Reflection of Internal Gravity Waves by Small Density Variations[1]

A. BARCILON, S. BLUMSACK AND J. LAU

Dept. of Meteorology, Dept. of Mathematics and Geophysical Fluid Dynamics Institute, Florida State University, Tallahassee 32306

(Manuscript received 30 July 1971, in revised form 27 September 1971)

ABSTRACT

The propagation of small-amplitude internal gravity waves through a finite layer of varying Brunt-Väisälä frequency is analyzed. A two-scale analysis is used and it is shown that the amount of reflection a wave undergoes is related to the correlation coefficient between the first harmonic of the wave and the variations of the medium. Therefore, with proper care one can extend the usual WKB technique to account for reflection.

1. Introduction

Stommel and Fedorov (1967) and, more recently, Fedorov (1970) describe the complicated spatial fluctuations found below the main thermocline—small inversions, extrema, and homogeneous layers separated by sharp gradients of both temperature and salinity have been observed at various depths. These thin, quasi-stationary homogeneous layers, sometimes called blinis, can extend from 2 to 20 km horizontally and have depths ranging from a few meters to a few hundred meters. Fedorov suggests that these step-like structures are similar to the stratification steps found in the experiments of Turner (1968). On the other hand, Phillips (1966, p. 199) and Orlanski and Bryan (1969) argue that internal gravity waves generate these structures. Whatever their origin, the presence of a step-like structure in the stratification could scatter and reflect internal gravity waves incident on such a "pancake stack" of these homogeneous layers.

Wave propagation in anisotropic inhomogeneous media has been successfully investigated by using WKB techniques. The application of these approximations to the propagation of acoustic gravity waves in the atmosphere has been discussed by Einaudi and Hines (1970). In the context of a linear analysis, the crucial assumption of these methods is that the wavelength of the wave is small compared to the scale length of the inhomogeneities. A physical consequence of this assumption is our inability to describe wave reflection (see Mahoney, 1967).

Although the WKB approximation is adequate when the medium properties are slowly varying, it fails to describe accurately two important classes of problems. The first is the description of wave scattering and dispersion by a region of the medium in which the scale length of the inhomogeneities is comparable to the wavelength of the incident wave, i.e., wave-scale variation. The second is the determination of the reflection of a wave train by a medium whose properties are slowly varying.

In this paper, the properties of the medium change significantly over a distance much larger than a wavelength. Superimposed on this long variation is a small-amplitude wave-scale variation. We shall see that many small wave-scale scatterers can cause significant reflection of a wave.

We can think of several other applications of the theory to be presented. The propagation of small-amplitude shallow water waves onto a region of variable topography is governed by a similar equation, as are a number of other physical problems in wave propagation in a non-uniform medium.

2. Formulation

Consider the propagation of an internal gravity wave in an unbounded, inviscid, stably stratified and incompressible fluid. Define y as the vertical coordinate and x as a horizontal coordinate. We allow variations of the Brunt-Väisälä frequency N with respect to y and assume that the lifetime of the irregularities is long compared to the wave period. We introduce the Boussinesq approximation to simplify the mathematics; if such an assumption were not made, we would have a more complicated "spring stiffness" in our governing one-dimensional Helmholtz equation. If

$$\Psi(x,y,t)=\mathrm{Re}[\psi(y)e^{i(kx-\omega t)}] \tag{2.1}$$

denotes the two-dimensional streamfunction, the equation for the complex amplitude $\psi(y)$ in the linearized Boussinesq approximation is

$$\frac{d^2\psi}{dy^2}+k^2\left[\frac{N^2(y)-\omega^2}{\omega^2}\right]\psi=0, \tag{2.2}$$

[1] Contribution No. 54 of the Geophysical Fluid Dynamics Institute, Florida State University.

