Elect. Comm. in Probab. **10** (2005), 207–217

ELECTRONIC COMMUNICATIONS in PROBABILITY

EXACT ASYMPTOTICS FOR BOUNDARY CROSSING PROBABILITIES OF BROWNIAN MOTION WITH PIECE-WISE LINEAR TREND

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Submitted 3 November 2004 , *accepted in final form* 12 August 2005

AMS 2000 Subject classification: Primary 60F10; secondary 60G15, 60G70.
Keywords: Brownian motion with trend, boundary crossing probability, large deviations, exact asymptotics.

Abstract

Let B be a standard Brownian motion and let b_γ be a piecewise linear continuous boundary function. In this paper we obtain an exact asymptotic expansion of $\boldsymbol{P}\{B(t) < b_\gamma(t), \forall t \in [0,1]\}$ provided that the boundary function satisfies $\lim_{\gamma\to\infty} b_\gamma(t^*) = -\infty$ for some $t^* \in (0,1]$.

Introduction

Let B denote a standard Brownian motion, and let $b : [0,1] \to \mathbb{R}$ be a deterministic boundary function. Several authors have studied the boundary crossing probability $\boldsymbol{P}\{\exists t \in [0,1] : B(t) > b(t)\}$. If the boundary function b is piecewise linear, then the boundary crossing probability can be calculated explicitly. When h is a straight line this probability is well known, see for example Borodin and Salminen (1996), p. 197. Scheike (1992) obtained an integral expression for trend functions consisting of two straight lines. Wang and Pötzelberger (1997), Novikov et al. (1999), Janssen and Kunz (2004) deal with the case of a general piecewise linear boundary. Wang and Pötzelberger (1997) gave an integral expression for the boundary crossing probability, while Janssen and Kunz (2004) have expanded this integral expression in a sum of multivariate normal distribution functions. For other related results see the recent articles of Benghin and Orsingher (1999), Pötzelberger and Wang (2001) and Abundo (2002).

In an asymptotic context the boundary function $b = b_\gamma$ may depend on γ. If $\lim_{\gamma\to\infty} b_\gamma(t^*) = -\infty$, with $t^* \in (0,1]$, then the boundary crossing probability tends to 1. For this case, it is interesting to find the speed of convergence to 1.
In various applications b_γ is of the form $u - \gamma h$ with h a trend function (or signal) and u a

given deterministic function with $u(0) > 0$. If h is positive at some point $t^* \in (0,1]$, then

$$\psi(\gamma h, u, B) = 1 - \boldsymbol{P}\{\exists t \in [0,1] : B(t) + \gamma h(t) > u(t)\} \to 0, \quad \gamma \to \infty.$$

The speed of convergence to 0 of $\psi(\gamma h, u, B)$ is of particular interest when dealing for instance with the asymptotic power of weighted Kolmogorov test (see e.g. Bischoff et al. (2004) for results concerning Brownian bridge). A large deviation type result for $\psi(\gamma h, u, B_0)$ with B_0 a Brownian bridge is derived in Bischoff et al. (2003a), whereas in Bischoff et al. (2003b) the exact asymptotic behaviour is obtained for h, u both piecewise linear continuous functions.
In an unrelated paper Lifshits and Shi (2002) showed in Lemma 2.3 the following asymptotic lower bound

$$\liminf_{\gamma \to \infty} \gamma^{-2} \log\Bigl(\boldsymbol{P}\Bigl\{\sup_{t \in [0,L]} (B(t) + \gamma h(t)) < a\gamma^{-b}\Bigr\}\Bigr) \quad \geq \quad -\frac{1}{2}\int_0^L (h'(t))^2\, dt, \qquad (1)$$

with h piecewise linear such that $h(0) = 0$ and $a > 0, b \geq 0, L > 0$ three constants. If $b = 0$ the logarithmic asymptotic above follows by large deviation theory (see e.g. Varadhan (1984), or Ledoux (1996)), we have

$$\liminf_{\gamma \to \infty} \gamma^{-2} \log\Bigl(\boldsymbol{P}\Bigl\{\sup_{t \in [0,L]} (B(t) + \gamma h(t)) < a\Bigr\}\Bigr) \quad = \quad -\frac{1}{2}\inf_{g \geq h}\int_0^L (g'(t))^2\, dt, \qquad (2)$$

with $g(t) = \int_0^t g'(s)\, ds, t \in [0,1]$.
For a given function $f : [0,\infty) \to \mathbb{R}$ with $f(0) \leq 0$ we denote throughout in the following by $\tilde{f}$ the smallest non-decreasing concave majorant of $\max(0, f)$ and by $\tilde{f}' = (\tilde{f})'$ the right-hand derivative of $\tilde{f}$ (if it exits).
In this article we obtain an exact asymptotic expansion for $\psi(h_\gamma, u_\gamma, B)$ where h_γ, u_γ are two piecewise linear continuous functions and the trend function h_γ becomes large as $\gamma \to \infty$. It turns out that the asymptotic is largely determined by $\tilde{h}_\gamma$ the smallest non-decreasing concave majorant of h_γ. In the special case that $h_\gamma = \gamma h$ is piecewise linear continuous with $h(0) \leq 0$ and $u_\gamma(t) = a\gamma^{-b}, \forall t \in [0,L], a > 0, b \geq 0$ we have (see Example 1)

$$\begin{aligned} &\boldsymbol{P}\Bigl\{\sup_{t \in [0,L]} (B(t) + \gamma h(t)) < a\gamma^{-b}\Bigr\} \\ &\quad = \quad (1 + o(1)) c_1 \gamma^{-c_2} \exp\Bigl(-\frac{\gamma^2}{2}\int_0^L (\tilde{h}'(t))^2\, dt + a\gamma^{1-b}\tilde{h}'(0+)\Bigr), \quad \gamma \to \infty, \end{aligned}$$

with c_1, c_2 two positive constants. An immediate consequence of the above result and (2) is

$$\inf_{g \geq \max(0,h)} \int_0^L (g'(t))^2\, dt \quad = \quad \int_0^L (\tilde{h}'(t))^2\, dt, \quad \text{with } g(t) = \int_0^t g'(s)\, ds, t \in [0,1]. \qquad (3)$$

Outline of the paper: In the next section we introduce several notation. In Section 3 we present the main result, and give its proof in Section 4.

