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A NOTE ON OCCUPATION TIMES OF STATIONARY PROCESSES

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Abstract

Consider a real valued stationary process $X = \{X_s : s \in \mathbb{R}\}$. For a fixed $t \in \mathbb{R}$ and a set D in the state space of X, let g_t and d_t denote the starting and the ending time, respectively, of an excursion from and to D (straddling t). Introduce also the occupation times I_t^+ and I_t^- above and below, respectively, the observed level at time t during such an excursion. In this note we show that the pairs (I_t^+, I_t^-) and $(t - g_t, d_t - t)$ are identically distributed. This somewhat curious property is, in fact, seen to be a fairly simple consequence of the known general uniform sojourn law which implies that conditionally on $I_t^+ + I_t^- = v$ the variable I_t^+ (and also I_t^-) is uniformly distributed on $(0, v)$. We also particularize to the stationary diffusion case and show, e.g., that the distribution of $I_t^- + I_t^+$ is a mixture of gamma distributions.

1 Introduction

Let $X = \{X_s : s \in \mathbb{R}\}$ be a stationary measurable process with the range $E \subset \mathbb{R}$. For a given $D \subset E$ let

$$M := \{s \in \mathbb{R} : X_s \in D\}^{cl}, \tag{1}$$

where cl means the closure of the set in the braces. Next define for fixed $t \in \mathbb{R}$

$$g_t := \sup\{s \le t : s \in M\}, \quad d_t := \inf\{s > t : s \in M\}, \tag{2}$$

and

$$I_t^+ := \int_{g_t}^{d_t} \mathbf{1}_{\{X_s > X_t\}}\, ds, \quad I_t^- := \int_{g_t}^{d_t} \mathbf{1}_{\{X_s < X_t\}}\, ds. \tag{3}$$

The main result of this note is

Theorem 1. *Let X be as above with the property*

$$\mathrm{Leb}\{s:\, X_s = X_0\} = 0 \quad \text{a.s.}$$

where Leb *stands for the Lebesgue measure. Then*

$$(I_t^+, I_t^-) \stackrel{d}{=} (t - g_t, d_t - t), \tag{4}$$

where $\stackrel{d}{=}$ means "is identical in law with". Moreover, conditioned on $V := I_t^+ + I_t^- = d_t - g_t$ the random variables $I_t^+, I_t^-, t - g_t$, and $d_t - t$ are identically distributed the common distribution being the uniform distribution on $(0, V)$.

The property (4) was observed in [16] to be valid for reflected Brownian motion on $\mathbb{R}_+$ with negative drift, RBM$^{\downarrow}$, for short, and for stationary excursions from 0 to 0. Later the authors of this note found (4) to be valid for all positively recurrent linear diffusions under smoothness assumptions on the scale function and the speed measure. Jim Pitman pointed out to us then the full generality (as stated in Theorem 1) of the result and remarked that it is a consequence of the results in [12] and [4]. However, because we have not found the identity (4) in the literature, we feel that it is worthwhile to discuss briefly this interesting but not widely known equality in law. The diffusion case is also very appealing with nice explicit formulae.
Clearly, from Theorem 1, it follws that

$$I_t^+ \stackrel{d}{=} I_t^-. \tag{5}$$

In the case X is a RBM$^{\downarrow}$ and stationary excursions from 0 to 0 are considered one would expect that the occupation time below the observed level is bigger (in some sense) than the time above, but the randomness of the level "balances" the random variables so that (5) holds. We refer also to [9], where (5) for a RBM$^{\downarrow}$ is shown to be a consequence of reversibility in space of the excursions.
The paper is organised so that in the next section we prove Theorem 1. The proof worked out from Pitman's remark relies on some results from [12] and [4] which are first recalled. In Section 3 we present an alternative proof of Theorem 1 in the case when X is a linear diffusion. The main tool in this proof is the Feynman-Kac formula. The common distribution of (I_t^+, I_t^-) and $(t - g_t, d_t - t)$ is also characterized via the Lévy measure of the inverse local time at the point where the excursions start and end. Applying Krein's spectral theory of strings the distribution of V (which determines the joint distribution of (I_t^+, I_t^-)) is shown to be a mixture of gamma distributions.