where k is the horizontal wavenumber, ω the wave frequency, and N the Brunt-Väisälä frequency based on the basic density structures ρ_0, i.e.,

$$N^2=-g\frac{d}{dy}(\ln\rho_0). \tag{2.3}$$

We assume that $N^2>\omega^2$, $k>0$, and $\omega>0$. We scale the problem as follows. Let L be a characteristic vertical scale of the internal gravity wave and write

$$\left.\begin{aligned} z&=y/L\\ \Lambda^2&=k^2L^2[N^2(y)-\omega^2]/\omega^2\end{aligned}\right\}, \tag{2.4}$$

where, for convenience, we take Λ to be a positive quantity. Eq. (2.2) becomes

$$\frac{d^2\psi}{dz^2}+\Lambda^2\psi=0. \tag{2.5}$$

Suppose Λ has wave-scale variation in a region $0<z<H$, called the scattering region, and is constant outside that region. We wish to find the reflection coefficient of an upward propagating wave which is generated somewhere in the region $z<0$. Let ϵ be a parameter characterizing the rate of change of Λ in the scattering region, i.e., ϵ represents the fractional change of Λ in a distance Λ^{-1}; then,

$$\Lambda^{-2}\frac{d\Lambda}{dy}\approx O(\epsilon). \tag{2.6}$$

In the analysis that follows, we take $\epsilon\ll 1$ and use asymptotic methods to determine the solution.

A simple change of variable leads to a useful form of (2.5). Let ψ be a function of the phase ζ, where

$$\zeta=\int_0^z \Lambda(z')dz'.$$

Then

$$\frac{d^2\psi}{d\zeta^2}+\left(\Lambda^{-2}\frac{d\Lambda}{dz}\right)\frac{d\psi}{d\zeta}+\psi=0. \tag{2.7}$$

In (2.7) the spring stiffness is normalized to unity when we use the phase variable ζ; the use of such a variable is dictated by the slow change in the spring stiffness with position.

3. Two-scale method

Consider $\Lambda(z)$ having the functional form

$$\Lambda(z)=\Lambda_0(\epsilon z)+\epsilon\Lambda_1(\zeta,\epsilon z). \tag{3.1}$$

The first term represents smooth long-scale variations in the Brunt-Väisälä frequency, while the second represents the small-amplitude, wave-scale perturbations; notice that the perturbations are modulated on the long-scale variations. In the ocean, $\Lambda(z)$ is the quantity that is measured and one could form Λ_0 and Λ_1 in a variety of ways. It will be convenient here to choose Λ_1 to be periodic when ζ varies by 2π.

We treat ζ and $Z=\epsilon z$ as independent variables and expand $\psi(\zeta,Z;\epsilon)$ as

$$\psi(\zeta,Z;\epsilon)=F_0(\zeta,Z)+\epsilon F_1(\zeta,Z)+\ldots. \tag{3.2}$$

Substituting (3.2) into (2.7), we find

$$\frac{\partial^2F_0}{\partial\zeta^2}+F_0=0, \tag{3.3}$$

$$\frac{\partial^2F_1}{\partial\zeta^2}+F_1=-G_1, \tag{3.4}$$

where G_1 contains the forcing of F_1 by F_0, namely

$$G_1=\left(\frac{2}{\Lambda_0}\right)\frac{\partial^2F_0}{\partial\zeta\partial Z}+\Lambda_0^{-2}\left(\frac{d\Lambda_0}{dZ}+\frac{\partial\Lambda_1}{\partial\zeta}\right)\frac{\partial F_0}{\partial\zeta}. \tag{3.5}$$