Preliminaries

Consider in the following the reproducing kernel Hilbert space which is naturally connected with $B(t), t \in [0,1]$ defined by

$$H_1 = \Bigl\{h \text{ absolute continuous}, \exists h' \in L^2([0,1]) : \ h(t) = \int_0^t h'(s)\, ds, \quad \forall t \in [0,1]\Bigr\}$$

furnished with the inner product and the corresponding norm

$$< h_1, h_2 >= \int_0^1 h_1'(s)h_2'(s)\, ds, \quad \forall h_1, h_2 \in H_1, \quad |h|^2 = \int_0^1 (h'(s))^2\, ds, \quad \forall h \in H_1.$$

We introduce next some notation needed in the sequel.
Throughout, let $k > 1$ be a fixed integer and put $t_0 = 0 < t_1 < \cdots < t_k = 1, \delta_i = t_{i+1} - t_i, i \le k-1$ and denote by Σ the covariance matrix of the random vector $(B(t_1), \ldots, B(t_k))^\top$. Its inverse matrix is given by

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{t_1-t_0} + \frac{1}{t_2-t_1} & -\frac{1}{t_2-t_1} & 0 & \cdots & 0 \\ -\frac{1}{t_2-t_1} & \frac{1}{t_2-t_1} + \frac{1}{t_3-t_2} & -\frac{1}{t_3-t_2} & & \vdots \\ 0 & -\frac{1}{t_3-t_2} & \frac{1}{t_3-t_2} + \frac{1}{t_4-t_3} & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\frac{1}{t_k-t_{k-1}} \\ 0 & \cdots & 0 & -\frac{1}{t_k-t_{k-1}} & \frac{1}{t_k-t_{k-1}} \end{pmatrix}. \tag{4}$$

Hence we have for any $\boldsymbol{x} = (x_1, \ldots, x_k)^\top, \boldsymbol{y} = (y_1, \ldots, y_k)^\top \in \mathbb{R}^k$

$$\boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{y} = \sum_{i=0}^{k-1} \frac{(y_{i+1} - y_i)}{t_{i+1} - t_i}(x_{i+1} - x_i), \quad \text{with } x_0 = y_0 = 0. \tag{5}$$

For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$ the relations $\boldsymbol{x} \ge \boldsymbol{y}$ and $\boldsymbol{x} > \boldsymbol{y}, \boldsymbol{x} < \boldsymbol{y}$ are understood componentwise.
Let I be an index subset of $\{1, \ldots, k\}$ with $|I| > 0$ elements. The subvector $\boldsymbol{x}_I = (x_i)_{i \in I}^\top \in \mathbb{R}^{|I|}$ of $\boldsymbol{x}$ consists of the components of $\boldsymbol{x}$ with indices in I and similarly, the square matrix Σ_I is obtained by deleting the rows and the columns of Σ with indices not in I. We write simply $\boldsymbol{x}_I^\top, \Sigma_I^{-1}$ instead of $(\boldsymbol{x}_I)^\top, (\Sigma_I)^{-1}$, respectively. Note in passing that both $\Sigma^{-1}, \Sigma_I^{-1}$ exist since Σ, Σ_I are positive definite matrices.
If L, M are two non-empty index sets, then LM is defined by

$$LM = \{i : i \in L, i+1 \in M\}.$$

In case that $|L||M| = 0$ then LM is the empty set.
Denote in the following by $\boldsymbol{g}$ the vector $(g(t_1), \ldots, g(t_k))^\top$ with $g : \mathbb{R} \to \mathbb{R}$ an arbitrary function.
Any polygonal line g discussed below is continuous with minimal representation given in terms of the nodes $(t_0, g(t_0)), (t_1, g(t_1)), \ldots, (t_k, g(t_k))$ so that $g(t^*) > 0$ for some $t^* \in (0, 1]$ and $g(0) \le 0$. Put next

$$K(g) = \{i \in \{1, \ldots, k\} : \ \tilde{g}(t_i) > g(t_i)\}$$

and let $I(g) \subseteq \{1, \ldots, k\}$ be the minimal index set such that the polygonal lines through $(t_i, \tilde{g}(t_i)), i = 0, \ldots, k$, and through $(t_i, \tilde{g}(t_i)), i \in I(g) \cup \{0\}$ are equal. Clearly, $I(g)$ exists, is unique and $|I(g)| \le k$. Since we assume that $g(t^*) > 0$ then we have $|I(g)| \ge 1$. For any $\boldsymbol{x} \in \mathbb{R}^k$ we get

$$\boldsymbol{x}_{I(g)}^\top \Sigma_{I(g)}^{-1} \tilde{\boldsymbol{g}}_{I(g)} = \boldsymbol{x}_{I(g)}^\top \Sigma_{I(g)}^{-1} \boldsymbol{g}_{I(g)}.$$