2 General case

2.1 On the distributions of $-g_0$ and d_0

Let $X = \{X_s : s \in \mathbb{R}\}$ be an arbitrary stationary process taking values in $E \subset \mathbb{R}$. It is assumed that the sample paths of X are right continuous and have left limits (cadlag). We consider X in the canonical space $(\Omega, \mathcal{F})$ of cadlag functions. Let $\{\theta_s : s \in \mathbb{R}\}$ denote the usual shift operator in this framework. For a set $D \subset E$ and $t = 0$ define M, d_0 and g_0 as in (1) and (2). Moreover, set

$$L := \{s:\, d_{s-} = 0\,,\, d_s > 0\}.$$

We now collect, following [12] (where, in fact, even more general case is considered), some formulae concerning the distributions of g_0 and $V := d_0 - g_0$. The crucial concept hereby is the Palm measure.

Definition 2. *The Palm measure* $\mathbf{Q}$ *associated with* X *is defined by*

$$\mathbf{Q}(B) := \mathbf{E}\left(| \{ s : 0 < s < 1, s \in L, \theta_s \in B \} | \right), \quad B \in \mathcal{F},$$

where $|\cdot|$ *denotes the number of points of the set in the braces.*

Proposition 3. *For a measurable function* $f : \mathbb{R} \times \Omega \to [0, \infty)$

$$\mathbf{E}\big(f(\theta_{g_0}, -g_0)\mathbf{1}_{\{-\infty < g_0 < 0\}}\big) = \int_\Omega \mathbf{Q}(d\omega) \int_0^{d_0} f(t, \omega)\, dt. \tag{6}$$

In particular,

$$\mathbf{P}(-\infty < g_0 < 0, \theta_{g_0} \in d\omega) = \mathbf{Q}(d\omega)\, d_0(\omega), \tag{7}$$

$$\mathbf{P}(-g_0 \in da) = \mathbf{Q}(d_0 > a)\, da, \quad a > 0. \tag{8}$$

Moreover,

$$\mathbf{P}(V \in dv) = v\, \mathbf{Q}(d_0 \in dv), \tag{9}$$

and conditionally on the paths $\{X_{g_0+s} : s \geq 0\}$ *the distribution of* $-g_0$ *depends only on* V *and is the uniform distribution on* $(0, V)$.

Proof. See [12] Theorem p. 290 and Corollary p. 298. □

Remark 4. (i) In [12] it is also proved that the Palm measure is a multiple of the Itô excursion law. Comparing formulae (8) and (9) with (16) and (17) in Proposition 8 gives an indication for this fact (in the diffusion case).
(ii) Proposition 3 yields also easily (cf. (18))

$$\mathbf{P}(-g_0 \in da\,,\, d_0 \in db) = da\, \pi(a, db),$$

where the measure π is characterized via

$$\pi(a, B) = \mu(a + B), \quad a + B := \{a + b : b \in B\}$$

with B a Borel set on $\mathbb{R}_+$ and $\mu(dv) := \mathbf{P}(V \in dv)$.

2.2 Occupation times for cyclically stationary processes

We consider now a cyclically stationary measurable process on finite time interval and its sojourn times above and below the initial level. Cyclically stationarity hereby means roughly that the periodic extension of the process is stationary.

Definition 5. *The measurable process* $\{X_t : 0 \leq t < l\}$*, where* $l > 0$ *is fixed, is called cyclically stationary if the process* $\{Y_t := X_{t|l} : t \in \mathbb{R}\}$*, where* $t|l$ *means* t *modulo* l*, is stationary in the usual sense, i.e., for any* $s \in \mathbb{R}$ *the processes* $\{Y_t\}$ *and* $\{Y_{s+t}\}$ *are identical in law.*

The important property of cyclically stationary processes needed in the proof of Theorem 1 is given in [4] Theorem 3.1. For the convenience of the reader we state and prove this result in the form directly applicable for our purpose; however, following closely [4].

Proposition 6. *Let $X = \{X_t : 0 \le t < 1\}$ be a measurable cyclically stationary process such that*

$$\mathrm{Leb}\{t : X_t = X_0\} = 0 \quad \text{a.s.} \tag{10}$$

Then the occupation times

$$\int_0^1 \mathbf{1}_{\{X_t \le X_0\}}\, dt \quad \textit{and} \quad \int_0^1 \mathbf{1}_{\{X_t \ge X_0\}}\, dt$$

are uniformly on $(0,1)$ distributed random variables.