The solution to (3.3) is

$$F_0(\zeta,Z)=A_0(Z)e^{i\zeta}+B_0(Z)e^{-i\zeta}, \tag{3.6}$$

where A_0, B_0 are the amplitudes of the waves having group velocities pointing in the downward and upward directions, respectively. When (3.6) and (3.5) are substituted into (3.4), terms proportional to $\exp[\pm i\zeta]$ appear on the right-hand side of (3.4). We remove the possible secularities by requiring that the forcing G_1 be orthogonal to the complementary solutions $\exp[\pm i\zeta]$, i.e.,

$$\int_0^{2\pi} d\zeta G_1(\zeta,Z)e^{\pm i\zeta}=0. \tag{3.7}$$

This condition yields two coupled equations for A_0 and B_0:

$$\frac{dA_0}{dZ}+\left(\frac{\Lambda_0'}{2\Lambda_0}\right)A_0=Q^*B_0, \tag{3.8}$$

$$\frac{dB_0}{dZ}+\left(\frac{\Lambda_0'}{2\Lambda_0}\right)B_0=QA_0, \tag{3.9}$$

where

$$Q(Z)=\frac{1}{4\pi\Lambda_0}\int_0^{2\pi} d\zeta\frac{\partial\Lambda_1}{\partial\zeta}(\zeta,Z)e^{2i\zeta}, \tag{3.10}$$

provided we choose Λ_1 such that $\Lambda_1(0,Z)=\Lambda_1(2\pi,Z)$.

In the usual WKB approach, $\Lambda_1\equiv 0$ and Eqs. (3.8) and (3.9) are uncoupled; the amplitudes A_0, B_0 change in such a way as to conserve $A_0\Lambda_0^{\frac{1}{2}}$ and $B_0\Lambda_0^{\frac{1}{2}}$. We write

$$a_0=A_0\Lambda_0^{\frac{1}{2}},\quad b_0=B_0\Lambda_0^{\frac{1}{2}}.$$

Then

$$\frac{da_0}{dZ}=q^*b_0, \tag{3.11}$$

$$\frac{db_0}{dZ}=qa_0, \tag{3.12}$$

with

$$q(Z)=\frac{1}{8\pi i\Lambda_0}\int_0^{2\pi} d\zeta\Lambda_1(\zeta,Z)e^{2i\zeta}, \tag{3.13}$$

representing the long-scale changes of the WKB invariants a_0, b_0. We see that reflection depends on there being a correlation between the small-scale structure Λ_1 and the first wave harmonic.

The solution of (3.11) and (3.12) requires the specification of boundary conditions. Suppose we have a wave incident at $Z=0$ with a known amplitude $b_0(0)$. The radiation condition at $Z=\epsilon H$ requires that $a_0(\epsilon H)=0$. Given Λ_1 we could calculate $q(Z)$ and therefore $a_0(Z)$, $b_0(Z)$.

As an illustration, suppose q is constant. It is a simple matter to solve (3.11) and, (3.12) for the reflection coefficient

$$\left|\frac{a_0(0)}{b_0(0)}\right|=\tanh(\epsilon H|q|). \tag{3.14}$$

The correlation coefficient between the *first harmonic* and the small oscillating irregularities provides the coupling between a_0 and b_0, i.e., it is a measure of the amount of reflection. If we write Λ^2 as

$$\Lambda^2=\delta+\epsilon\cos(2\sqrt{\delta}z),\quad \delta=\text{constant},$$

our governing equation (2.5) reduces to Mathieu's equation. That equation was discussed in the context of wave propagation by Brillouin (1953) and Benjamin (1968). If we consider the (δ,ϵ) plane (see, for example, Cole, 1968), we find that a wedge centered at $\delta=1$ represents attenuation, i.e., the solutions decay with z. The problem being inviscid, the attenuation is not dissipative; rather, it expresses the fact that the incoming wave amplitude decreases at the expense of the reflected wave. Therefore, in the Mathieu equation formulation the wedge near $\delta=1$ denotes a region in which reflection is important. From the classical results pertaining to the solution of Mathieu's equation, we can represent the bounding lines for the first few wedges in the (δ,ϵ) plane for which reflection is important as

$$\delta=\begin{cases}-2\epsilon^2, & \text{wedge near } \delta=0\\ 1\mp 2\epsilon, & \text{wedge near } \delta=1\\ 4-\epsilon^2/3,\ 4+5\epsilon^2/3, & \text{wedge near } \delta=4.\end{cases}$$

For infinitesimal ϵ, only the wedge near $\delta=1$ is important, all others having higher order contacts at their apex.