If for some $m \in \{0, 1, \ldots, k\}$ the points $((t_m, g(t_m)), (t_{m+1}, g(t_{m+1})), (t_{m+2}, g(t_{m+2})))$ lie in a line, then

$$\frac{g(t_{m+1}) - g(t_m)}{t_{m+1} - t_m} = \frac{g(t_{m+2}) - g(t_{m+1})}{t_{m+2} - t_{m+1}} = \frac{g(t_{m+2}) - g(t_m)}{t_{m+2} - t_m},$$

hence by the definition of the index set $I(g)$ and (5) we have for any $\boldsymbol{x} \in \mathbb{R}^k$

$$\boldsymbol{x}^\top \Sigma^{-1} \tilde{\boldsymbol{g}} = \boldsymbol{x}_{I(g)}^\top \Sigma_{I(g)}^{-1} \boldsymbol{g}_{I(g)} \tag{6}$$

and consequently for any polygonal line $g \in H_1$ we get

$$|\tilde{g}|^2 = \int_0^1 (\tilde{g}'(s))^2 \, ds = \tilde{\boldsymbol{g}}^\top \Sigma^{-1} \tilde{\boldsymbol{g}} = \boldsymbol{g}_{I(g)}^\top \Sigma_{I(g)}^{-1} \boldsymbol{g}_{I(g)}. \tag{7}$$

Main Result

We consider in the following a piecewise linear continuous boundary functions $b_\gamma = u_\gamma - h_\gamma$ with nodes in $(t_i, b_\gamma(t_i)), i \le k$ with $b_\gamma(0) > 0, \gamma > 0$. As mentioned above $\psi(h_\gamma, u_\gamma, B)$ can be calculated explicitly (see e.g. Janssen and Kunz (2004))

$$\begin{aligned} \psi(h_\gamma, u_\gamma, B) \quad = \quad & \Big[(2\pi)^k \prod_{i=0}^{k-1} (t_{i+1} - t_i)\Big]^{-1/2} \int_{\boldsymbol{x} < \boldsymbol{u}_\gamma - \boldsymbol{h}_\gamma} \exp(-\boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x}/2) \\ & \times \prod_{i=0}^{k-1} \Big[1 - \exp\big(-2(b_\gamma(t_i) - x_i)(b_\gamma(t_{i+1}) - x_{i+1})/(t_{i+1} - t_i)\big)\Big] \, d\boldsymbol{x}, \end{aligned} \tag{8}$$

with $x_0 = 0$. In order to deal with the asymptotic behaviour of the above probability we suppose that

$$\lim_{\gamma \to \infty} |\tilde{h}_\gamma| = \infty, \quad \lim_{\gamma \to \infty} \frac{\tilde{h}_\gamma}{|\tilde{h}_\gamma|} \quad = \quad \tilde{h}, \tag{9}$$

$$\forall i \le k, \lim_{\gamma \to \infty} u_\gamma(t_i) \quad = \quad u(t_i) \in [0, \infty), \tag{10}$$

$$\forall i \le k \quad \lim_{\gamma \to \infty} \frac{\tilde{h}_\gamma(t_i) - h_\gamma(t_i)}{|\tilde{h}_\gamma|} = a_i \in [0, \infty), \lim_{\gamma \to \infty} (h_\gamma(t_i) - \tilde{h}_\gamma(t_i)) \quad = \quad c_i \in [-\infty, 0] \tag{11}$$

are fulfilled. It is not easy to see, form the integral representation, the asymptotic behaviour of $\psi(h_\gamma, u_\gamma, B)$ if $|h_\gamma|$ becomes large and u_γ satisfies (9). In the main theorem below it is shown that the first dominating term in the asymptotic is $\exp(-|\tilde{h}_\gamma|^2/2)$. As noted in Lifshits and Shi (2002) even obtaining that term (in a logarithmic asymptotic) cannot be done by applying directly known results from large deviation theory. We present next the main result:

Theorem 1 *Let $b_\gamma = u_\gamma - h_\gamma, \gamma > 0$ and h, u be given continuous polygonal lines such that $\tilde{h}_\gamma, \tilde{h} \in H_1, \tilde{h} \ne 0$ and $\tilde{h}_\gamma \ne 0, b_\gamma(0) > 0$ hold for any $\gamma > 0$. Assume that b_γ is linear in each interval $[t_i, t_{i+1}], 0 \le i \le k-1$ and for any $\gamma > 0$*

$$I(h_\gamma) = I(h) =: I, \quad K(h_\gamma) = K(h) =: K. \tag{12}$$

If (9),(10),(11) hold and further

$$\lim_{\gamma \to \infty} q_\gamma^{-1} \Bigg[1 - \exp\bigg(-\frac{2b_\gamma(0)}{t_1}\Big(\frac{x_1}{1 + \mathbf{1}(1 \in I)|\tilde{h}_\gamma|} + \tilde{h}_\gamma(t_1) - h_\gamma(t_1)\Big)\bigg)\Bigg] \quad = \quad q(x_1), \ \forall x_1 \ge c_1 \tag{13}$$

is satisfied with $q_\gamma > 0, q(x_1)$ *positive, then as* $\gamma \to \infty$

$$\begin{aligned}
&\psi(h_\gamma, u_\gamma, B)\\
= \;& (1+o(1))Cq_\gamma|\tilde{h}_\gamma|^{-|I\cup IJ\cup JI|-2|II|}\prod_{i\in IK}\Big[\mathbf{1}(a_{i+1}>0)+\frac{2}{|\tilde{h}_\gamma|(t_{i+1}-t_i)}\\
&\times\Big(\mathbf{1}(a_{i+1}=0,c_{i+1}=-\infty)[\tilde{h}_\gamma(t_{i+1})-h_\gamma(t_{i+1})]+\mathbf{1}(a_{i+1}=0,c_{i+1}>-\infty)\Big)\Big]\\
&\times\prod_{i\in KI}\Big[\mathbf{1}(a_i>0)+\frac{2}{|\tilde{h}_\gamma|(t_{i+1}-t_i)}\Big(\mathbf{1}(a_i=0,c_i=-\infty)[\tilde{h}_\gamma(t_i)-h_\gamma(t_i)]\\
&+\mathbf{1}(a_i=0,c_i>-\infty)\Big)\Big]\exp\Big(-\frac{1}{2}|\tilde{h}_\gamma|^2+(\boldsymbol{u}_\gamma)_I^\top\Sigma_I^{-1}\boldsymbol{h}_I\Big), \qquad (14)
\end{aligned}$$