Proof. To start with, recall Tucker's extension of the Glivenko-Cantelli theorem (see [18]): if $Z = \{Z_n\}$ is a stationary sequence of random variables and $\mathcal{I}^Z$ is the invariant σ-field determined by Z (for this concept see, e.g., [5]) then a.s.

$$\sup_x \left| \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{Z_i \le x\}} - \mathbf{P}(Z_1 \le x \,|\, \mathcal{I}^Z) \right| \to 0 \quad \text{as } n \to \infty\,. \tag{11}$$

Let Y be the stationary process obtained by a periodic continuation of X as introduced in Definition 5 and let $\mathcal{I}^Y$ be the invariant σ-field of Y. Then for all n the sequence $\{Z_k^{(n)} := Y_{\frac{k}{2^n}}\}$ is stationary and we have for all x and positive integers m a.s.

$$\int_0^1 \mathbf{1}_{\{X_s \le x\}}\, ds = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} \mathbf{1}_{\{Z_k^{(n)} \le x\}} = \lim_{n\to\infty} \frac{1}{m\,2^n} \sum_{k=0}^{(2^n-1)m} \mathbf{1}_{\{Z_k^{(n)} \le x\}}.$$

Notice that by the measurability assumption the integral above is well defined. From (11) it follows that a.s.

$$\frac{1}{2^n} \sum_{k=0}^{2^n-1} \mathbf{1}_{\{Z_k^{(n)} \le x\}} = \lim_{m\to\infty} \frac{1}{m\,2^n} \sum_{k=0}^{(2^n-1)m} \mathbf{1}_{\{Z_k^{(n)} \le x\}} = \mathbf{P}(Y_0 \le x \,|\, \mathcal{I}^n),$$

where $\mathcal{I}^n$ is the invariant σ-algebra determined by $Z^{(n)}$. By the martingale convergence theorem, since $\mathcal{I}^Y = \sigma\{\mathcal{I}^1, \mathcal{I}^2, \dots\}$, we have a.s.

$$\begin{aligned} \int_0^1 \mathbf{1}_{\{X_s \le x\}}\, ds &= \lim_{n\to\infty} \mathbf{P}(Y_0 \le x \,|\, \mathcal{I}^n) \\ &= \lim_{n\to\infty} \mathbf{E}(\mathbf{P}(Y_0 \le x \,|\, \mathcal{I}^Y) \,|\, \mathcal{I}^n) \\ &= \mathbf{P}(Y_0 \le x \,|\, \mathcal{I}^Y). \end{aligned} \tag{12}$$

Next define for any Borel set B

$$\eta(B) := \mathbf{P}(Y_0 \in B \,|\, \mathcal{I}^Y).$$

From (12) it follows that a.s.

$$\int_0^1 \mathbf{1}_{\{X_s \le X_0\}}\, ds = \eta((-\infty, Y_0]),$$

and, from the assumption (10), $x \mapsto \eta((-\infty, x])$ is continuous. Using the tower property we have a.s.

$$\mathbf{P}(Y_0 \in B \,|\, \eta) = \mathbf{E}(\mathbf{P}(Y_0 \in B \,|\, \mathcal{F}_Y) \,|\, \eta) = \mathbf{E}(\eta(B) \,|\, \eta) = \eta(B)$$

showing that η is the regular version of $\mathbf{P}(Y_0 \in \cdot \,|\, \eta)$. Therefore, by the continuity of η, it holds that $\eta((-\infty, Y_0])$ is uniformly distributed on $(0,1)$, as claimed. □

We have the following surprisingly general corollary covering, e.g., all excursion and other bridges.

Corollary 7. *Let* $Z = \{Z_t : 0 \le t < l\}$ *be a measurable process and* U *uniformly on* $(0,l)$ *distributed random variable independent of* Z*. Assume that*

$$\mathrm{Leb}\{t : Z_t = Z_U\} = 0 \quad \text{a.s.}$$

Then the occupation times

$$\int_0^l \mathbf{1}_{\{Z_t < Z_U\}}\, dt \quad \textit{and} \quad \int_0^l \mathbf{1}_{\{Z_t > Z_U\}}\, dt$$

are uniformly distributed on $(0,l)$.