Finally, wave trapping can occur if a wave of frequency close but smaller than the local Brunt-Väisälä frequency is generated inside the scattering region. For such a wave δ is negative and small and, if $\delta<-2\epsilon^2$, the wave considered could propagate unattenuated in the scattering region. If N^2 is equal to the same constant for $z<0$, $z>H$ and if $\delta<0$, that wave cannot propagate outside the scattering region and the scattering layer then acts as a waveguide.

4. Numerical method

The principal assumption made in the preceeding section concerns the form of Λ_1 in Eq. (3.1). The two-scale method tells us to decompose Λ_1 into its Fourier components in each region where the phase changes by 2π. We have tacitly assumed that the Fourier coefficients are functions of Z, i.e., the correlation coefficients do not change appreciably from one cycle of the wave to the next. There are functions Λ_1 that do not satisfy this property.

However, for a stepwise varying Λ in $0<z<H$, Eq. (2.5) can be solved exactly within each step. The solution in the ith step with $\Lambda=\Lambda_i$ can be matched to that of the $(i-1)$th step, and can also provide initial conditions for the $(i+1)$th step. Hence, the solution at any step can be related to the incident and reflected components in the region $z<0$ (denoted by subscript 1) by

$$\begin{pmatrix}A_i\\ B_i\end{pmatrix}=\left(\prod_{j=1}^{i-1}\mathbf{T}^{(j)}\right)\begin{pmatrix}A_1\\ B_1\end{pmatrix}=\begin{pmatrix}\tau_{11}^{(i)} & \tau_{12}^{(i)}\\ \tau_{21}^{(i)} & \tau_{22}^{(i)}\end{pmatrix}\begin{pmatrix}A_1\\ B_1\end{pmatrix}, \tag{4.1}$$

where

$$\mathbf{T}^{(j)}=\frac{1}{2}\begin{Bmatrix}\left(1+\dfrac{\Lambda_j}{\Lambda_{j+1}}\right)\exp[i\Lambda_j(z_j-z_{j-1})] & \left(1-\dfrac{\Lambda_j}{\Lambda_{j+1}}\right)\exp[-i\Lambda_j(z_j-z_{j-1})]\\ \left(1-\dfrac{\Lambda_j}{\Lambda_{j+1}}\right)\exp[i\Lambda_j(z_j-z_{j-1})] & \left(1+\dfrac{\Lambda_j}{\Lambda_{j+1}}\right)\exp[-i\Lambda_j(z_j-z_{j-1})]\end{Bmatrix}.$$

Here $z_0=z_1=0$. We can approximate any wave-scale variation in $0<z<H$ by a stepwise varying function by dividing the scattering region into a large number M of small steps. Since only the transmitted waves can be observed at the Mth step, the reflection coefficient is found by setting $A_M=\tau_{11}^{(M)}A_1+\tau_{12}^{(M)}B_1$ equal to

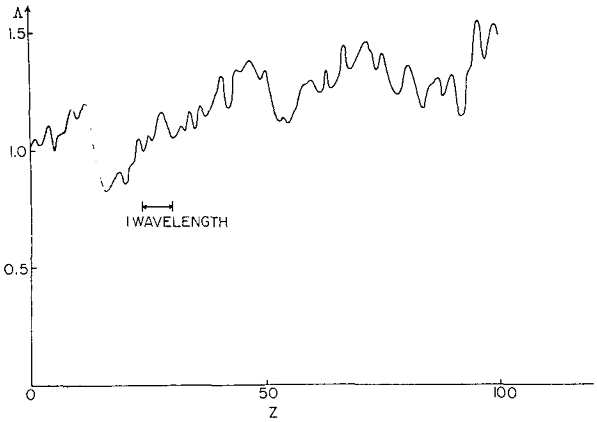

FIG. 1. An example of a pseudo-random "continuous" Λ. The associated reflection coefficient is 0.23.

zero, i.e.,

$$\left|\frac{A_1}{B_1}\right| = \left|\frac{\tau_{12}^{(M)}}{\tau_{11}^{(M)}}\right|. \tag{4.2}$$

When Λ satisfies the restrictions of the previous section, the results obtained from (3.14) and (4.2) are in good agreement.