with $J=\{1,\ldots,k\}\setminus(I\cup K)$ *and* C *a positive constant defined by*

$$\begin{aligned}
C \;=\;& \Big[(2\pi)^k\prod_{i=0}^{k-1}(t_{i+1}-t_i)\Big]^{-1/2}\Big[\prod_{i\in II\cup IJ\cup JI}2/(t_{i+1}-t_i)\Big]\\
&\times\int_{\boldsymbol{x}\geq\boldsymbol{c}}\exp\big(-(\boldsymbol{x}^*-\boldsymbol{u})^\top\Sigma^{-1}(\boldsymbol{x}^*-\boldsymbol{u})/2-\boldsymbol{x}_I^\top\Sigma_I^{-1}\boldsymbol{h}_I\big)q(x_1)\\
&\times\prod_{i\in II\cup IJ\cup JI}x_ix_{i+1}\prod_{i\in JJ}\Big[1-\exp(-2x_ix_{i+1}/(t_{i+1}-t_i))\Big]\\
&\times\prod_{i\in IK}\Big[\mathbf{1}(a_{i+1}=0,c_{i+1}=-\infty)x_i+\mathbf{1}(a_{i+1}=0,c_{i+1}>-\infty)x_i(x_{i+1}-c_{i+1})\\
&+\mathbf{1}(a_{i+1}>0)\Big[1-\exp(-2a_{i+1}x_i/(t_{i+1}-t_i))\Big]\Big]\prod_{i\in KI}\Big[\mathbf{1}(a_i=0,c_i=-\infty)x_{i+1}\\
&+\mathbf{1}(a_i=0,c_i>-\infty)x_{i+1}(x_i-c_i)+\mathbf{1}(a_i>0)\Big[1-\exp(-2a_ix_{i+1}/(t_{i+1}-t_i))\Big]\Big]\\
&\times\prod_{i\in KK\cup JK\cup KJ}\Big[1-\exp\big(-2(x_i-c_i)(x_{i+1}-c_{i+1})/(t_{i+1}-t_i)\big)\Big]\,d\boldsymbol{x}, \qquad (15)
\end{aligned}$$

where $\boldsymbol{x}_I^*=(0,\ldots,0)^\top\in\mathbb{R}^{|I|}, \boldsymbol{x}_{J\cup K}^*=\boldsymbol{x}_{J\cup K}, \boldsymbol{c}=(c_1,\ldots,c_k)^\top, c_i\leq 0, i=1,\ldots,k,\ a_i=c_i=0, i\notin K,\ \mathbf{1}(\cdot)$ *is the boolean indicator function, and* $\prod_{i\in\{\phi\}}=:1, \exp(-\infty)=:0$.

Corollary 2 *Let* $h\in H_1$ *be a piecewise linear continuous function with nodes at* $(t_i,h(t_i)), 0\leq i\leq k$ *and* $h(0)=0$. *Then we have*

$$\min_{\boldsymbol{x}\geq\boldsymbol{h}}\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}=\min_{g\geq h,g\in H_1}|g|^2=\tilde{\boldsymbol{h}}^\top\Sigma^{-1}\tilde{\boldsymbol{h}}=\boldsymbol{h}_{I(h)}^\top\Sigma_{I(h)}^{-1}\boldsymbol{h}_{I(h)}=|\tilde{h}|^2. \qquad (16)$$

Remarks: 1) If $h_\gamma=\gamma h$ with $h(0)\leq 0, \tilde{h}\in H_1$ and $h(t^*)>0, t^*\in(0,1]$, then $\tilde{h}_\gamma=\gamma\tilde{h}\in H_1$ with norm $|\tilde{h}_\gamma|=\gamma|\tilde{h}|\to\infty$ as $\gamma\to\infty$. Further $I(h_\gamma)=I(h), K(h_\gamma)=K(h)$ holds for any $\gamma>0$ and

$$a_i=\frac{\tilde{h}(t_i)-h(t_i)}{|\tilde{h}|}>0,\quad c_i=-\infty,\quad \forall i\in K(h),\quad a_i=c_i=0,\quad \forall i\notin K(h).$$

Condition (13) can be checked easily. For instance if $1 \in I$ and $\lim_{\gamma\to\infty} b_\gamma(0)/(\gamma|\tilde{h}|) = a_0 > 0$, then we can take $q_\gamma = 1$, hence $q(x_1) = 1 - \exp(-2a_0 t_1^{-1} x_1)$. If $a_0 = 0$ then we put $q_\gamma = 2b_\gamma(0)/(\gamma|\tilde{h}|t_1)$ which implies $q(x_1) = x_1$.
If further $u_\gamma(t) = d_\gamma u(t), t \in [0,1], \gamma > 0$ with $\lim_{\gamma\to\infty} d_\gamma = d \in [0,\infty)$, then clearly (10) holds.

2) In the case $h_\gamma = \tilde{h}_\gamma \in H_1$ we have $I(h_\gamma) = \{1,\ldots,k\}, |K(h_\gamma)| = 0, |J(h_\gamma)| \geq 0$. If $\tilde{h}_\gamma$ is strictly concave then $|J(h_\gamma)| = 0$.

3) The assumption that the boundary function b_γ is continuous can be easily dropped using the result of Janssen and Kunz (2004).

4) Exact asymptotic expansion for $\boldsymbol{P}\{B(t) + h_\gamma(t) < u_\gamma(t), t \in [0,L]\}$ with $L \in (0,\infty]$ can be shown along the same lines of the proof of the main result above.