Proof. For all $s \in [0,l]$, the random variable $U'(s) := U + s$ modulo l is also uniformly distributed on $(0,l)$, and, thus, $Y = \{Y_t : 0 \le t < l\}$, where $Y_t := Z_{U'(t)}$, is cyclically stationary. We have

$$\begin{aligned} \int_0^l \mathbf{1}_{\{Z_t < Z_U\}}\, dt &= \int_0^l \mathbf{1}_{\{Z_{U'(t)} < Z_U\}}\, dt \\ &= \int_0^l \mathbf{1}_{\{Y_t < Y_0\}}\, dt. \end{aligned}$$

Consequently, the claim follows from Proposition 6. □

2.3 Proof of Theorem 1

Let $\{X_s : s \in \mathbb{R}\}$ be a measurable stationary process as defined in Section 15. We consider the case $t = 0$. Because

$$I_0^+ + I_0^- = d_0 - g_0 =: V$$

it is enough to show that, e.g., the conditional distributions of I_0^- and d_0 given V coincide. From Proposition 3 we know that d_0 given V is uniformly distributed on $(0,V)$. To prove that this is also the case for I_0^- define for $0 \le t < V$

$$Z_t := X_{g_0+t}$$

and consider

$$I_0^- := \int_{g_0}^{d_0} \mathbf{1}_{\{X_s < X_0\}}\, ds = \int_0^V \mathbf{1}_{\{Z_t < Z_{-g_0}\}}\, dt.$$

By Proposition 3, given V the random variable $-g_0$ is uniformly distributed on $(0,V)$ but otherwise independent of Z. Consequently, combining this with the result in Corollary 7 concludes the proof. □

3 Diffusion case

3.1 Proof of Theorem 1 via the Feynman-Kac formula

We prove Theorem 1 for a stationary diffusion $X = \{X_s : s \in \mathbb{R}\}$ living in an interval $[0, r)$ or $[0, r]$ where 0 is a reflecting boundary and in the case of the half open interval r is either natural or entrance-not-exit and in the other case r is reflecting. It is also assumed that $D = \{0\}$ in (1), i.e.,

$$M = \{t : X_t = 0\}.$$

The cases when the state space of X is the whole $\mathbb{R}$ or D is an interval can be treated similarly. The generator of X is denoted by

$$\mathcal{G} = \frac{d}{dm}\frac{d}{dS}\ ,$$

where S is the scale function and m is the speed measure. We assume that

$$m(dx) = m(x)\, dx \quad \text{and} \quad S(x) = \int_0^x S'(y)\, dy$$

with continuous $m(x)$ and $S'(x)$. Recall that the stationary distribution of X is given by

$$\mu(dx) := m(dx)/m(E)$$

with $m(E) < \infty$. Fix $y \in E$ and introduce

$$u(x) := \mathbf{E}_x \left(\exp\Big(-\alpha \int_0^{H_0} \mathbf{1}_{\{0 \le X_s \le y\}} ds - \beta \int_0^{H_0} \mathbf{1}_{\{X_s > y\}} ds \Big) \right),$$

where $\mathbf{E}_x$ denote the expectation associated with X given that $X_0 = x$ and

$$H_0 := \inf\{t > 0 :\ X_t = 0\}.$$

From the Feynman-Kac formula it follows that $u(x)$, $x > 0$, is the unique bounded smooth solution of the generalized differential equation

$$\mathcal{G}u(x) = \begin{cases} \alpha\, u(x), & 0 < x < y, \\ \beta\, u(x), & x > y \end{cases}$$

satisfying the condition $u(0) = 1$. For $x = y$ we have

$$u(y) = \frac{\psi_\alpha^+(0)\, \varphi_\beta(y)}{\psi_\alpha^+(y)\, \varphi_\beta(y) - \psi_\alpha(y)\, \varphi_\beta^+(y)}\ ,$$

where ψ_α and φ_α are the increasing and the decreasing fundamental solution, respectively, of the equation

$$\mathcal{G}u(x) = \alpha u(x), \quad x > 0. \tag{13}$$

For ψ_α the killing condition $\psi_\alpha(0+) = 0$ must be imposed. The notation φ_β^+, for instance, means the derivative with respect to the scale function. Next noting that

$$\frac{d}{dm} r(y) := \frac{d}{dm}\big(\psi_\alpha^+(y)\, \varphi_\beta(y) - \psi_\alpha(y)\, \varphi_\beta^+(y)\big) = (\alpha - \beta)\psi_\alpha(y)\, \varphi_\beta(y)$$

