T}_{\max}$$

Theorem 7. *The Lindenbaum property is interpreted on the ontological level as a fact that each macroscopic object obeying the classical logic possesses a given property (F) or its negation $(\widetilde{F})$.*

Definition 8. The naming predicate "to be an x" corresponding to definite description of x is an *atom* in the lattice of all possible predicates. Namely, this predicate is given by $N^{(v)} = F_1^{(v)} \wedge F_2^{(v)} \wedge ... \wedge F_n^{(v)}$ with $F_i^{(v)} \in \{F_i, \neg F_i\}$. The lattice of all possible predicates is *Boolean* in the case of classical objects. This lattice is denoted by $\mathcal{L}_B$.

Corollary 9. *The Lindenbaum property is equivalent to the Ultrafilter Lemma stating that each filter can be extended to the ultrafilter.*

Corollary 10. *If in the context of a Boolean lattice $\mathcal{L}_B$ objects x_i are represented by definite descriptions, i.e., $a_i = (\exists!x)N_{a_i}(x)$ where $N_{a_i}(.)$ are naming predicates then the Leibniz law holds for these objects as a theorem. Hence, each object can be individuated uniquely by its naming predicate. It is the case that for any other predicate A corresponding to the property (A) we have that:*

$$N \leq A \text{ or } N \leq \neg A.$$

All theorems presented abov will be precisely considered.

References

[1] Kant, I. (1929). *Critique of Pure Reason*, Macmillan, New York, p. B600.

[2] Ketland, J. (2006). Structuralism and the identity of indiscernibles, *Analysis* **66.4**, 303-315.

[3] Krause, D., de Araujo Feitosa H. Algebraic Aspects of Quantum Indiscernibility, *preprint*, 2008.

[4] Mittelstaedt, P. (1989). The Leibniz Principle in Quantum Logic. Internet. *J. Theoret. Phys.* **28**, 159-168.

[5] Quine, W.V., (1969). *Ontological Relativity and Other Essays*, New York, Columbia Un. Press.

[6] Wilczek, P. (2008). Constructible Models of Orthomodular Quantum Logics. *Electronic J. Theoret. Phys.* **19** (2008) 9–32.

[7] Wilczek, P. (2008). Large Cardinals, Model of Set Theory and Leibniz Law. *Vol. of Abstracts, 6 European Congress of Analytic Philosophy*, p. 117.