5) The asymptotic behaviour of $\psi(h_\gamma, u_\gamma, B_0)$ with h_γ, u_γ piecewise linear functions and B_0 a Brownian bridge can be derived by our main result using further a time transformation that transforms a Brownian bridge to a Brownian motion.
Alternatively the Brownian bridge case can be shown directly using similar arguments as for the Brownian motion. Note that the reproducing kernel Hilbert space connected to B_0 is the subspace of H_1 with functions $h \in H_1 : h(0) = h(1) = 0$. See Bischoff et al. (2003b) for the case $h_\gamma = \gamma h, u_\gamma = u, \gamma > 0$. Note further that the main term in the asymptotic will be determined by the smallest concave majorant of h_γ. (In the Brownian motion case it is the smallest non-decreasing concave majorant $\tilde{h}_\gamma$).

Example 1. Consider h and u continuous functions being further linear on each interval $[t_i, t_{i+1}], i \leq k-1$ such that $h(0) = 0$ and $u(0) > 0$. We discuss briefly the asymptotic behaviour of $\psi(\gamma h, d_\gamma u, B)$ with d_γ positive such that $\lim_{\gamma\to\infty} d_\gamma = d \in [0,\infty)$.

Let $\tilde{h}$ be the smallest concave non-decreasing majorant of h. Clearly, $\tilde{h}$ exists and $\tilde{h} \in H_1$. The constants a_i, c_i and the index sets I, K can be defined as in remark 1) above. Further we can take $q_\gamma = 2d_\gamma u(0)/(t_1\gamma|h|)$ and $q(x_1) = x_1$ for $1 \in I$. We consider for simplicity only the case I, K are non-empty disjoint index sets such that $I \cup K = \{1,\ldots,k\}$. Hence $|J| = |IJ| = |JI| = |JK| = |KJ| = 0$. By the above theorem we get as $\gamma \to \infty$

$$\psi(h_\gamma, d_\gamma u, B) = (1+o(1)) 2C d_\gamma t_1^{-1} u(0) (\gamma|\tilde{h}|)^{-|I|-2|II|-1} \exp\Big(-\frac{\gamma^2}{2}|\tilde{h}|^2 + \gamma d_\gamma \boldsymbol{u}_I^\top \Sigma_I^{-1} \boldsymbol{h}_I\Big),$$

with C a positive constant (see below (17)). If the function u is constant in t, say $u(t) = 1, \forall t \in [0,1]$ then

$$\boldsymbol{u}^\top \Sigma^{-1} \tilde{\boldsymbol{h}} = \boldsymbol{u}_I^\top \Sigma_I^{-1} \boldsymbol{h}_I = \tilde{h}(t_1)/t_1 = \tilde{h}'(0+) \geq 0,$$

consequently

$$\psi(h_\gamma, d_\gamma u, B) = (1+o(1)) 2C d_\gamma t_1^{-1} (\gamma|\tilde{h}|)^{-|I|-2|II|-1} \Big(-\frac{\gamma^2}{2}|\tilde{h}|^2 + \gamma d_\gamma \tilde{h}'(0+)\Big), \quad \gamma \to \infty.$$

It is interesting to note that if $\tilde{h}(t_1) = 0$, which means that $\tilde{h}$ is zero in the segment $[0, t_1]$, and h is non-positive in $[0, t_1]$, then the second asymptotic term above vanishes.

Next, we give an explicit formula for the constant C. In view of (15) we have

$$
\begin{aligned}
C &= \Big[(2\pi)^k \prod_{i=0}^{k-1}(t_{i+1}-t_i)\Big]^{-1/2}\Big[\prod_{i\in II} 2/(t_{i+1}-t_i)^{-1}\Big] \\
&\quad \times \int_{\boldsymbol{x}\geq \boldsymbol{c}} \exp\Big(-(\boldsymbol{x}^*-\boldsymbol{u})^\top \Sigma^{-1}(\boldsymbol{x}^*-\boldsymbol{u})/2 - \boldsymbol{x}_I^\top \Sigma_I^{-1}\boldsymbol{h}_I\Big) x_1 \prod_{i\in II} x_i x_{i+1} \\
&\quad \times \prod_{i\in IK}\Big[1-\exp(-2a_{i+1}x_i/(t_{i+1}-t_i))\Big] \prod_{i\in KI}\Big[1-\exp(-2a_i x_{i+1}/(t_{i+1}-t_i))\Big]\, d\boldsymbol{x} \\
&= (2\pi)^{-|I|/2}|\Sigma_I|^{-1/2}\exp(-d^2\boldsymbol{u}_I^\top\Sigma_I^{-1}\boldsymbol{u}_I/2)\Big[\prod_{i\in II} 2/(t_{i+1}-t_i)^{-1}\Big]\int_{\boldsymbol{x}_I\geq \boldsymbol{0}_I}\exp(-\boldsymbol{x}_I^\top\Sigma_I^{-1}\boldsymbol{h}_I) \\
&\quad \times x_1 \prod_{i\in II} x_i x_{i+1} \prod_{i\in IK}\Big[1-\exp(-2a_{i+1}x_i/(t_{i+1}-t_i))\Big] \\
&\quad \times \prod_{i\in KI}\Big[1-\exp(-2a_i x_{i+1}/(t_{i+1}-t_i))\Big]\, d\boldsymbol{x}_I \\
&= (2\pi)^{-|I|/2}|\Sigma_I|^{-1/2}\exp(-d^2\boldsymbol{u}_I^\top\Sigma_I^{-1}\boldsymbol{u}_I/2)\Big[\prod_{i\in II} 2/(t_{i+1}-t_i)^{-1}\Big] \\
&\quad \times \prod_{i\in \hat{I}II}\int_0^\infty x_i^2\exp(-x_i v_i)\,dx_i \prod_{i\in\hat{I}IK}\int_0^\infty x_i\exp(-x_i v_i)\Big[1-\exp(-2a_{i+1}x_i/(t_{i+1}-t_i))\Big]\,dx_i \\
&\quad \times \prod_{i\in KII}\int_0^\infty x_i\exp(-x_i v_i)\Big[1-\exp(-2a_{i-1}x_i/(t_{i+1}-t_i))\Big]\,dx_i \\
&\quad \times \prod_{i\in KIK}\int_0^\infty \exp(-x_i v_i)\Big[1-\exp(-2a_{i-1}x_i/(t_{i+1}-t_i))\Big] \\
&\quad \times \Big[1-\exp(-2a_{i+1}x_i/(t_{i+1}-t_i))\Big]\,dx_i \\
&= 2^{|II|+|\hat{I}II|+3|KIK|-|I|/2}(\pi^{|I|}|\Sigma_I|)^{-1/2}\exp(-d^2\boldsymbol{u}_I^\top\Sigma_I^{-1}\boldsymbol{u}_I/2)\Big[\prod_{i\in II}(t_{i+1}-t_i)^{-1}\Big] \\
&\quad \times \Big[\prod_{i\in\hat{I}II}\frac{1}{v_i^3}\Big]\Big[\prod_{i\in\hat{I}IK}\Big(\frac{1}{v_i^2}-\frac{(t_{i+1}-t_i)^2}{(2a_{i+1}+(t_{i+1}-t_i)v_i)^2}\Big)\Big] \\
&\quad \Big[\prod_{i\in KII}\Big(\frac{1}{v_i^2}-\frac{(t_{i+1}-t_i)^2}{(2a_{i-1}+(t_{i+1}-t_i)v_i)^2}\Big)\Big]\Big[\prod_{i\in KIK}\frac{a_{i-1}a_{i+1}(a_{i-1}+a_{i+1}+(t_{i+1}-t_i)v_i)}{v_i(2a_{i-1}+(t_{i+1}-t_i)v_i)} \\
&\quad \times \frac{1}{(2a_{i+1}+(t_{i+1}-t_i)v_i)(2a_{i-1}+2a_{i+1}+(t_{i+1}-t_i)v_i)}\Big], \qquad (17)
\end{aligned}
$$