and using the time reversibility of stationary diffusions we have

$$\begin{aligned}
&\mathbf{E}\left(\exp(-\alpha I_t^- - \beta I_t^+)\right)\\
&\quad= \int_E \mathbf{E}\left(\exp(-\alpha I_t^- - \beta I_t^+) \,|\, X_t = y\right) \mathbf{P}(X_t \in dy)\\
&\quad= \int_E \left(u(y)\right)^2 \mu(dy)\\
&\quad= \frac{\left(\psi_\alpha^+(0)\right)^2}{\alpha-\beta} \int_E \mu(dy)\, \frac{\varphi_\beta(y)}{\psi_\alpha(y)}\, \frac{d}{dm}\left(-\frac{1}{r(y)}\right)\\
&\quad= \frac{1}{m(E)\,(\alpha-\beta)} \left(\frac{\varphi_\beta^+(0)}{\varphi_\beta(0)} - \frac{\varphi_\alpha^+(0)}{\varphi_\alpha(0)}\right)\\
&\quad= \frac{1}{m(E)\,(\alpha-\beta)} \left(\frac{1}{G_\alpha(0,0)} - \frac{1}{G_\beta(0,0)}\right), \qquad (14)
\end{aligned}$$

where, in the next to the last step, we have integrated by parts and $G_\alpha(0,0)$ denotes the Green kernel at $(0,0)$ for X (for more information about Green kernels see [1]).
It is seen in the similar way or by using the Chapman-Kolmogorov equation (see [8] Proposition 3.4) that the joint Laplace transform of $(t-g_t, d_t-t)$ is also given by the right-hand side of (14).
From the special form (14) of the Laplace transform of (I_t^+, I_t^-) it follows that

$$(I_t^+, I_t^-) \stackrel{d}{=} (UV, (1-U)V),$$

where $V = I_t^+ + I_t^-$ and U is a uniformly on $(0,1)$ distributed random variable independent of V (see [8] Proposition 5.7) proving the latter statement of the Theorem.

3.2 Density of $(-g_0, d_0)$ in terms of a Lévy measure

Let X and M be as in Section 3.1. Clearly, the distribution of $(t-g_t, d_t-t)$ (and of (I_t^+, I_t^-)) does not depend on t; therefore, to simplify the notation, we take $t=0$. Let $A = \{A_s : s \geq 0\}$ be the right-continuous inverse of the local time of $\{X_s : s \geq 0\}$ at 0 (taken with respect to the speed measure). As is well known, A is a subordinator and under the assumption $X_0 = 0$

$$\begin{aligned}
\mathbf{E}_0\left(\exp(-\alpha A_s)\right) &= \exp\left(-s \int_0^\infty \left(1-e^{-\alpha t}\right) n^+(dt)\right)\\
&= \exp\left(-s \int_0^\infty \alpha\, e^{-\alpha t} n^+(t,\infty)\, dt\right), \qquad (15)
\end{aligned}$$

where the Lévy measure n^+ is given by (see [2] p. 214)

$$n^+(dt) = \frac{d}{dS(x)} \mathbf{P}_x(H_0 \in dt)\big|_{x=0+}$$

Proposition 8. *With the notation as above,*

$$\mathbf{P}(-g_0 \in dt) = \mathbf{P}(d_0 \in dt) = \frac{n^+(t,\infty)}{m(E)}\, dt, \qquad (16)$$

$$\mathbf{P}(V \in dv) = \frac{v}{m(E)}\, n^+(dv) \quad \textit{with } V := d_0 - g_0; \tag{17}$$

$$\mathbf{P}(d_0 \in dt, -g_0 \in ds)/dt\,ds = -\frac{1}{m(E)}\frac{d}{dv}n^+(v,\infty)\Big|_{v=t+s}. \tag{18}$$

Consequently, given V the random variable $-g_0$ (and also d_0) is uniformly distributed on $(0,V)$.