Next, we generate a pseudo-random continuous function for Λ as follows. Suppose $\Lambda \approx O(1)$ and divide the interval $[0,H]$ into many small intervals of width h; typically, we take $h=0.1$ so that there are over 60 sub-intervals per wavelength. The function Λ is defined in each interval by means of a random walk. At $z=0$ we take $\Lambda=1$. The change in Λ from one interval to the next is $\epsilon h^{\frac{1}{2}}R$, where R is a pseudo-random number uniformly distributed between -1 and $+1$; notice that the change in Λ over one wavelength is $O(\epsilon)$ since there are about $2\pi/h$ steps in our random walk over this distance. The incident wave sees a continuous Λ such as that shown in Fig. 1. The reflection coefficient computed by using (4.2) is 0.23. As expected, this is considerably smaller than we would obtain from (3.14) using $\epsilon=0.1$, $H=100$, $q=1$. Although the generation of such a Λ does not pretend to model the actual Brunt-Väisälä frequency found in the oceans, this example contains many features of the actual step-like structure found by Fedorov (1970; see Fig. 2 therein).

5. Summary

If an internal gravity wave impinges onto a region containing wave-scale irregularities superimposed upon large-scale variations of the Brunt-Väisälä frequency, one could obtain significant reflection if that region is of considerable extent. Care must be exercised when using the WKB approximation in wave propagation problems. One should estimate the correlation coefficient between the wave's first harmonic and the medium and, if small, the "usual" WKB technique can be used. Otherwise, reflection must be accounted for, say, in the manner described here.

Acknowledgments. This work was supported in part at Florida State University by the Office of Naval Research under Contract N-00014-68-A-0159. Two of the authors, A. Barcilon and J. Lau, conducted part of this research at the Naval Research Laboratory in Washington, D. C.

REFERENCES

Benjamin, T. B., 1968: Comments on Dr. Phillips' paper: "Internal wave interactions." *Sixth Symp. Naval Hydrogr.*, Office of Naval Research, Washington, D. C.

Brillouin, L., 1953: *Wave Propagation in Periodic Structures.* New York, Dover Publ., 255 pp.

Cole, J. D., 1968: *Perturbation Methods in Applied Mathematics.* New York, Blaisdell Publ., p. 94.

Einaudi, F. and C. O. Hines, 1970: WKB approximation in the application to acoustic-gravity waves. *Can. J. Phys.*, **48**, 1458–1471.

Fedorov, K. N., 1970: On the step-like structure of temperature inversions in the ocean. *Izv. Atmos. Oceanic Phys.*, **6**, 1178–1188.

Mahoney, J. J., 1967: The reflection of short waves in a variable medium. *Quart. Appl. Math*, **25**, 313–316.

Orlanski, I., and K. Bryan, 1969: Formation of the thermocline step structure by large-amplitude internal gravity waves. *J. Geophys. Res.*, **74**, 6975–6983.

Phillips, O. M., 1966: *The Dynamics of the Upper Ocean.* Cambridge University Press, p. 199.

Stommel, H., and K. N. Fedorov, 1967: Small scale structure in temperature and salinity near Timor and Mindanao. *Tellus*, **19**, 306–325.

Turner, J. S., 1968: The behavior of a stable salinity gradient heated from below. *J. Fluid. Mech.*, **33**, 183–200.