with $v_i = \boldsymbol{e}_i^\top\Sigma_I^{-1}\boldsymbol{h}_I > 0, i\in I$ where $\boldsymbol{e}_i$ is the ith unit vector in $\mathbb{R}^{|I|}$, $\boldsymbol{0}=(0,\ldots,0)^\top\in\mathbb{R}^k$ and

$$\hat{I}II=\{i-1\in\{0\}\cup I, i\in II\},\quad \hat{I}IK=\{i-1\in\{0\}\cup I, i\in IK\},$$

$$KII=\{i-1\in K, i\in II\},\quad KIK=\{i-1\in K, i\in IK\}.$$

Proofs

Proof of Theorem 1. Assume for simplicity that I, J and K are non-empty index sets. By the definition $I \cup J \cup K = \{1, \ldots, k\}$ with $J = \{i \in 1, \ldots k : \tilde{h}_\gamma(t_i) = h_\gamma(t_i), i \notin I\}$. Define in the following $\gamma_i = |\tilde{h}_\gamma|$ if $i \in I$ and $\gamma_i = 1$ otherwise and put $\boldsymbol{x}_\gamma = (x_1/\gamma_1, \ldots, x_k/\gamma_k)^\top$. Using (6) and (7) we get for any $\boldsymbol{x} \in \mathbb{R}^k$

$$\begin{aligned}&(\boldsymbol{x}_\gamma + \tilde{\boldsymbol{h}}_\gamma - \boldsymbol{u}_\gamma)^\top \Sigma^{-1} (\boldsymbol{x}_\gamma + \tilde{\boldsymbol{h}}_\gamma - \boldsymbol{u}_\gamma)\\ &\quad = \quad (\boldsymbol{x}_\gamma - \boldsymbol{u}_\gamma)^\top \Sigma^{-1} (\boldsymbol{x}_\gamma - \boldsymbol{u}_\gamma) + |\tilde{h}_\gamma|^2 + 2\boldsymbol{x}_I^\top \Sigma_I^{-1} (\boldsymbol{h}/|\tilde{h}_\gamma|)_I - 2(\boldsymbol{u}_\gamma)_I^\top \Sigma_I^{-1} \boldsymbol{h}_I.\end{aligned}$$

By the definition of the index set I (recall $\tilde{h}_\gamma(t_i) = h_\gamma(t_i), \forall i \in I$) and the fact that $\tilde{h}_\gamma$ is concave non-decreasing we get $\Sigma_I^{-1}(\tilde{\boldsymbol{h}}_\gamma)_I > \boldsymbol{0}_I$. Furthermore $(\Sigma^{-1}\tilde{\boldsymbol{h}}_\gamma)_i \neq 0, \forall i \in I$ is satisfied, hence Lemma A.1 of Bischoff et al. (2003b) implies

$$\begin{aligned}&(\boldsymbol{x}_\gamma + \tilde{\boldsymbol{h}}_\gamma - \boldsymbol{u}_\gamma)^\top \Sigma^{-1} (\boldsymbol{x}_\gamma + \tilde{\boldsymbol{h}}_\gamma - \boldsymbol{u}_\gamma)\\ &\quad \geq \quad |\tilde{h}_\gamma|^2 - 2(\boldsymbol{u}_\gamma)_I^\top \Sigma_I^{-1} \boldsymbol{h}_I + 2\boldsymbol{x}_I^\top \Sigma_I^{-1} (\boldsymbol{h}/|\tilde{h}_\gamma|)_I + (\boldsymbol{x} - \boldsymbol{u}_\gamma)_{J\cup K}^\top (\Sigma_{J\cup K})^{-1} (\boldsymbol{x} - \boldsymbol{u}_\gamma)_{J\cup K}.\end{aligned}$$