Proof: Formula (16) is obtained by inverting the corresponding Laplace transform. Indeed,

$$\mathbf{E}\,(\exp(-\alpha d_0)) = -\frac{\varphi_\alpha^+(0)}{m(E)\,\alpha\,\varphi_\alpha(0)}, \tag{19}$$

and from here the inversion can be done as in [2] p. 215, see also [13, 14]. Notice that for the right hand end point r of I it holds

$$\lim_{x\to r} \varphi_\alpha^+(x) = 0$$

since r is either natural or entrance-not-exit or regular and reflecting. Next consider formulae (17) and (18). Because

$$(-g_0, d_0) \overset{d}{=} (UV, (1-U)V),$$

where $V = d_0 - g_0$ and U is a uniformly on $(0,1)$ distributed random variable independent of V it follows (see [17] Proposition 2.4) that the density f_V of V is obtained from the density f_{g_0} of $-g_0$ by the rule

$$f_V(v) = v\,\frac{d}{dv}f_{g_0}(v)$$

yielding (17). Moreover, the joint density f_{g_0,d_0} of $(-g_0,d_0)$ is given by

$$f_{g_0,d_0}(u,v) = f_V(u+v)/(u+v)$$

and this is equivalent with (18). □

Remark 9. Let $\widehat{X}$ denote the diffusion obtained from $\{X_s : s \geq 0\}$ by killing at the first hitting time of 0, and $\widehat{p}(t;x,y)$ the transition density (with respect to the speed measure) of $\widehat{X}$. Then (see [2], p. 154)

$$\mathbf{P}_x(H_0 \in dt)/dt = \frac{d}{dS(y)}\widehat{p}(t;x,y)\Big|_{y=0+} =: \widehat{p}^+(t;x,0).$$

Hence, we may derive the density f_{g_0,d_0} by proceeding informally via the Chapman-Kolmogorov equation

$$\begin{aligned}
&\mathbf{P}(d_0 \in dt, -g_0 \in ds)/dt\,ds \\
&= \int_0^\infty \mu(dx)\widehat{p}^+(t;x,0)\widehat{p}^+(s;x,0) \\
&= \frac{1}{m(E)}\left(\frac{d}{dS(y_1)}\frac{d}{dS(y_2)}\int_I m(dx)\,\widehat{p}(t;x,y_1)\widehat{p}(s;x,y_2)\right)\Big|_{y_1,y_2=0+} \\
&= \frac{1}{m(E)}\left(\frac{d}{dS(y_1)}\frac{d}{dS(y_2)}\widehat{p}(t+s;y_1,y_2)\right)\Big|_{y_1,y_2=0+}.
\end{aligned}$$

3.3 Spectral representations for d_0 and V

In this section we show that the common distribution of $d_0, -g_0, I_0^+$, and I_0^- is a mixture of exponential distributions and the distribution of

$$V := d_0 - g_0 = I_0^+ + I_0^-$$

is a mixture of gamma distributions. The mixing measures are the same and closely related to the so called principal spectral measure of X, as defined in Krein's theory of strings, see [3, 7, 10]. Our starting point is the result in [6] which states that there exists a unique measure Δ such that

$$\nu(t) := n^+(dt)/dt = \int_0^\infty e^{-zt}\,\Delta(dz). \tag{20}$$

Moreover, Δ has the properties

$$\int_0^\infty \frac{\Delta(dz)}{z(z+1)} < \infty \tag{21}$$

and

$$\int_0^\infty \frac{\Delta(dz)}{z} = \infty. \tag{22}$$

We remark (cf. [6]) that (21) is equivalent with the defining property of the Lévy measure of a subordinator, i.e.,

$$\int_0^\infty (1 \wedge t)\, n^+(dt) < \infty.$$

For the property in (22) see [3] p. 82. and [11].

Proposition 10. *Let Δ be the measure introduced above. Then the measure*

$$\widetilde{\Delta}(dz) = \Delta(dz)/(m(E)\, z^2),$$

is a probability measure. Moreover,

$$\mathbf{P}(d_0 \in dt)/dt = \int_0^\infty z e^{-zt}\,\widetilde{\Delta}(dz), \tag{23}$$

and

$$\mathbf{P}(V \in dv)/dv = \int_0^\infty z^2 v\, e^{-zv}\,\widetilde{\Delta}(dz). \tag{24}$$

Proof: To prove that $\widetilde{\Delta}$ is a probability measure recall (see [15]) first that the Green kernel G_α of X has the property

$$\lim_{\alpha \searrow 0} \alpha G_\alpha(x,x) = 1/m(E), \quad \text{for all } x \in I. \tag{25}$$