By the assumptions we obtain further as $\gamma \to \infty$

$$\begin{aligned}&(\boldsymbol{x}_\gamma + \tilde{\boldsymbol{h}}_\gamma - \boldsymbol{u}_\gamma)^\top \Sigma^{-1} (\boldsymbol{x}_\gamma + \tilde{\boldsymbol{h}}_\gamma - \boldsymbol{u}_\gamma) - \Big(|\tilde{h}_\gamma|^2 - 2(\boldsymbol{u}_\gamma)_I^\top \Sigma_I^{-1} \boldsymbol{h}_I\Big)\\ &\quad \to \quad 2\boldsymbol{x}_I^\top \Sigma_I^{-1} \boldsymbol{h}_I + (\boldsymbol{x}^* - \boldsymbol{u})^\top \Sigma^{-1} (\boldsymbol{x}^* - \boldsymbol{u}) ,\end{aligned}$$

with $\boldsymbol{x}_I^* = \boldsymbol{0}_I, \boldsymbol{x}_{J\cup K}^* = \boldsymbol{x}_{J\cup K}$ and for any $i \geq 1$ (recall $\delta_i = t_{i+1} - t_i$)

$$\begin{aligned}&1 - \exp\Big(-2\delta_i^{-1}\big(x_i/\gamma_i + \tilde{h}_\gamma(t_i) - h_\gamma(t_i)\big)\big(x_{i+1}/\gamma_{i+1} + \tilde{h}_\gamma(t_{i+1}) - h_\gamma(t_{i+1})\big)\Big)\\ &\quad = \quad (1+o(1)) \begin{cases} 2(\delta_i|\tilde{h}_\gamma|^2)^{-1} x_i x_{i+1}, & i \in II,\\ 2(\delta_i|\tilde{h}_\gamma|)^{-1} x_i x_{i+1}, & i \in IJ \text{ or } i \in JI,\\ 1 - \exp(-2\delta_i^{-1} x_i x_{i+1}), & i \in JJ,\\ 1 - \exp(-2a_{i+1}\delta_i^{-1} x_i), & i \in IK,\ a_{i+1} > 0\\ 2(|\tilde{h}_\gamma|\delta_i)^{-1}(\tilde{h}_\gamma(t_{i+1}) - h_\gamma(t_{i+1}))x_i, & i \in IK,\ a_{i+1} = 0, c_{i+1} = -\infty\\ 2(|\tilde{h}_\gamma|\delta_i)^{-1}(x_{i+1} - c_{i+1})x_i, & i \in IK,\ a_{i+1} = 0, c_{i+1} > -\infty\\ 1 - \exp(-2a_i\delta_i^{-1} x_{i+1}), & i \in KI,\ a_i > 0\\ 2(|\tilde{h}_\gamma|\delta_i)^{-1}(\tilde{h}_\gamma(t_i) - h_\gamma(t_i))x_{i+1}, & i \in KI,\ a_i = 0, c_i = -\infty\\ 2(|\tilde{h}_\gamma|\delta_i)^{-1}(x_i - c_i)x_{i+1}, & i \in KI,\ a_i = 0, c_i > -\infty\\ 1 - \exp\big(-2\delta_i^{-1}(x_i - c_i)(x_{i+1} - c_{i+1})\big), & i \in KK \cup JK \cup KJ.\end{cases}\end{aligned}$$

Note in passing that the terms in the right-hand side above are non-negative for any $x_i \geq c_i, x_{i+1} \geq c_{i+1}$ and $\exp(-\infty) =: 0$. Further, $q(x_1)$ is either bounded by 1 or is linear in x_1 and by the definition of the index set K we have $\boldsymbol{c}_{I\cup J} = \boldsymbol{0}_{I\cup J}$ and $c_k \leq 0, \forall k \in K$. Write for notation simplicity in the following $\tilde{h}_{\gamma,i}, h_{\gamma,i}$ instead of $\tilde{h}_\gamma(t_i), h_\gamma(t_i)$, respectively. In light of (8), changing variables $\boldsymbol{x} \to -\boldsymbol{x}_\gamma - \tilde{\boldsymbol{h}}_\gamma + \boldsymbol{u}_\gamma$ and applying further Lebesgue's Bounded