Because (cf. (14))

$$G_\alpha(0,0) = -\varphi_\alpha(0)/\varphi_\alpha^+(0),$$

it follows from (25), (19), (20), and Fubini's theorem,

$$\begin{aligned} m(E) &= \lim_{\alpha \searrow 0} \frac{-\varphi_\alpha^+(0)}{\alpha\varphi_\alpha(0)} = \int_0^\infty n^+(t,\infty)\,dt \\ &= \int_0^\infty dt \int_t^\infty \nu(s)\,ds = \int_0^\infty dt \int_0^\infty \Delta(dz)\frac{e^{-zt}}{z} \\ &= \int_0^\infty \frac{\Delta(dz)}{z^2}, \end{aligned}$$

and, therefore, $\widetilde{\Delta}$ is a probability measure. Formulae (23) and (24) follow now from (17) in Proposition 8 and spectral representation (20). □

Remark 11. From the proof of Proposition 10 a new test for positive recurrence emerges: a recurrent diffusion X is positively recurrent if and only if

$$\int_0^\infty \frac{\Delta(dz)}{z^2} < \infty.$$

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References

[1] A. N. Borodin and P. Salminen. *Handbook of Brownian Motion – Facts and Formulae, 2nd edition.* Birkhäuser, Basel, Boston, Berlin, 2002.

[2] K. Itô and H.P. McKean. *Diffusion Processes and Their Sample Paths.* Springer Verlag, Berlin, Heidelberg, 1974.

[3] I. S. Kac and M. G. Krein. On the spectral functions of the string. *Amer. Math. Soc. Transl.*, II Ser 103:19–102, 1974.

[4] O. Kallenberg. Ballot theorems and sojourn laws for stationary processes. *Ann. Probab.*, 27(4):2011–2019, 1999.

[5] O. Kallenberg. *Foundations of Modern Probability, 2nd edition.* Springer, 2002.

[6] F. Knight. Characterization of the Lévy measures of inverse local times of gap diffusions. In *Seminar on Stochastic Processes*, pages 53–78, Boston, Bassel, Stuttgart, 1981. Birkhäuser.

[7] S. Kotani and S. Watanabe. Krein's spectral theory of strings and generalized diffusion processes. In J. Azéma, P. Meyer, and M. Yor, editors, *Functional Analysis and Markov Processes*, number 923 in Lecture Notes in Mathematics, Springer Verlag, 1981.

[8] M. Kozlova and P. Salminen. Diffusion local time storage. *Stoch. Proc. Appl.*, 114(2):211–229, 2004.

[9] M. Kozlova and P. Salminen. On occupation time identity for reflecting Brownian motion with drift. To appear in *Periodica Math. Hung.*, (Special volume in honor of Endre Csáki and Pál Révész), 2005.

[10] U. Küchler. On sojourn times, excursions and spectral measures connected with quasi diffusions. *J. Math. Kyoto Univ*, 26(3):403–421, 1986.

[11] U. Küchler and P. Salminen. On spectral measures of strings and excursions of quasi diffusions. In J. Azéma, P. A. Meyer, and M. Yor, editors, *Séminaire de Probabilités XXIII*, number 1372 in Springer Lecture Notes in Mathematics, pages 490–502, Springer Verlag, 1989.

[12] J. Pitman. Stationary excursions. In J. Azéma, P. A. Meyer, and M. Yor, editors, *Séminaire de Probabilités XXI*, Lecture notes in Math. 1247, pages 289–302. Springer, 1986.

[13] J. Pitman and M. Yor. On the lengths of excursions of some Markov processes. In J. Azéma, M. Émery, and M. Yor, editors, *Séminaire de Probabilités XXXI*, Lecture Notes in Math. 1655, pages 272–286. Springer, 1997.

[14] J. Pitman and M. Yor. Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches. *Bernoulli*, 9(1):1–24, 2003.

[15] P. Salminen. On the distribution of diffusion local time. *Statistics and Probability Letters*, 18:219–225, 1993.

[16] P. Salminen and I. Norros. On busy periods of the unbounded Brownian storage. *Queueing Systems*, 39:317–333, 2001.

[17] P. Salminen and P. Vallois. On first range times for linear diffusions. To appear in *J. Theor. Probab.*, 2005.

[18] H. G. Tucker. A generalization of the Glivenko–Cantelli theorem. *Ann. Math. Stat.*, 30:828–830, 1959.