Convergence Theorem we obtain

$$
\begin{aligned}
&\psi(h_\gamma, u_\gamma, B)\\
&= \Big[(2\pi)^k \prod_{i=0}^{k-1}\delta_i\Big]^{-1/2}\int_{\boldsymbol{x}<\boldsymbol{u}_\gamma-\boldsymbol{h}_\gamma}\exp(-\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}/2)\\
&\quad\times\prod_{i=0}^{k-1}\Big[1-\exp\Big(-2\delta_i^{-1}(b_\gamma(t_i)-x_i)(b_\gamma(t_{i+1})-x_{i+1})\Big)\Big]\,d\boldsymbol{x}\\
&= \Big[(2\pi)^k \prod_{i=0}^{k-1}\delta_i\Big]^{-1/2}|\tilde{h}_\gamma|^{-|I|}\int_{\boldsymbol{x}>\boldsymbol{h}-\tilde{\boldsymbol{h}}_\gamma}\varphi(\boldsymbol{x}_\gamma+\tilde{\boldsymbol{h}}_\gamma-\boldsymbol{u}_\gamma)\\
&\quad\times\Big[1-\exp\Big(-2b_\gamma(0)t_1^{-1}(x_1/\gamma_1+\tilde{h}_{\gamma,1}-h_{\gamma,1})\Big)\Big]\\
&\quad\times\prod_{i=1}^{k-1}\Big[1-\exp\Big(-2\delta_i^{-1}(x_i/\gamma_i+\tilde{h}_{\gamma,i}-h_{\gamma,i})(x_{i+1}/\gamma_{i+1}+\tilde{h}_{\gamma,i+1}-h_{\gamma,i+1})\Big)\Big]\,d\boldsymbol{x}\\
&= \Big[(2\pi)^k \prod_{i=0}^{k-1}\delta_i\Big]^{-1/2}|\tilde{h}_\gamma|^{-|I|}\exp\Big(-|\tilde{h}_\gamma|^2/2+(\boldsymbol{u}_\gamma)_I^\top\Sigma_I^{-1}\boldsymbol{h}_I\Big)\\
&\quad\times\int_{\boldsymbol{x}>\boldsymbol{h}-\tilde{\boldsymbol{h}}_\gamma}\exp\Big(-(\boldsymbol{x}_\gamma-\boldsymbol{u}_\gamma)^\top\Sigma^{-1}(\boldsymbol{x}_\gamma-\boldsymbol{u}_\gamma)/2-\boldsymbol{x}_I^\top\Sigma_I^{-1}(\tilde{\boldsymbol{h}}_\gamma/|\tilde{h}_\gamma|)_I\Big)\\
&\quad\times\Big[1-\exp\Big(-2b_\gamma(0)t_1^{-1}(x_1/\gamma_1+\tilde{h}_{\gamma,1}-h_{\gamma,1})\Big)\Big]\\
&\quad\times\prod_{i=1}^{k-1}\Big[1-\exp\Big(-2\delta_i^{-1}(x_i/\gamma_i+\tilde{h}_{\gamma,i}-h_{\gamma,i})(x_{i+1}/\gamma_{i+1}+\tilde{h}_{\gamma,i+1}-h_{\gamma,i+1})\Big)\Big]\,d\boldsymbol{x}\\
&= (1+o(1))\Big[(2\pi)^k\prod_{i=0}^{k-1}\delta_i\Big]^{-1/2}\Big[\prod_{i\in II\cup IJ\cup JI}2/\delta_i\Big]|\tilde{h}_\gamma|^{-|I\cup IJ\cup JI|-2|II|}\prod_{i\in IK}\Big[\mathbf{1}(a_{i+1}>0)\\
&\quad+2(|\tilde{h}_\gamma|\delta_i)^{-1}\Big(\mathbf{1}(a_{i+1}=0,c_{i+1}=-\infty)(\tilde{h}_{\gamma,i+1}-h_{\gamma,i+1})+\mathbf{1}(a_{i+1}=0,c_{i+1}>-\infty)\Big)\Big]\\
&\quad\times\prod_{i\in KI}\Big[\mathbf{1}(a_i>0)+2(|\tilde{h}_\gamma|\delta_i)^{-1}\Big(\mathbf{1}(a_i=0,c_i=-\infty)(\tilde{h}_{\gamma,i}-h_{\gamma,i})+\mathbf{1}(a_i=0,c_i>-\infty)\Big)\Big]q_\gamma\\
&\quad\times\exp(-|\tilde{h}_\gamma|^2/2+(\boldsymbol{u}_\gamma)_I^\top\Sigma_I^{-1}\boldsymbol{h}_I)\int_{\boldsymbol{x}\geq\boldsymbol{c}}\exp\Big(-(\boldsymbol{x}^*-\boldsymbol{u})^\top\Sigma^{-1}(\boldsymbol{x}^*-\boldsymbol{u})/2-\boldsymbol{x}_I^\top\Sigma_I^{-1}\boldsymbol{h}_I\Big)q(x_1)\\
&\quad\times\prod_{i\in II\cup IJ\cup JI}x_ix_{i+1}\prod_{i\in JJ}\Big[1-\exp(-2\delta_i^{-1}x_ix_{i+1})\Big]\times\prod_{i\in IK}\Big[\mathbf{1}(a_{i+1}=0,c_{i+1}=-\infty)x_i\\
&\quad+\mathbf{1}(a_{i+1}=0,c_{i+1}>-\infty)x_i(x_{i+1}-c_{i+1})+\mathbf{1}(a_{i+1}>0)\Big[1-\exp(-2a_{i+1}\delta_i^{-1}x_i)\Big]\Big]\\
&\quad\times\prod_{i\in KI}\Big[\mathbf{1}(a_i=0,c_i=-\infty)x_{i+1}+\mathbf{1}(a_i=0,c_i>-\infty)x_{i+1}(x_i-c_i)+\mathbf{1}(a_i>0)\\
&\quad\times\Big[1-\exp(-2a_i\delta_i^{-1}x_{i+1})\Big]\Big]\prod_{i\in KK\cup JK\cup KJ}\Big[1-\exp\Big(-2\delta_i^{-1}(x_i-c_i)(x_{i+1}-c_{i+1})\Big)\Big]\,d\boldsymbol{x}.
\end{aligned}
$$

Since $\Sigma_I^{-1}\tilde{\boldsymbol{h}}_I = \Sigma_I^{-1}\boldsymbol{h}_I > \boldsymbol{0}_I$ then $\boldsymbol{x}_I^\top \Sigma_I^{-1}\boldsymbol{h}_I > 0$ holds for any $\boldsymbol{x} \in (0,\infty)^k$. Hence it follows easily that the last integral above is positive and finite, thus the proof is complete. $\square$

Proof of Corollary 2. Following the proof of the above theorem it is easy to see that the dominating term (as $\gamma \to \infty$) of $\psi(\gamma h, 1, B)$ is the same as the leading term of the discrete boundary crossing probability $\boldsymbol{P}\{\max_{1\le i\le k}(B(t_i) + \gamma h(t_i)) < 1\}$. Large deviation theory implies that the leading term of the latter is $\exp(-\frac{\gamma^2}{2}\min_{\boldsymbol{x}\ge\boldsymbol{h},\boldsymbol{x}\in\mathbb{R}^k} \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x})$, hence the proof follows using further (3). $\square$

Acknowledgement: I am extremely thankful to the Referee and Professor Martin Barlow for several corrections and suggestions and to both Professor Enzo Orsingher and Professor Mario Abundo for kindly providing some relevant articles.

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